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# Self-dual double cyclic codes over $\mathbb{Z}_2$

H. Movahedi[a], L. Pourfaraj[b,*]

[a] *Department of Mathematics, Faculty of Science, Central Tehran Branch, Islamic Azad University, Tehran, Iran.*

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**Abstract.** A *double cyclic code* (or *DC code*) of length $n = k + l$ over $\mathbb{Z}_2$ is a binary linear code, where any cyclic shift of the first $k$ coordinates and the last $l$ coordinates of a codeword is also a codeword. In this paper, we study the relationship between separability and self-duality of these codes. Also, we obtain the shadow code by determining the generator polynomials of the doubly even subcode of the self-dual code.



**Keywords:** Canonical projections, double cyclic codes, self-dual codes, shadow codes.

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## 1. Introduction and preliminaries

Linear codes and cyclic codes play important roles in algebraic coding theory and error-correcting codes. Many researchers have been interested in these codes over finite rings. Let $R = \mathbb{Z}_2$ and $R^n$ be the set of all $n-$tuples vectors over $R$. A *binary linear code* $\Psi$ of length $n$ is an additive subgroup of $R^n$. The *dual code* of $\Psi$ is $\Psi^{\perp} = \{\nu \in R^n |\ \upsilon \cdot \nu = 0 \pmod 2, \forall \upsilon \in \Psi\}$ where $\upsilon \cdot \nu = \sum_{i=0}^{n-1} \upsilon_i \nu_i \pmod 2$ is the *inner product* for $\upsilon = (\upsilon_0, \upsilon_1, \ldots, \upsilon_{n-1}), \nu = (\nu_0, \nu_1, \ldots, \nu_{n-1}) \in R^n$. A code $\Psi$ is *self-orthogonal*, if $\Psi \subseteq \Psi^{\perp}$. If $\Psi = \Psi^{\perp}$, $\Psi$ is *self-dual*. A binary linear code $\Psi$ is *cyclic*, if the cyclic shift of a codeword is also a codeword. Among the cyclic codes, the self-dual codes have been specially investigated. Sloane and Thompson have given the shortest length of nontrivial self-dual binary cyclic codes [7]. The construction of binary self-dual cyclic

*Corresponding author.
E-mail address: hoda.movahedi@chmail.ir (H. Movahedi); l.pourfaraj@iauctb.ac.ir (L. Pourfaraj).

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codes of a large minimal distance has been studied via Heijne and Top [5].
A *binary double cyclic code* (or *DC code* in short form) $\Psi$ of length $n = k+l$ is a binary linear code if $(\alpha_0, ..., \alpha_{k-1}|\alpha'_0, ..., \alpha'_{l-1}) \in \Psi$, then $(\alpha_{k-1}, \alpha_0, ..., \alpha_{k-2}|\alpha'_{l-1}, \alpha'_0, .., \alpha'_{l-2}) \in \Psi$. If $\Psi_k = \{(\alpha_0, .., \alpha_{k-1}) \in R^k |\ (\alpha_0, .., \alpha_{k-1}|\alpha'_0, ..., \alpha'_{l-1}) \in \Psi\}$ and $\Psi_l = \{(\alpha'_0, .., \alpha'_{l-1}) \in R^l |\ (\alpha_0, .., \alpha_{k-1}|\alpha'_0, .., \alpha'_{l-1}) \in \Psi\}$, then $\Psi_k$, $\Psi_l$ are some cyclic linear codes and $\Psi$ is a subcode of $\Psi_k \times \Psi_l$. Also, if a subcode $\Psi$ of $\Psi_k \times \Psi_l$ coincides with $\Psi_k \times \Psi_l$, we say that $\Psi$ is a *separable* code.
Abualrub et al. introduced that $\mathbb{Z}_2\mathbb{Z}_4-$additive cyclic codes are as $\mathbb{Z}_4-$sub modules and obtained a set of generator polynomials for these codes [1]. Also, properties for $\mathbb{Z}_2\mathbb{Z}_4-$additive cyclic codes were found [1, 3]. The ring $R[u]/(u^k - 1) \times R[u]/(u^l - 1)$ with $\eta(u) * (\alpha(u)|\alpha'(u)) = (\eta(u)\alpha(u)|\eta(u)\alpha'(u))$ where $\eta(u) \in R[u]$ is an $R[u]-$module which is denoted by $R_{k,l}$. Borges et al. studied $R-$DC codes as $R[u]-$submodules of $R_{k,l}$ and determined the structure of these codes [4]. In this note, we study and classify the self-dual $R-$DC codes for finding the shadow codes.

## 2. Self-duality

In this section, we investigate some properties of self-dual $R-$DC codes. We determine the relationship between self-duality and separability of these codes. If $\Psi$ is a self-dual $R-$DC code of length $n = k + l$, then $n$ is even and $r, s$ have the same parity. Clearly if $k$ and $l$ are odd, then $\Psi_k$ and $\Psi_l$ are not self-dual. The following lemma states when an $R-$DC code is self-orthogonal.

**Lemma 2.1** Let $\Psi$ be an $R-$DC code of length $n = k + l$. If $\Psi_k$ and $\Psi_l$ are self-orthogonal, then $\Psi_k \times \Psi_l$ and $\Psi$ are both self-orthogonal.

**Proof.** For any codewords

$$\alpha = (\alpha_0, .., \alpha_{k-1}|\alpha'_0, .., \alpha'_{l-1}), \beta = (\beta_0, .., \beta_{k-1}|\beta'_0, .., \beta'_{l-1}) \in \Psi_k \times \Psi_l,$$

we have $(\alpha_0, .., \alpha_{k-1}), (\beta_0, .., \beta_{k-1}) \in \Psi_k$ and $(\alpha'_0, .., \alpha'_{l-1}), (\beta'_0, .., \beta'_{l-1}) \in \Psi_l$. Since $\Psi_k$ and $\Psi_l$ are self-orthogonal, $\sum_{i=0}^{k-1} \alpha_i\beta_i = 0 \pmod 2$ and $\sum_{j=0}^{l-1} \alpha'_j\beta'_j = 0 \pmod 2$. Then $\alpha \cdot \beta = \sum_{i=0}^{k-1} \alpha_i\beta_i + \sum_{j=0}^{l-1} \alpha'_j\beta'_j = 0 \pmod 2$ and so $\Psi_k \times \Psi_l$ is self-orthogonal.
Also since $\Psi \subseteq \Psi_k \times \Psi_l$, for every $\alpha, \beta \in \Psi$ we have $\alpha \cdot \beta = 0 \pmod 2$. Therefore $\Psi$ is self-orthogonal. ■

**Lemma 2.2** If $\Psi$ is an $R-$DC code of length $n = k + l$, then $(\Psi_k)^\perp \times (\Psi_l)^\perp \subseteq \Psi^\perp$. Moreover if $\Psi$ is separable, then $(\Psi_k)^\perp \times (\Psi_l)^\perp = \Psi^\perp$.

