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Construction of frame relative to n-Hilbert space

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Abstract. In this paper, our aim is to introduce the concept of a frame in n-Hilbert space and describe some of its properties. We further discuss tight frame relative to n-Hilbert space. At the end, we study the relationship between frame and bounded linear operator in n-Hilbert space.

Keywords: n-inner product space, n-normed space, pseudo-inverse, frame, tight frame.

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1. Introduction and preliminaries

In the study of vector spaces, one of the most fundamental concept is that of a basis. A basis provides us with an expansion of all vectors in terms of its elements. In infinite-dimensional Hilbert space, we are forced to work with infinite series and so depending on the work on infinite series, different concepts of basis has been established which may contain infinitely many elements namely, Schauder basis, orthonormal basis etc. In fact, in a separable Hilbert space every element can be expressed as a infinite linear combination of an orthonormal basis.The condition linearly independentness is not being assumed to define such Schauder basis or orthonormal basis but Schauder basis or orthonormal basis automatically becomes linearly independent. A frame is also spanning set of a Hilbert space but it is a redundant or linearly dependent system for a Hilbert space. So, frame can be considered as a generalization of orthonormal basis. In fact, frames play important

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role in theoretical research of wavelet analysis, signal denoising, feature extraction, robust signal processing etc.

In 1946, Gabor [7] first initiated a technique for rebuilding signals using a family of elementary signals. In 1952, Duffin and Schaeffer abstracted Gabor's method to define frame for Hilbert space in their fundamental paper [4]. Later on, frame theory was popularized by Daubechies et al. [5]. The concept of 2-inner product space was first introduced by Diminnie et al. [6] in 1970's. In 1989, Misiak [12] developed the generalization of a 2-inner product space for $n \geqslant 2$.

In this paper, our focus is to study and characterize various properties of frame and tight frame in n-Hilbert space. Finally, we shall established that an image of a frame under a bounded linear operator will be a frame if and only if the operator is invertible and give a characterization of frame in terms of its pre-frame operator in n-Hilbert space.

Throughout this paper, H will denote separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $l^2(\mathbb{N})$ denote the space of square summable scalar-valued sequences with index set of natural numbers $\mathbb{N}$.

We recall some basic definitions and results.

Theorem 1.1 [3] Let H_1, H_2 be two Hilbert spaces and $U : H_1 \to H_2$ be a bounded linear operator with closed range $\mathcal{R}_U$. Then there exists a bounded linear operator $U^\dagger : H_2 \to H_1$ such that $UU^\dagger x = x$ for all $x \in \mathcal{R}_U$.

The operator $U^\dagger$ defined in Theorem 1.1 is called the pseudo-inverse of U.

Theorem 1.2 [11] The set $\mathcal{S}(H)$ of all self-adjoint operators on H is a partially ordered set with respect to the partial order $\leqslant$ which is defined as for $T, S \in \mathcal{S}(H)$:

$$T \leqslant S \Leftrightarrow \langle Tf, f \rangle \leqslant \langle Sf, f \rangle \quad \forall f \in H.$$

Definition 1.3 [11] A self-adjoint operator $U : H \to H$ is called positive if $\langle Ux, x \rangle \geqslant 0$ for all $x \in H$. In notation, we can write $U \geqslant 0$. A self-adjoint operator $V : H \to H$ is called a square root of U if $V^2 = U$. If, in addition $V \geqslant 0$, then V is called positive square root of U and is denoted by $V = U^{1/2}$.

Theorem 1.4 [11] The positive square root $V : H \to H$ of an arbitrary positive self-adjoint operator $U : H \to H$ exists and is unique. Further, the operator V commutes with every bounded linear operator on H which commutes with U.

Definition 1.5 [3] A sequence $\{f_i\}_{i=1}^{\infty} \subseteq H$ is said to be a frame for H if there exist positive constants A and B such that

$$A\|f\|^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leqslant B\|f\|^2 \quad \forall f \in H. \tag{1}$$

The constants A and B are called frame bounds. If the collection $\{f_i\}_{i=1}^{\infty}$ satisfies only the right inequality of (1), then it is called a Bessel sequence with bound B.

Theorem 1.6 [3] Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in H and $B > 0$ be given. Then $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence with Bessel bound B if and only if the operator $T : l^2(\mathbb{N}) \to H$ defined by $T\{c_i\} = \sum\limits_{i=1}^{\infty} c_i f_i$ is bounded and $\|T\| \leqslant \sqrt{B}$.

Definition 1.7 [3] Let $\{f_i\}_{i=1}^{\infty}$ be a frame for H. Then the bounded linear operator

$T : l^2(\mathbb{N}) \to H$ defined by $T\{c_i\} = \sum\limits_{i=1}^{\infty} c_i f_i$ is called pre-frame operator and its adjoint operator $T^* : H \to l^2(\mathbb{N})$, given by $T^* f = \{\langle f, f_i \rangle\}_{i=1}^{\infty}$ is called the analysis operator. The operator $S : H \to H$ defined by

$$Sf = TT^* f = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i \quad \forall f \in H$$

is called the frame operator.

