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Inverse eigenvalue problem for bordered diagonal matrices

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Abstract. In this paper, the inverse eigenvalue problem for the bordered diagonal matrices are reconsidered whose elements are equal to zero except for the first row, the first column and the diagonal elements. The necessary and sufficient conditions for existence of a symmetric bordered diagonal matrix from special spectral data have been determined. A new algorithm to make such matrices is derived and some numerical examples are given to illustrate the efficiency of the method.

Keywords: Inverse eigenvalue problem, bordered diagonal matrix, leading principal submatrix, minimal eigenvalue, maximal eigenvalue.

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1. Introduction and preliminaries

Inverse eigenvalue problems arise in several applications, including system and control theory, neural networks, physics, vibration analysis, and so on [2–4]. With respect to the spectral information arise from the problem at hand and which kind of matrix we are looking for, there are various methods in inverse eigenvalue problems.

Here we consider the problem of constructing a symmetric bordered diagonal matrix

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of the following form from special spectral data:

$$A = \begin{pmatrix} a_1 & b_1 & b_2 & \cdots & b_{n-1} \\ b_1 & ma_1 & 0 & \cdots & 0 \\ b_2 & 0 & ma_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & 0 & 0 & \cdots & ma_1 \end{pmatrix}, \ a_1 \in R \ , \ b_j > 0 \ , \ m \in R. \quad (1)$$

This class of matrices appear in certain symmetric inverse eigenvalue and inverse Sturm-Liouville problems, for instance, in the movement equation with some perturbations in regulation systems in control theory [1, 5, 6]

Throughout this paper, we denote the $j \times j$ leading principal submatrix of A by A_j, the characteristic polynomial of A_j by $\phi_j(\lambda)$ and the eigenvalues of A_j by $\lambda_1^{(j)} \leqslant \lambda_2^{(j)} \leqslant \cdots \leqslant \lambda_j^{(j)}$ for $j = 1, \ldots, n$. This work is motivated by the results in [6, 7] where two special cases of inverse eigenvalue problems are introduced in order to construct a symmetric tridiagonal matrix with constant diagonal elements. Peng et al. [6] and Pickman et al. [7] considered the following cases:

Proposition 1.1 There exists a matrix of the form:

$$A = \begin{pmatrix} 0 & b_1 & b_2 & \cdots & b_{n-1} \\ b_1 & 0 & 0 & \cdots & 0 \\ b_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad b_j \neq 0 \ \ , 1 \leqslant j \leqslant n-1$$

if and only if

$$\lambda_1^{(1)} = 0 \ , \ \lambda_1^{(j)} = -\lambda_j^{(j)} \ , \ j = 2, 3, \ldots, n$$

such that $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ are the minimal and maximal eigenvalue of A_j , $j = 1, \ldots, n$, respectively.

Proposition 1.2 [2] There exists a unique matrix of the form:

$$A = \begin{pmatrix} a & b_1 & b_2 & \cdots & b_{n-1} \\ b_1 & a & 0 & \cdots & 0 \\ b_2 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & 0 & 0 & \cdots & a \end{pmatrix}, \ a \in R \ , \ b_j \neq 0 \ , 1 \leqslant j \leqslant n-1$$

if and only if

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)}$$

and

$$\lambda_1^{(j)} + \lambda_j^{(j)} = 2\lambda_1^{(1)} \ , \ j = 2, \ldots, n.$$

Here we develop the case in which there exists a unique matrix, A, of the form as defined in (1), if and only if

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)} \tag{2}$$

and

$$\lambda_1^{(j)} + \lambda_j^{(j)} = (m+1)\lambda_1^{(1)} \ \ , \ j = 2, \ldots, n \ , \ m \in R. \tag{3}$$

Note that the cases presented in [7] are obtained for $m = -1$ and $m = 1$ in the relation (3). We also show that in computation of matrix A, the characteristic polynomial of every leading principal submatrix of A is independent of other sub-matrices and hence every element of A can be computed independently. This leads to an algorithm for constructing a matrix of the form (1) which is much simpler than the algorithm in [6], besides fewer calculations are required. The rest of the paper is organised as follows. Section 2 sets the framework of the inverse eigenvalue problem for bordered diagonal matrices. In Section 3 we present some experimental results and finally, the conclusion is presented in the last section.