**Proof.** Suppose that $(\beta_0, \beta_1, \ldots, \beta_{k-1}|\beta'_0, \beta'_1, \ldots, \beta'_{l-1}) \in (\Psi_k)^\perp \times (\Psi_l)^\perp$. Since $\Psi \subseteq \Psi_k \times \Psi_l$, for every $(\alpha_0, \alpha_1, \ldots, \alpha_{k-1}|\alpha'_0, \alpha'_1, \ldots, \alpha'_{l-1}) \in \Psi$, we have $\sum_{i=0}^{k-1} \alpha_i\beta_i = 0 \pmod 2$ and $\sum_{j=0}^{l-1} \alpha'_j\beta'_j = 0 \pmod 2$. Hence $\sum_{i=0}^{k-1} \alpha_i\beta_i + \sum_{j=0}^{l-1} \alpha'_j\beta'_j = 0 \pmod 2$. Therefore, $(\beta_0, \beta_1, \ldots, \beta_{r-1}|\beta'_0, \beta'_1, \ldots, \beta'_{s-1}) \in \Psi^\perp$ (i.e., $(\Psi_k)^\perp \times (\Psi_l)^\perp \subseteq \Psi^\perp$). Now, if $\Psi$ is separable, $dim(\Psi = \Psi_k \times \Psi_l) = dim(\Psi_k) + dim(\Psi_l)$. So

$$\begin{aligned} dim(\Psi^\perp) &= n - dim(\Psi) = k + l - dim(\Psi_k) - dim(\Psi_l) \\ &= [k - dim(\Psi_k)] + [l - dim(\Psi_l)] = dim((\Psi_k)^\perp) + dim((\Psi_l)^\perp). \end{aligned}$$

Therefore $(\Psi_k)^\perp \times (\Psi_l)^\perp = \Psi^\perp$. ■

**Proposition 2.3** Let $\Psi$ be an $R-$DC code of length $n = k + l$ and let $\Psi_k$ and $\Psi_l$ be self-dual, then
$(i)$ $\Psi_k \times \Psi_l$ is self-dual;
$(ii)$ $\Psi$ is separable if and only if $\Psi$ is self-dual;
$(iii)$ $\Psi$ is non separable if and only if $\Psi \subsetneqq \Psi^{\perp}$.

**Proof.** Since $\Psi_k$ and $\Psi_l$ are self-dual, $\Psi_k = (\Psi_k)^{\perp}$, $\Psi_l = (\Psi_l)^{\perp}$ and so

$$\Psi_k \times \Psi_l = (\Psi_k)^{\perp} \times (\Psi_l)^{\perp}. \tag{1}$$

$(i)$ We have $\Psi_k \times \Psi_l \subseteq (\Psi_k \times \Psi_l)^{\perp}$ by Lemma 2.1. Self-duality of $\Psi_k$ and $\Psi_l$ implies that $dim(\Psi_k) = \frac{k}{2} = dim((\Psi_k)^{\perp})$ and $dim(\Psi_l) = \frac{l}{2} = dim((\Psi_l)^{\perp})$. Then $dim(\Psi_k \times \Psi_l) = dim(\Psi_k) + dim(\Psi_l) = \frac{k+l}{2}$, so

$$dim((\Psi_k \times \Psi_l)^{\perp}) = n - dim(\Psi_k \times \Psi_l) = \frac{k+l}{2} = dim(\Psi_k \times \Psi_l).$$

Therefore $\Psi_k \times \Psi_l$ is self-dual.
$(ii)$ If $\Psi$ is separable, $\Psi = \Psi_k \times \Psi_l$. Then by $(i)$, $\Psi$ is self-dual.
Conversely, if $\Psi$ is self-dual, then $\Psi = \Psi^{\perp}$. By Lemma 2.2 and Eq. (1), we conclude that $\Psi \subseteq \Psi_k \times \Psi_l = (\Psi_k)^{\perp} \times (\Psi_l)^{\perp} \subseteq \Psi^{\perp} = \Psi$. Hence $\Psi = \Psi_k \times \Psi_l$.
$(iii)$ If $\Psi$ is non separable, $\Psi \subsetneqq \Psi_k \times \Psi_l = (\Psi_k)^{\perp} \times (\Psi_l)^{\perp}$ by Eq. (1) and $(\Psi_k)^{\perp} \times (\Psi_l)^{\perp} \subsetneqq \Psi^{\perp}$ by Lemma 2.2. Hence $\Psi \subsetneqq \Psi^{\perp}$.
Conversely, Suppose that $\Psi \subsetneqq \Psi^{\perp}$. If $\Psi$ is a separable code, $\Psi = \Psi^{\perp}$ by $(ii)$, which is a contradiction. ■

We denote the *reciprocal polynomial* of a polynomial $g(u)$ by $g^*(u) = u^{deg(g(u))}g(u^{-1})$.

**Corollary 2.4** Let $\Psi$ be a separable $R-$DC code of length $n = k + l$, then $\Psi_k$ and $\Psi_l$ are self-dual codes if and only if the code $\Psi$ is self-dual.

**Proof.** Suppose that $\Psi_k$ and $\Psi_l$ are self-dual codes, then $\Psi = \Psi_k \times \Psi_l$ is a self-dual code by Proposition 2.3.
Conversely, Suppose that $\Psi$ is self-dual and $\Psi_k$ or $\Psi_l$ are not self-dual codes. Then $\Psi_k \neq (\Psi_k)^{\perp}$ or $\Psi_l \neq (\Psi_l)^{\perp}$ and so $\Psi_k \times \Psi_l \neq (\Psi_k)^{\perp} \times (\Psi_l)^{\perp}$. On the other hand the self-duality of $\Psi$ implies that $\Psi_k \times \Psi_l = (\Psi_k)^{\perp} \times (\Psi_l)^{\perp}$ by Lemma 2.2. That is a contradiction. ■

**Corollary 2.5** If $\Psi$ is a separable self-dual $R-$DC code of length $n = k+l$ where $k$ and $l$ are even, then

$$\left[\begin{array}{cc|cc} I_{k/2} & M & 0 & 0 \\ 0 & 0 & N & I_{l/2} \end{array}\right]$$

is the generator matrix of $\Psi$ where $\left[I_{k/2}\ M_{k/2\times k/2}\right]$ and $\left[N_{l/2\times l/2}\ I_{l/2}\right]$ are generator matrices of $\Psi_k$ and $\Psi_l$ in standard forms, respectively.

**Proof.** Suppose that $\Psi = \Psi_k \times \Psi_l$. Since $\Psi_k$ and $\Psi_l$ are self-dual codes, we have

$dim(\Psi_k) = \frac{k}{2} = dim((\Psi_k)^{\perp})$ and $dim(\Psi_l) = \frac{l}{2} = dim((\Psi_l)^{\perp}))$. Then

$$\begin{bmatrix} I_{k/2} & M & | & 0 & 0 \\ 0 & 0 & | & N & I_{l/2} \end{bmatrix}.$$

is a generator matrix for $\Psi_k \times \Psi_l = \Psi$. ■

Consider $(g(u), h(u))$ as the greatest common divisor of polynomials $g(u) \neq 0$ and $h(u) \neq 0$. If $\Psi = \langle (p(u)|0), (q(u)|r(u)) \rangle$ is a non separable self-dual $R-$DC code, then
$(i)$ $p(u) = \dfrac{(u^k - 1)}{(p(u), q(u))^*}$,
$(ii)$ $r(u) = \dfrac{(u^l - 1)(p(u), q(u))^*}{r^*(u)p^*(u)}$,
$(iii)$ $q(u) = \dfrac{u^k - 1}{p^*(u)}\mu(u)$ where $deg(\mu(u)) < deg(p(u)) - deg((p(u), q(u)))$ and
$\mu(u)\mu^*(u) = u^{t-deg(r(u))+deg((p(u),q(u)))+deg(\mu(u)))} \pmod{\dfrac{p^*(u)}{(p(u), q(u))^*}}$ for the least common multiple $t$ of $k$ and $l$ (see [4], Theorem 4.20). This notification and the following conditions are useful for giving some examples.

**Proposition 2.6** If $\Psi = \langle (p(u)|0), (q(u)|r(u)) \rangle$ is a non separable self-dual $R-$DC code of even length $n = k + l$, then
$(i)$ $\Psi_k$ and $\Psi_l$ are not self-dual codes;
$(ii)$ $deg(p(u)) > \lfloor \frac{k}{2} \rfloor$ and $wt(p(u)) = 0 \pmod 2$;
$(iii)$ $wt(q(u)) = wt(r(u)) \pmod 2$.

**Proof.** $(i)$ Clearly if $k$ and $l$ are odd, then $\Psi_k$ and $\Psi_l$ are not self-dual. Suppose that $k$ and $l$ are even, since $\Psi$ is a self-dual code, we show that $\Psi_k$ is self-dual if and only if $\Psi_l$ is self-dual. $\Psi_k = \langle (p(u), q(u)) \rangle$ is self-dual if and only if

$$(p(u), q(u)) = \frac{u^k - 1}{(p(u), q(u))^*} = p(u)$$

if and only if

$$r(u) = \frac{(u^l - 1)(p(u), q(u))^*}{r^*(u)p^*(u)} = \frac{u^l - 1}{r^*(u)}$$

if and only if $\Psi_l = \langle r(u) \rangle$ is self-dual (see [7]). Therefore by Proposition 2.3, $\Psi$ is separable, which is a contradiction.
$(ii)$ Suppose that $\alpha = (\alpha_0, \ldots, \alpha_{r-1}), \beta = (\beta_0, \ldots, \beta_{r-1}) \in \langle p(u) \rangle$ and $\gamma = (\alpha|0), \gamma' = (\beta|0) \in \langle (p(u)|0) \rangle$, then $\gamma, \gamma' \in \Psi$. Since $\Psi$ is self-dual, $\gamma \cdot \gamma' = \alpha \cdot \beta = 0 \pmod 2$, i.e., $\langle p(u) \rangle \subseteq \langle p(u) \rangle^{\perp}$. Hence $dim(\langle p(u) \rangle^{\perp}) = k - dim(\langle p(u) \rangle) \geqslant dim(\langle p(u) \rangle)$ and so $k \geqslant 2dim(\langle p(u) \rangle) = 2(k - deg(p(u)))$, thus $deg(p(u)) \geqslant \lceil \frac{k}{2} \rceil$.
We know that $dim(\Psi) = k + l - deg(p(u)) - deg(r(u))$ and $dim(\Psi) = \dfrac{k+l}{2}$. On the other hand $\Psi_l$ is not self-dual code, then $deg(r(u)) \neq \dfrac{l}{2}$ and so $deg(p(u)) \neq \dfrac{k}{2}$ where

$k, l$ are even. Therefore $deg(p(u)) > \lfloor \frac{k}{2} \rfloor$. Also self-duality implies that $wt((p(u)|0)) = 0 \pmod 2$, then $wt(p(u)) = 0 \pmod 2$.
(iii) By self-duality of $\Psi$, we have $wt((q(u)|r(u))) = 0 \pmod 2$. Then $wt(r(u)) = wt(q(u)) \pmod 2$. ■

**Proposition 2.7** If $\Psi = \langle (p(u)|0), (q(u)|r(u)) \rangle$ is a non separable self-dual $R-$DC code of even length $n = k + l$ and $p(u) \neq 0$, then $wt(q(u)) = wt(r(u)) = 0 \pmod 2$.