2. Symmetric bordered diagonal matrices from special spectral data

In this section we construct a symmetrical bordered diagonal matrix A of the form (1), from the minimal and maximal eigenvalues $\lambda_1^{(j)}, \lambda_j^{(j)}$ of all its leading principal submatrices $A_j, j = 1, 2, ..., n$. For convenience of discussion, let us define $b_0 = 1$, $\phi_0(\lambda) = 1$ and $\phi_j(\lambda) = det(\lambda I_j - A_j)$.

Lemma 2.1 For a given matrix A of the form:

$$A = \begin{pmatrix} a_1 & b_1 & b_2 & \cdots & b_{n-1} \\ b_1 & a_2 & 0 & \cdots & 0 \\ b_2 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & 0 & 0 & \cdots & a_n \end{pmatrix} \ , \ a_j \in R \ , \ b_j > 0 \ , \tag{4}$$

the sequence $\{\phi_j(\lambda)\}$ satisfies the recurrence relation

$$\phi_j(\lambda) = (\lambda - a_j)\phi_{j-1}(\lambda) - b_{j-1}^2 \prod_{i=2}^{j-1} (\lambda - a_i) \quad , \quad 1 \leqslant j \leqslant n. \tag{5}$$

Proof. It is easy to verify by expending the determinant. ∎

Theorem 2.2 Let $2n-1$ real numbers $\lambda_1^{(j)}, \lambda_j^{(j)}$ $, j = 1, \cdots, n$ be given. Then there exists a unique $n \times n$ matrix A of the form (4) such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are respectively the minimal and the maximal eigenvalues of each leading principal submatrix $A_j, j = 1, \ldots, n$ of A, if and only if

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)}. \tag{6}$$

Proof. See [7]. ■

From the above theorem the following corollary is deduced.

Corollary 2.3 If $2n-1$ real numbers $\lambda_1^{(j)}, \lambda_j^{(j)}$, $j = 1, \cdots, n$ satisfy (6), then the elements of unique matrix A are in the following form:

$$a_1 = \lambda_1^{(1)}$$

and for $j = 2, \ldots, n$, we have

$$a_j = \frac{\lambda_1^{(j)} \phi_{j-1}(\lambda_1^{(j)}) \prod_{i=2}^{j-1}(\lambda_j^{(j)} - a_i) - \lambda_j^{(j)} \phi_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1}(\lambda_1^{(j)} - a_i)}{\phi_{j-1}(\lambda_1^{(j)}) \prod_{i=2}^{j-1}(\lambda_j^{(j)} - a_i) - \phi_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1}(\lambda_1^{(j)} - a_i)}, \tag{7}$$

$$b_{j-1}^2 = \frac{(\lambda_j^{(j)} - \lambda_1^{(j)}) \phi_{j-1}(\lambda_1^{(j)}) \phi_{j-1}(\lambda_j^{(j)})}{\phi_{j-1}(\lambda_1^{(j)}) \prod_{i=2}^{j-1}(\lambda_j^{(j)} - a_i) - \phi_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1}(\lambda_1^{(j)} - a_i)}. \tag{8}$$

Lemma 2.4 For a given matrix A of the form (1), the sequence $\{\phi_j(\lambda)\}$ satisfies the following recurrence relation:

$$\phi_j(\lambda) = (\lambda - ma_1)^{j-2} \left[(\lambda - a_1)(\lambda - ma_1) - \sum_{i=1}^{j-1} b_i^2 \right] \ , \quad 2 \leqslant j \leqslant n. \tag{9}$$

Proof. The result can be easily verified by the direct expansion of the determinant of A. ■

Corollary 2.5 If A is a matrix of the form (1), then ma_1 is the repeated eigenvalue of order $j-2$ of leading principal submatrix A_j for $2 \leqslant j \leqslant n$. (See the equation (9)).