**Proof.** The code $\Psi$ is self-dual, then $(p(u)|0) \cdot (q(u)|r(u)) = 0 \pmod 2$ and so $p(u) \cdot q(u) = 0 \pmod 2$. Since $wt(p(u)) = 0 \pmod 2$, then $wt(q(u)) = 0 \pmod 2$ or $p(u) = 0$. By Proposition 2.6, $(iii)$, we have $wt(r(u)) = wt(q(u)) = 0 \pmod 2$ for $p(u) \neq 0$. ■

**Proposition 2.8** If $\Psi = \langle (p(u)|0), (q(u)|r(u)) \rangle$ is a non separable self-dual $R-$DC code of even length $n = k + l$ and $p(u) = 0$, then
$(i)$ $l \geqslant k$;
$(ii)$ $wt(q(u)) = wt(r(u)) = 1 \pmod 2$;
$(iii)$ If $k$ and $l$ are odd, then $k = l$.

**Proof.** $(i)$ $p(u) = 1 + u^k = 0$ implies that $(p(u), q(u)) = 1$ and so

$$r(u)r^*(u) = \frac{u^l - 1}{u^k - 1}. \tag{2}$$

We have $r(u) \in R[u]/(u^l - 1)$, then $l \geqslant k$.
$(ii)$ Suppose that $wt(q(u)) = 0 \pmod 2$, then

$$\begin{aligned} q(u) &= u^{j_1} + u^{j_2} + ... + u^{j_{2k'-1}} + u^{j_{2k'}} \\ &= (1+u)\Big[u^{j_1}(1 + u + ... + u^{j_2 - j_1 - 1}) + \\ &\quad ... + u^{j_{2k'-1}}(1 + u + ... + u^{j_{2k'} - j_{2k'-1} - 1})\Big] \end{aligned}$$

where $j_1, j_2, ..., j_{2k'-1}, j_{2k'}, k' \in \mathbb{Z}^+$ such that $0 \leqslant j_1 < j_2 < ... < j_{2k'-1} < j_{2k'} < deg(p(u)) = k$. Hence $(p(u), q(u)) \neq 1$ and so $p(u) \neq 0$, this is a contradiction. Therefore $wt(r(u)) = wt(q(u)) = 1 \pmod 2$ by Proposition 2.6, $(iii)$.
$(iii)$ Let $k$ and $l$ be odd and $k \neq l$, by $(i)$, $k = 2k' + 1 < l$. By Eq. (2), we conclude that $l = (2k' + 1)2k''$ for $k'' \in \mathbb{Z}^+$ which is a contradiction. Then $k = l$ for $k, l$ odd. ■

## 3. Types of self-dual codes

Consider a code $\Psi$, if the Hamming weights of all codewords are even, $\Psi$ is *even* and otherwise is *odd*. For even code $\Psi$, if the Hamming weights of all codewords are divisible by 4, $\Psi$ is *doubly-even* and it is *singly-even*, if there exists at least one codeword $\alpha$ such that $wt(\alpha) = 2 \pmod 4$. The *Euclidean weight* $wt_E(\alpha)$ of a vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{2k'}^n$ is $\sum_{i=1}^n min\left\{\alpha_i^2, (2k' - \alpha_i)^2\right\}$. For a self-dual code $\Psi$, if the Euclidean weights of all codewords are multiple of $4k'$, then $\Psi$ is *Type II*, otherwise is *Type I*. If $k' = 1$, the Euclidean weight is equivalent to the Hamming weight. Also the self-dual $R-$singly even codes are equivalent to Type I codes and the self-dual $R-$doubly even codes are equivalent to Type II codes.

**Lemma 3.1** Let $\Psi$ be an $R-$DC code of length $n = r + s$.
$(i)$ If $\Psi_k$ and $\Psi_l$ are even codes, then $\Psi_k \times \Psi_l$ and $\Psi$ are even codes;
$(ii)$ If $\Psi_k$ and $\Psi_l$ are singly even codes, then $\Psi_k \times \Psi_l$ is a singly even code;
$(iii)$ If $\Psi_k$ and $\Psi_l$ are doubly even codes, then $\Psi_k \times \Psi_l$ and $\Psi$ are doubly even codes.

**Proof.** $(i)$ For every $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{k-1}|\alpha'_0, \alpha'_1, \ldots, \alpha'_{l-1}) \in \Psi_k \times \Psi_l$, we have $(\alpha_0, \alpha_1, \ldots, \alpha_{k-1}) \in \Psi_k$ and $(\alpha'_0, \alpha'_1, \ldots, \alpha'_{l-1}) \in \Psi_s$. Since $\Psi_k$ and $\Psi_l$ are even codes, $\sum_{i=0}^{k-1} \alpha_i = 0 \pmod 2$, $\sum_{j=0}^{l-1} \alpha'_j = 0 \pmod 2$ and so $wt(\alpha) = (\sum_{i=0}^{k-1} \alpha_i + \sum_{j=0}^{l-1} \alpha'_j) = 0 \pmod 2$, then $\Psi_k \times \Psi_l$ is even. On the other hand, since $\Psi \subseteq \Psi_k \times \Psi_l$, $wt(\alpha) = 0 \pmod 2$ for every $\alpha \in \Psi$. We conclude that $\Psi$ is even.
$(ii)$ Let $\Psi_k$ and $\Psi_l$ be singly even, then $\Psi_k$, $\Psi_l$ and $\Psi_k \times \Psi_l$ are even. Also there exists $(\alpha_0, \ldots, \alpha_{r-1}) \in \Psi_k$ such that $\sum_{i=0}^{k-1} \alpha_i = 2 \pmod 4$. For $\alpha = (\alpha_0, \ldots, \alpha_{k-1}|0, \ldots, 0) \in \Psi_k \times \Psi_l$, we have $wt(\alpha) = \sum_{i=0}^{k-1} \alpha_i = 2 \pmod 4$, thus $\Psi_k \times \Psi_l$ is singly even.
$(iii)$ Let $\Psi_k$ and $\Psi_l$ be doubly even, then $\Psi_k$, $\Psi_l$ and $\Psi_k \times \Psi_l$ are even. Also for every $= (\alpha_0, \ldots, \alpha_{k-1}|\alpha'_0, \ldots, \alpha'_{l-1}) \in \Psi_k \times \Psi_l$, we have $(\alpha_0, \ldots, \alpha_{l-1}) \in \Psi_k$ and $(\alpha'_0, \ldots, \alpha'_{l-1}) \in \Psi_l$. Then $\sum_{i=0}^{k-1} \alpha_i = 0 \pmod 4$, $\sum_{j=0}^{l-1} \alpha'_j = 0 \pmod 4$. So $wt(\alpha) = (\sum_{i=0}^{k-1} \alpha_i + \sum_{j=0}^{l-1} \alpha'_j) = 0 \pmod 4$. Then $\Psi_k \times \Psi_l$ is doubly even. On the other hand, if $\alpha \in \Psi$, $wt(\alpha) = 0 \pmod 4$ since $\Psi \subseteq \Psi_k \times \Psi_l$. Therefore $\Psi$ is doubly even. ■

Notice that if $\Psi_k$ and $\Psi_l$ are self-dual codes, then $\Psi_k$ and $\Psi_l$ are Type I (see [5], Lemma 4.1). Hence $\Psi_k \times \Psi_l$ is Type I by Proposition 2.3 and Lemma 3.1. Moreover, if $\Psi_k$ and $\Psi_l$ are self-dual, then the $R-$DC code $\Psi$ is Type I if and only if $\Psi$ is separable. Also, if $\Psi$ is a separable self-dual $R-$DC code of length $n = k + l$ where $k, l$ are even, then $\Psi$ is Type I.

**Proposition 3.2** Let $\Psi$ be an $R-$DC code of length $n = k + l$, where $n$ is a multiple of 8. If $\Psi$ is Type II, then $\Psi$ is non separable.

**Proof.** Since $\Psi$ is Type II, $\Psi$ is self-dual. If $\Psi$ is separable, then by Corollary 2.4 and Lemma 3.1, $\Psi$ is Type I. This is a contradiction. ■

**Proposition 3.3** Let $\Psi = \langle (p(u)|0), (q(u)|r(u)) \rangle$ be a non separable self-dual $R-$DC code of even length $n = k + l$.
$(i)$ If $wt(p(u)) = 2 \pmod 4$ or $wt((q(u)|r(u))) = 2 \pmod 4$, then $\Psi$ is Type I;
$(ii)$ If $wt(p(u)) = 0 \pmod 4$ and $wt((q(u)|r(u))) = 0 \pmod 4$, then $\Psi$ is Type II.

**Proof.** $(i)$ If $wt(p(u)) = 2 \pmod 4$, we have $wt((p(u)|0)) = 2 \pmod 4$. Then the self-dual code $\Psi$ with $wt((p(u)|0)) = 2 \pmod 4$ or $wt((q(u)|r(u))) = 2 \pmod 4$ is Type I.
$(ii)$ If $wt(p(u)) = 0 \pmod 4$, we have $wt((p(u)|0)) = 0 \pmod 4$. Since $\Psi$ is self-dual, for every $c, c' \in \Psi$ we have $\alpha \cdot \alpha' = 0 \pmod 2$ and so $wt(\alpha + \alpha') = wt(\alpha) + wt(\alpha') = 0 \pmod 4$. Therefore the self-dual code $\Psi$ is Type II. ■

***Example* 3.4** $(i)$ Suppose that $k = 2$, $\Psi_k = \langle 1 + u \rangle = \langle 11 \rangle$ and $l = 4$, $\Psi_s = \langle 1 + u^2 \rangle = \langle 1010, 0101 \rangle$. It is clear that $\Psi_k$ and $\Psi_l$ are trivial Type I cyclic codes. Then the separable code

$$\Psi = \Psi_k \times \Psi_l = \langle (1 + u|0), (0|1 + u^2) \rangle = \langle (11|0000), (00|1010), (00|0101) \rangle.$$

is Type I.
$(ii)$ Consider the trivial binary self-dual cyclic code $\Psi = \langle 1 + u^{n/2} \rangle$.
Suppose that $k = l = n/2$, $p(u) = 1 + u^k = 0$, $q(u) = 1$ and $r(u) = 1$. Then $\Psi = \langle (q(u)|r(u)) \rangle$ is a non separable Type I $R-$DC code. In addition, $\Psi_k = \langle (1 + u^k, 1) \rangle = \langle 1 \rangle$

Table 1. Types of the non separable self-dual $R-$DC codes of length $n = k + l$ where $1 \leqslant k, l \leqslant 10$.

| $k$ | $p(u)$ | $l$ | $r(u)$ | $q(u)$ | Type |
|---|---|---|---|---|---|
| 1 | $1+u=0$ | 1 | 1 | 1 | I |
| 2 | $1+u^2=0$ | 2 | 1 | $u^i$ $(i \in \mathbb{Z}_2)$ | I |
| | | 6 | $1+u+u^2$ | $u^i$ $(i \in \mathbb{Z}_2)$ | II |
| | | 10 | $1+u+u^2$ $+u^3+u^4$ | $u^i$ $(i \in \mathbb{Z}_2)$ | I |
| 3 | $1+u^3=0$ | 3 | 1 | $u^i$ $(i \in \mathbb{Z}_3)$ | I |
| 4 | $1+u^4=0$ | 4 | 1 | $u^i$ $(i \in \mathbb{Z}_4)$ | I |
| | $(1+u)^3$ | 4 | $1+u$ | $(1+u)u^i$ $(i \in \mathbb{Z}_2)$ | II |
| | | 8 | $(1+u)^3$ | $(1+u)u^i$ $(i \in \mathbb{Z}_2)$ | I |
| 5 | $1+u^5=0$ | 5 | 1 | $u^i$ $(i \in \mathbb{Z}_5)$ | I |
| 6 | $1+u^6=0$ | 6 | 1 | $u^i$ $(i \in \mathbb{Z}_6)$ | I |
| | $(1+u)(1+u+u^2)^2$ | 6 | $1+u$ | $(1+u)u^i$ $(i \in \mathbb{Z}_4)$ | I |
| | $(1+u)^2(1+u+u^2)$ | 4 | $1+u$ | - | - |
| | | 8 | $(1+u)^3$ | - | - |
| 7 | $1+u^7=0$ | 7 | 1 | $u^i$ $(i \in \mathbb{Z}_7)$ | I |
| | $(1+u)(1+u+u^3)$ | - | - | - | - |
| | $(1+u)(1+u^2+u^3)$ | - | - | - | - |
| 8 | $1+u^8=0$ | 8 | 1 | $u^i$ $(i \in \mathbb{Z}_8)$ | I |
| | $(1+u)^7$ | 8 | $1+u$ | $(1+u)u^i$ $(i \in \mathbb{Z}_6)$ | II |
| | $(1+u)^6$ | 8 | $(1+u)^2$ | $(1+u)^2u^i$ $(i \in \mathbb{Z}_4)$ | II |
| | $(1+u)^5$ | 4 | $1+u$ | $(1+u)^3u^i$ $(i \in \mathbb{Z}_2)$ | I |
| | | 8 | $(1+u)^3$ | $(1+u)^3u^i$ $(i \in \mathbb{Z}_2)$ | II |
| 9 | $1+u^9=0$ | 9 | 1 | $u^i$ $(i \in \mathbb{Z}_9)$ | I |
| | $(1+u)(1+u^3+u^6)$ | - | - | - | - |
| 10 | $1+u^{10}=0$ | 10 | 1 | $u^i$ $(i \in \mathbb{Z}_{10})$ | I |
| | $(1+u)\times$ $(1+u+u^2+u^3+u^4)^2$ | 10 | $1+u$ | $(1+u)u^i$ $(i \in \mathbb{Z}_8)$ | I |
| | $(1+u)^2\times$ $(1+u+u^2+u^3+u^4)$ | 4 | $1+u$ | - | - |
| | | 8 | $(1+u)^3$ | - | - |