Now we prove the following theorem:

Theorem 2.6 Let $2n-1$ real numbers $\lambda_1^{(j)}, \lambda_j^{(j)}$, $j = 1, \ldots, n$ be given. Then there exists a unique $n \times n$ matrix A of the form (1) such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are respectively the minimal and the maximal eigenvalues of the leading principal submatrix A_j , $j = 1, \ldots, n$ of A, if and only if

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)}, \tag{10}$$

$$\lambda_1^{(j)} + \lambda_j^{(j)} = (m+1)\lambda_1^{(1)} \ , \ j = 2, \ldots, n \ , \ m \in R. \tag{11}$$

Proof. Suppose that, the conditions (10) and (11) are satisfied. From Theorem 2.2, there is a unique matrix A of the form (4) such that a_j, b_{j-1}^2 are obtained from (7) and (8), respectively. Also the characteristic polynomial of the leading principal submatrix A_j wil be obtained from (5).
For convenience, let

$$\lambda_j := \lambda_j^{(j)}, \quad j = 2, \ldots, n.$$

Hence

$$\lambda_1^{(j)} = (m+1)a_1 - \lambda_j \quad , \quad j = 2, \dots, n.$$

By induction on j it can be deduced that:

$$a_j = ma_1 \quad , \quad j = 2, \dots, n, \tag{12}$$

$$b_{j-1}^2 = (\lambda_j - a_1)(\lambda_j - ma_1) - (\lambda_{j-1} - a_1)(\lambda_{j-1} - ma_1) \quad , \quad j = 2, \dots, n. \tag{13}$$

Therefore the unique matrix A is of the form (1).

Now suppose the matrix A is of the form (1). From Theorem 2.2, the minimal and the maximal eigenvalues of the leading principal submatrix A_j , $j = 1, \dots, n$ satisfy in (10). We show with the induction on j, the relation (11) will be verified. For $j = 2$ we have:

$$\phi_2(\lambda) = (\lambda - a_1)(\lambda - ma_1) - b_1^2 = \lambda^2 - (m+1)a_1\lambda - b_1^2,$$

therefore sum of the roots of $\phi_2(\lambda)$ is $(m+1)a_1$ and hence the relation (11) is satisfied for $j = 2$. Now by assuming that the relation is true for $j = k$, we show that the statement is satisfied for $j = k+1$. From Lemma 2.4 and Corollary 2.5, we have:

$$\phi_{k+1}(\lambda) = (\lambda - ma_1)^{k-1}\left[(\lambda - a_1)(\lambda - ma_1) - \sum_{i=1}^{k} b_i^2\right]$$

Therefore $\phi_{k+1}(\lambda)$ has the repeated root $\lambda = ma_1$ of order $k-1$ and the other two roots are the zeros of the following polynomial.

$$P_{k+1}(\lambda) = (\lambda - a_1)(\lambda - ma_1) - \sum_{i=1}^{k} b_i^2 = \lambda^2 - (m+1)a_1\lambda + ma_1^2 - \sum_{i=1}^{k} b_i^2$$

Clearly, sum of the roots of $P_{k+1}(\lambda)$ is $(m+1)a_1$. These two roots will be the minimal and maximal eigenvalues of A_{k+1}. Otherwise if these roots are smaller or bigger than ma_1, then we must have $P_{k+1}(ma_1) > 0$ which can not be true (since $P_{k+1}(ma_1) = -\sum_{i=1}^{k} b_i^2 < 0$) and also if these two roots are smaller or bigger than a_1, then a_1 falls outside the roots of $P_{k+1}(\lambda)$, so $P_{k+1}(a_1) > 0$ which again can not be true (since $P_{k+1}(a_1) = -\sum_{i=1}^{k} b_i^2 < 0$). Therefore the minimal and maximal eigenvalues of A_{k+1} are the roots of $P_{k+1}(\lambda)$ and stisfy in the following inequalities:

$$\lambda_1^{(k+1)} < a_1 < \lambda_{k+1}^{(k+1)},$$

$$\lambda_1^{(k+1)} < ma_1 < \lambda_{k+1}^{(k+1)},$$

and

$$\lambda_1^{(k+1)} + \lambda_{k+1}^{(k+1)} = (m+1)a_1.$$

Hence relations (10) and (11) are satified for $j = 2, ..., n$, thus the proof is complete. ■