and $\Psi_l = \langle 1 \rangle$ are not self-dual.
$(iii)$ Suppose that $k = 12$, $l = 6$, then the code

$$\Psi = \langle (1+u+u^2+u^6+u^7+u^8|0), (1+u^2+u^3+u^5|1+u) \rangle,$$

is a non separable $R-$DC code of Type I. On the other hand $\Psi_l = \langle 1+u \rangle$ and $\Psi_k = \langle (1+u)^2(1+u+u^2) \rangle$ are not self-dual codes.
$(v)$ Consider $k = 4$, $l = 12$. The non separable $R-$DC code

$$\Psi = \langle (1+u+u^2+u^3|0), (1+u|1+u+u^2+u^3+u^4+u^5) \rangle$$

is Type II, although $\Psi_k = \langle (1+u) \rangle$ and $\Psi_l = \langle (1+u)(1+u^2+u^4) \rangle$ are not self-dual.

## 4. Shadow codes

Let $\Psi$ be a self-dual $R-$code, then the subset $\Psi_0$ of $\Psi$ consisting of all doubly even codewords is a subcode of $\Psi$ (see [6]). Suppose $\Psi_2 := \Psi - \Psi_0$. The set $S(\Psi)$ consists of all vectors $\alpha$ such that $\alpha \cdot \beta = 0$ for all $\beta \in \Psi_0$ and $\alpha \cdot \beta = 1$ for all $\beta \in \Psi_2$ is called a *shadow code*. For a Type II code $\Psi$, we have $\Psi_2 = 0$ and $S(\Psi) = \Psi$. For a Type I code $\Psi$, we have $S(\Psi) = \Psi_0^{\perp} - \Psi$ and so $S(\Psi)$ is not a subcode of $\Psi_0^{\perp}$. We determine the generators of the doubly even subcode of a self-dual $R-$DC code based on the generators of the code to find the shadow code.

**Lemma 4.1** Let $\Psi = \langle (p(u)|0), (q(u)|r(u)) \rangle$ be an $R-$DC code of length $n = k + l$. Consider the subcode $\Psi' = \langle (p'(u)|0), (q'(u)|r'(u)) \rangle$ of $\Psi$, then $p(u) \mid p'(u)$ and $r(u) \mid r'(u)$. Moreover, if $\Psi$ is a separable code, then $p(u) \mid q'(u)$.

**Proof.** Since $(p'(u)|0) \in \Psi' \subseteq \Psi$, there are $\gamma_1(u), \nu_1(u) \in R[u]$ such that

$$\begin{aligned}(p'(u)|0) &= \nu_1(u) * (p(u)|0) + \gamma_1(u) * (q(u)|r(u)) \\ &= (\nu_1(u)p(u) + \gamma_1(u)q(u)|\gamma_1(u)r(u)).\end{aligned}$$

So $p'(u) = \nu_1(u)p(u) + \gamma_1(u)q(u)$ and $\gamma_1(u)r(u) = 0$ (i.e., $\gamma_1(u) = 0$ or $(u^l - 1) \mid \gamma_1(u)r(u)$). We have $p(u) \mid \dfrac{u^l - 1}{r(u)} q(u)$ (see [4], Proposition 3.6) and so $p(u) \mid \gamma_1(u)q(u)$. Hence

$$p(u) \mid \nu_1(u)p(u) + \gamma_1(u)q(u) = p'(u).$$

Since $(q'(u)|r'(u)) \in C' \subseteq C$, there are $\gamma_2(u), \nu_2(u) \in R[u]$ such that

$$\begin{aligned}(q'(u)|r'(u)) &= \nu_2(u) * (p(u)|0) + \gamma_2(u) * (q(u)|r(u)) \\ &= (\nu_2(u)p(u) + \gamma_2(u)q(u)|\gamma_2(u)r(u)).\end{aligned}$$

Thus $r'(u) = \gamma_2(u)r(u)$.
If $\Psi$ is separable, we conclude that

$$(q'(u)|r'(u)) \in \Psi' \subseteq \Psi = \langle (p(u)|0), (0|r(u)) \rangle.$$

Hence there are $\gamma_3(u), \nu_3(u) \in R[u]$ such that

$$(q'(u)|r'(u)) = \nu_3(u) * (p(u)|0) + \gamma_3(u) * (0|r(u)) = (\nu_3(u)p(u)|\gamma_3(u)r(u)).$$

Therefore $q'(u) = \nu_3(u)p(u)$. ■

We know that for an $R-$DC code $\Psi$ of Type I, $|\Psi| = 2^{(k+l)/2}$. Therefore the generator polynomials of the subcode $\Psi_0$ consisting of all doubly even codewords and the generator polynomials of its dual, help us to find the shadow code, when $k, l$ are large.