Note that if some b_j, $i = 1, \dots, n-1$ are negative, the relations (10) and (11) still hold. As a result, if A is a matrix of the form (1) such that $b_j \neq 0$ then following relations hold:

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)}$$

and

$$\lambda_1^{(j)} + \lambda_j^{(j)} = (m+1)\lambda_1^{(1)} \quad , \; j = 2, \ldots, n \; , \; m \in R.$$

This means that if all b_j are positive then matrix A will be unique. Furthermore, by Lemma 2.4 and Theorem 2.6 we demonstrated the elements of matrix (1) and all the characteristic polynomials of its leading pricipal submatrices can be computed independently of other elements and submatrices respectively. Whereas, for determining the elements of matrix (1) by (7) and (8) and the characteristic polynomials of its leading pricipal submatrices by (5), we need more computations and also every element a_j, b_j and the characteristic polynomial of A_j depend on the smaller leading pricipal submatrices.

3. Numerical Results

***Example* 3.1** Given nine real numbers

$$\begin{array}{ccccccccc} \lambda_1^{(5)} & \lambda_1^{(4)} & \lambda_1^{(3)} & \lambda_1^{(2)} & \lambda_1^{(1)} & \lambda_2^{(2)} & \lambda_3^{(3)} & \lambda_4^{(4)} & \lambda_5^{(5)} \\ -16.6140 & -16.4464 & -15.2780 & -14.5005 & 2.3 & 5.3005 & 6.0780 & 7.2464 & 7.4140 \end{array}.$$

These numbers satisfy relations (10) and (11) of Theorem 2.6, i.e.

$$\lambda_1^{(j)} + \lambda_j^{(j)} = (-5+1)\lambda_1^{(1)} \quad , \; j = 2, \ldots, 5.$$

Therefore with the given spectral data, the symmetric bordered diagonal matrix is as follows:

$$A = \begin{pmatrix} 2.3 & 7.1 & 4 & 5.13 & 2 \\ 7.1 & -11.5 & 0 & 0 & 0 \\ 4 & 0 & -11.5 & 0 & 0 \\ 5.13 & 0 & 0 & -11.5 & 0 \\ 2 & 0 & 0 & 0 & -11.5 \end{pmatrix}.$$

Now suppose the matrix A is given in the above form. Clearly, A is in consonance with (1), with $m = -5$ and the eigenvalues of leading principal submatrices A_j of A are as follows:
$\sigma(A_1) = \{2.3\}$,
$\sigma(A_2) = \{-14.5005 \; , \; 5.3005\}$,
$\sigma(A_3) = \{-15.2780 \; , \; -11.5 \; , \; 6.0780\}$,
$\sigma(A_4) = \{-16.4464 \; , \; -11.5 \; , \; -11.5 \; , \; 7.2464\}$,
$\sigma(A_5) = \{-16.6140 \; , \; -11.5 \; , \; -11.5 \; , \; -11.5 \; , \; 7.4140\}$.
We observe relations (10) and (11) hold and

$$\lambda_1^{(j)} + \lambda_j^{(j)} = (-5+1)\lambda_1^{(1)} = -9.2 \quad , \; j = 2, \ldots, 5.$$

***Example* 3.2** Suppose the following eleven real numbers are given:

$$\begin{array}{ccccccccccc} \lambda_1^{(6)} & \lambda_1^{(5)} & \lambda_1^{(4)} & \lambda_1^{(3)} & \lambda_1^{(2)} & \lambda_1^{(1)} & \lambda_2^{(2)} & \lambda_3^{(3)} & \lambda_4^{(4)} & \lambda_5^{(5)} & \lambda_6^{(6)} \\ -21.151 & -9.117 & -8.13 & -6 & -5.9 & -4.7 & -3.1898 & -3.0898 & -0.9598 & 0.0272 & 12.0612 \end{array}.$$

These numbers satisfy (10) and (11) of the Theorem 2.6, i.e.

$$\lambda_1^{(j)} + \lambda_j^{(j)} = (0.934+1)\lambda_1^{(1)} \quad , \ j = 2, \dots, 6.$$