**Proposition 4.2** Let $\Psi_0 = \langle (p_0(u)|0), (q_0(u)|r_0(u)) \rangle$ be the subcode of an $R-$DC code $\Psi = \langle (p(u)|0), (q(u)|r(u)) \rangle$ of Type I and even length $n = k + l$ such that $\Psi_0$ consists of all doubly even codewords.
$(i)$ If $wt(p(u)) = 0 \pmod 4$, then $p_0(u) = p(u)$ and $r_0(u) = (u + 1)r(u)$;
$(ii)$ If $wt(p(u)) = 2 \pmod 4$, then $p_0(u) = (u + 1)p(u)$ and $r_0(u) = r(u)$.

**Proof.** We have $|\Psi_0| = \frac{|\Psi|}{2} = \frac{2^{dim(\Psi)}}{2} = 2^{dim(\Psi)-1}$ (see [2], Lemma 2.3) and so $dim(\Psi_0) = dim(\Psi) - 1$. Then

$$dim(\Psi_0) = n - deg(p_0(u)) - deg(r_0(u)) = n - deg(p(u)) - deg(r(u)) - 1.$$

Therefore

$$deg(p_0(u)) + deg(r_0(u)) = deg(p(u)) + deg(r(u)) + 1. \tag{3}$$

$(i)$ If $wt(p(u)) = 0 \pmod 4$, then $(p(u)|0) \in \Psi_0$. Since $\Psi$ is Type I,

$$(q(u)|r(u)) \notin \Psi_0.$$

Hence $r_0(u) \neq r(u)$. Lemma 4.1 and Eq. (3) imply that $p_0(u) = p(u)$ and $r_0(u) = (u+1)r(u)$.
$(ii)$ If $wt(p(u)) = 2 \pmod 4$, then $(p(u)|0) \notin \Psi_0$ and so $p_0(u) \neq p(u)$. By Lemma 4.1 and Eq. (3), we have $p_0(u) = (u+1)p(u)$ and $r_0(u) = r(u)$. ■

**Theorem 4.3** Let $\Psi_0 = \langle (p_0(u)|0), (q_0(u)|r_0(u)) \rangle$ be the subcode of a separable self-dual $R-$DC code $\Psi = \langle (p(u)|0), (q(u)|r(u)) \rangle$ of even length $n = k + l$ such that $\Psi_0$ consists of all doubly even codewords with the dual code $\Psi_0^{\perp} = \langle (\bar{p}_0(u)|0), (\bar{q}_0(u)|\bar{r}_0(u)) \rangle$, then
$(i)$ $p_0(u) = (u+1)p(u)$, $r_0(u) = r(u)$, $q_0(u) = p(u)$ and $\Psi_0$ is non separable;
$(ii)$ $\bar{p}_0(u) = p(u)$, $\bar{r}_0(u) = \frac{r(u)}{u+1}$, $\bar{q}_0(u) = \frac{p(u)}{u+1}$ and $\Psi_0^{\perp}$ is non separable.

**Proof.** The separable self-dual $R-$DC code $\Psi$ is a code of Type I (see Sect. 3).
$(i)$ By Corollary 2.4, $\Psi_k = \langle p(u) \rangle$ is self-dual and so $\Psi_k$ is Type I (see [5], Lemma 4.1), i.e., $wt(p(u)) = 2 \pmod 4$. By Proposition 4.2, $p_0(u) = (u+1)p(u)$ and $r_0(u) = r(u)$. By Lemma 4.1, $p(u) \mid q_0(u)$, then $deg(p(u)) \leqslant deg(q_0(u))$. Since $deg(q_0(u)) < deg(p_0(u)) = deg(p(u)) + 1$, we have $deg(p(u)) \leqslant deg(q_0(u)) < deg(p(u)) + 1$. Hence $deg(p(u)) = deg(q_0(u))$ and so $p(u) = q_0(u)$. Therefore $\Psi_0$ is non separable.
$(ii)$ By Corollary 2.4, $\Psi_k = \langle p(u) \rangle$ and $\Psi_l = \langle r(u) \rangle$ are self-dual. Then $p(u)p^*(u) = u^k - 1$ and $r(u)r^*(u) = u^l - 1$ (see [7]). Hence we have

$$\bar{p}_0(u) = \frac{u^k - 1}{(p_0(u), q_0(u))^*} = \frac{u^k - 1}{p^*(u)} = p(u),$$

$$\bar{r}_0(u) = \frac{(u^l - 1)(p_0(u), q_0(u))^*}{r_0^*(u)p_0^*(u)} = \frac{u^l - 1}{(u+1)r^*(u)} = \frac{r(u)}{u+1}.$$

There exists $\gamma(u) \in R[u]$ such that

$$\bar{q}_0(u) = \frac{u^k - 1}{p_0^*(u)}\gamma(u) = \frac{u^k - 1}{(u+1)p^*(u)}\gamma(u) = \frac{p(u)}{u+1}\gamma(u),$$

then $\bar{p}_0(u) = p(u) \mid (u+1)\bar{q}_0(u)$ and so $deg(\bar{p}_0(u)) \leqslant deg((u+1)\bar{q}_0(u))$. Since $deg(\bar{q}_0(u)) < deg(\bar{p}_0(u))$, we have $\bar{q}_0(u) = \frac{p(u)}{u+1}$. Therefore $\Psi_0^{\perp}$ is non separable. ■

Now by applying Proposition 3.3, the following cases for the subcodes $\Psi_0$ of the non separable $R-$DC codes $\Psi$ of Type I are considered.

**Theorem 4.4** Let $\Psi_0 = \langle (p_0(u)|0), (q_0(u)|r_0(u))\rangle$ be the subcode of a non separable $R-$DC code $\Psi = \langle (p(u)|0), (q(u)|r(u))\rangle$ of Type I and even length $n = k + l$ such that $\Psi_0$ consists of all doubly even codewords.
$(i)$ If $wt(p(u)) = 0 \pmod 4$, $wt((q(u)|r(u))) = 2 \pmod 4$ and $deg(q(u))+1 < deg(p(u))$, then $p_0(u) = p(u)$, $r_0(u) = (u + 1)r(u)$ and $q_0(u) = (u + 1)q(u)$;
$(ii)$ If $wt(p(u)) = 2 \pmod 4$ and $wt((q(u)|r(u))) = 0 \pmod 4$, then $p_0(u) = (u+1)p(u)$, $r_0(u) = r(u)$ and $q_0(u) = q(u)$;
$(iii)$ If $wt(p(u)) = 2 \pmod 4$ and $wt((q(u)|r(u))) = 2 \pmod 4$, then $p_0(u) = (u + 1)p(u)$, $r_0(u) = r(u)$ and $q_0(u) = p(u) + q(u)$.