Therefore with the given spectral data, the symmetric bordered diagonal matrix is as follows:

$$A = \begin{pmatrix} -4.7000 & 1.3462 & 0.5301 & 3.2765 & 2.8375 & 15.9643 \\ 1.3462 & -4.3898 & 0 & 0 & 0 & 0 \\ 0.5301 & 0 & -4.3898 & 0 & 0 & 0 \\ 3.2765 & 0 & 0 & -4.3898 & 0 & 0 \\ 2.8375 & 0 & 0 & 0 & -4.3898 & 0 \\ 15.9643 & 0 & 0 & 0 & 0 & -4.3898 \end{pmatrix}.$$

Now suppose the matrix A is given in the above form. Clearly, A is in consonance with (1), with $m = 0.934$ and the eigenvalues of leading principal submatrices A_j of A are:
$\sigma(A_1) = \{-4.7\}$,
$\sigma(A_2) = \{-5.9000\ ,\ -3.1898\}$,
$\sigma(A_3) = \{-6.000\ ,\ -4.3898\ ,\ -3.0898\}$,
$\sigma(A_4) = \{-8.1300\ ,\ -4.3898\ ,\ -4.3898\ ,\ -0.9598\}$,
$\sigma(A_5) = \{-9.1170\ ,\ -4.3898\ ,\ -4.3898\ ,\ -4.3898\ ,\ 0.0272\}$,
$\sigma(A_6) = \{-21.1510\ ,\ -4.3898\ ,\ -4.3898\ ,\ -4.3898\ ,\ -4.3898\ ,\ 12.0612\}$.
We observe relations (10) and (11) hold and

$$\lambda_1^{(j)} + \lambda_j^{(j)} = (0.934+1)\lambda_1^{(1)} = -9.0898\ , \quad j = 2, \dots, 6.$$

Now we show that the conditions of Proposition 1.2 for the uniqueness of matrix A are not sufficient.

***Example* 3.3** Given seven real numbers

$$\begin{array}{ccccccc} \lambda_1^{(4)} & \lambda_1^{(3)} & \lambda_1^{(2)} & \lambda_1^{(1)} & \lambda_2^{(2)} & \lambda_3^{(3)} & \lambda_4^{(4)} \\ -7.9161 & -7.8310 & -5 & -2 & 1 & 3.8310 & 3.9161 \end{array}.$$

where these numbers satisfy the conditions of Proposition 1.2, i.e.

$$\lambda_1^{(j)} + \lambda_j^{(j)} = 2\lambda_1^{(1)} \quad , \ j = 2, \dots, 4.$$

However, the following matrices also satisfy Proposition 1.2.

$$A_1 = \begin{pmatrix} -2 & 3 & -5 & -1 \\ 3 & -2 & 0 & 0 \\ -5 & 0 & -2 & 0 \\ -1 & 0 & 0 & -2 \end{pmatrix} \qquad \text{and} \qquad A_2 = \begin{pmatrix} -2 & 3 & 5 & 1 \\ 3 & -2 & 0 & 0 \\ 5 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}$$

with the eigenvalues of leading principal submatrices A_j of both A_1 , A_2 as follows:
$\sigma(A_1) = \{-2.0000\}$,
$\sigma(A_2) = \{1.0000\ ,\ -5.0000\}$,
$\sigma(A_3) = \{-7.8310\ ,\ -2.0000\ ,\ 3.8310\}$,
$\sigma(A_4) = \{-7.9161\ ,\ -2.0000\ ,\ -2.0000\ ,\ 3.9161\}$.

This example shows that if $b_j \neq 0$ but with different signs, then resulting matrix is not unique, therefore, for A to be unique, it is required that all $b_j \neq 0$ and of the same sign(either all positive or all negative). In this paper we have assumed $b_j > 0$ everywhere.

4. Conclusion

In this study, we developed the inverse eigenvalue problem for the symmetric bordered diagonal matrices and the necessary and sufficient conditions for existence such matrices with special spectral data have been investigated. Furthermore, we showed that the characteristic polynomial of every leading principal submatrix of the original bordered diagonal matrix can be computed independently from other sub-matrices. Moreover a simple algorithm with few calculations has been introduced to compute elements of the bordered diagonal matrix independently of other elements of the matrix.

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