**Proof.** $(i)$ By Proposition 4.2, $p_0(u) = p(u)$ and $r_0(u) = (u + 1)r(u)$. We know that $(q_0(u)|r_0(u)) \in \Psi$. So, there are $\nu_1(u), \gamma_1(u) \in R[u]$ such that

$$\begin{aligned}(q_0(u)|r_0(u)) &= \nu_1(u) * (p(u)|0) + \gamma_1(u) * (q(u)|r(u))\\ &= (\nu_1(u)p(u) + \gamma_1(u)q(u)|\gamma_1(u)r(u)).\end{aligned}$$

Hence $\gamma_1(u) = \dfrac{r_0(u)}{r(u)} = u + 1$ and so $q_0(u) = \nu_1(u)p(u) + (u + 1)q(u)$. Since $deg(q_0(u)) < deg(p_0(u)) = deg(p(u))$ and $deg(q(u)) + 1 < deg(p(u))$, $\nu_1(u) = 0$. Then $q_0(u) = (u + 1)q(u)$.
$(ii)$ By Proposition 4.2, $p_0(u) = (u+1)p(u)$ and $r_0(u) = r(u)$. So there are $\nu_2(u), \gamma_2(u) \in R[u]$ such that

$$\begin{aligned}(q_0(u)|r_0(u)) &= \nu_2(u) * (p(u)|0) + \gamma_2(u) * (q(u)|r(u))\\ &= (\nu_2(u)p(u) + \gamma_2(u)q(u)|\gamma_2(u)r(u)).\end{aligned}$$

Hence $\gamma_2(u) = \dfrac{r_0(u)}{r(u)} = 1$ and so $q_0(u) = \nu_2(u)p(u) + q(u)$. Since $deg(q_0(u)) < deg(p_0(u)) = deg(p(u)) + 1$ and $deg(q(u)) < deg(p(u))$, $\nu_2(u) \in \{0, 1\}$. By self-duality of $\Psi$, $(p(u)|0) \cdot (q(u)|r(u)) = p(u) \cdot q(u) = 0 \pmod 2$ and so

$$wt(p(u) + q(u)) = wt(p(u)) + wt(q(u)) \pmod 4.$$

If $wt(q(u)) = 2 \pmod 4$ and $wt(r_0(u)) = wt(r(u)) = 2 \pmod 4$, then $wt(q_0(u)) = 2 \pmod 4$ and $wt(p(u) + q(u)) = 0 \pmod 4$.
If $wt(q(u)) = 0 \pmod 4$ and $wt(r_0(u)) = wt(r(u)) = 0 \pmod 4$, then $wt(q_0(u)) = 0 \pmod 4$ and $wt(p(u) + q(u)) = 2 \pmod 4$.
Therefore $q_0(u) \neq p(u) + q(u)$ and we conclude that $q_0(u) = q(u)$ by this fact that $\nu_2(u) = 0$.
$(iii)$ Similar (ii): $p_0(u) = (u + 1)p(u)$ and $r_0(u) = r(u)$ and there exists $\nu_3(u) \in \{0, 1\}$ such that $q_0(u) = \nu_3(u)p(u) + q(u)$.
If $wt(q(u)) = 2 \pmod 4$ and $wt(r_0(u)) = wt(r(u)) = 0 \pmod 4$, then $wt(q_0(u)) = 0 \pmod 4$.
If $wt(q(u)) = 0 \pmod 4$ and $wt(r_0(u)) = wt(r(u)) = 2 \pmod 4$, then $wt(q_0(u)) = 2 \pmod 4$.
Therefore $q_0(u) \neq q(u)$ and so $\nu_3(u) = 1$, i.e., $q_0(u) = p(u) + q(u)$. ■

***Example* 4.5** (*i*) Consider Example 3.4,(i): The separable self-dual code

$$\Psi = \Psi_k \times \Psi_l = \langle (1+u|0), (0|1+u^2)\rangle = \langle (11|0000), (00|1010), (00|0101)\rangle.$$

Hence $\Psi_0 = \langle (p(u)(u+1)|0), (p(u)|r(u))\rangle = \langle (1+u|1+u^2)\rangle$,

$$\begin{aligned}\Psi_0^{\perp} &= \langle (p(u)|0), (\frac{p(u)}{u+1}|\frac{r(u)}{u+1})\rangle = \langle (1+u|0), (1|1+u)\rangle \\ &= \langle (11|0000), (10|1100), (01|0110), (10|0011)\rangle, \\ S(\Psi) = \Psi_0^{\perp} - \Psi &= \{(10|1100), (01|0110), (10|0011), (10|1010), \\ &\qquad (01|1100), (10|0110), (01|0011), (01|1010)\}.\end{aligned}$$

(*ii*) Consider Example 3.4,(iii): The non separable Type I code

$$\Psi = \langle (1+u+u^2+u^6+u^7+u^8|0), (1+u^2+u^3+u^5|1+u)\rangle,$$

thus

$$\begin{aligned}\Psi_0 &= \langle (p(u)(u+1)|0), (p(u)+q(u)|r(u))\rangle \\ &= \langle (1+u^3+u^6+u^9|0), (u+u^3+u^5+u^6+u^7+u^8|1+u)\rangle.\end{aligned}$$

Hence $\Psi_0^{\perp} = \langle (1+u+u^2+u^6+u^7+u^8|0), (1+u+u^4+u^6|1)\rangle$.
Therefore we can find $S(\Psi) = \Psi_0^{\perp} - \Psi$.

## 5. Conclusion

In this work, binary self-dual DC codes are studied and the structure of them is determined. Also the relationship between the self-duality and the separability is investigated and the Types of these codes are shown. Further, their shadow codes are obtained. The shadow code will be helpful for determining possible minimal distance for each length of binary self-dual DC codes. Therefore the highest minimal distance and the extremal code can consider for each length. On the other hand, a similar method can generalize for self-dual additive cyclic codes over some other rings.

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