The frame operator S is bounded, positive, self-adjoint and invertible [3].

Definition 1.8 [9] A n-norm on a linear space X (over the field $\mathbb{K}$ of real or complex numbers) is a function

$$(x_1, x_2, \cdots, x_n) \longmapsto \|x_1, x_2, \cdots, x_n\|, \quad x_1, x_2, \cdots, x_n \in X$$

from X^n to the set $\mathbb{R}$ of all real numbers such that for every $x_1, x_2, \cdots, x_n \in X$ and $\alpha \in \mathbb{K}$,

(1) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent,
(2) $\|x_1, x_2, \cdots, x_n\|$ is invariant under permutations of $x_1, x_2, \cdots, x_n$,
(3) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \, \|x_1, x_2, \cdots, x_n\|$,
(4) $\|x + y, x_2, \cdots, x_n\| \leqslant \|x, x_2, \cdots, x_n\| + \|y, x_2, \cdots, x_n\|$.

A linear space X, together with a n-norm $\|\cdot, \cdots, \cdot\|$, is called a linear n-normed space.

Definition 1.9 [12] Let $n \in \mathbb{N}$ and X be a linear space of dimension greater than or equal to n over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field. A n-inner product on X is a map

$$(x, y, x_2, \cdots, x_n) \longmapsto \langle x, y | x_2, \cdots, x_n \rangle, \quad x, y, x_2, \cdots, x_n \in X$$

from X^{n+1} to the set $\mathbb{K}$ such that for every $x, y, x_1, x_2, \cdots, x_n \in X$ and $\alpha \in \mathbb{K}$,

(1) $\langle x_1, x_1 | x_2, \cdots, x_n \rangle \geqslant 0$ and $\langle x_1, x_1 | x_2, \cdots, x_n \rangle = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent,
(2) $\langle x, y | x_2, \cdots, x_n \rangle = \langle x, y | x_{i_2}, \cdots, x_{i_n} \rangle$ for every permutations $(i_2, \cdots, i_n)$ of $(2, \cdots, n)$,
(3) $\langle x, y | x_2, \cdots, x_n \rangle = \overline{\langle y, x | x_2, \cdots, x_n \rangle}$,
(4) $\langle \alpha x, y | x_2, \cdots, x_n \rangle = \alpha \, \langle x, y | x_2, \cdots, x_n \rangle$,
(5) $\langle x + y, z | x_2, \cdots, x_n \rangle = \langle x, z | x_2, \cdots, x_n \rangle + \langle y, z | x_2, \cdots, x_n \rangle$.

A linear space X together with n-inner product $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle$ is called n-inner product space.

Theorem 1.10 [8] For n-inner product space $(X, \langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle)$,

$$|\langle x, y | x_2, \cdots, x_n \rangle| \leqslant \|x, x_2, \cdots, x_n\| \, \|y, x_2, \cdots, x_n\|$$

hold for all $x, y, x_2, \cdots, x_n \in X$, which is called Cauchy-Schwarz inequality.

Theorem 1.11 [12] For every n-inner product space $(X, \langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle)$,

$$\|x_1, x_2, \cdots, x_n\| = \sqrt{\langle x_1, x_1 | x_2, \cdots, x_n \rangle}$$

defines a n-norm for which

$$\langle x, y | x_2, \cdots, x_n \rangle = \frac{1}{4} \left(\|x + y, x_2, \cdots, x_n\|^2 - \|x - y, x_2, \cdots, x_n\|^2 \right)$$

and

$$\|x + y, x_2, \cdots, x_n\|^2 + \|x - y, x_2, \cdots, x_n\|^2 = 2 \left(\|x, x_2, \cdots, x_n\|^2 + \|y, x_2, \cdots, x_n\|^2 \right)$$

hold for all $x, y, x_1, x_2, \cdots, x_n \in X$.

Definition 1.12 [9] Let $(X, \|\cdot, \cdots, \cdot\|)$ be a linear n-normed space. A sequence $\{x_k\}$ in X is said to converge to some $x \in X$ if

$$\lim_{k \to \infty} \|x_k - x, e_2, \cdots, e_n\| = 0$$

for every $e_2, \cdots, e_n \in X$ and it is called a Cauchy sequence if

$$\lim_{l,k \to \infty} \|x_l - x_k, e_2, \cdots, e_n\| = 0$$

for every $e_2, \cdots, e_n \in X$. The space X is said to be complete if every Cauchy sequence in this space is convergent in X. A n-inner product space is called n-Hilbert space if it is complete with respect to its induce norm. 2-Hilbert space [2] is a particular case of n-Hilbert space for $n = 2$.

Definition 1.13 [1] Let $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-Hilbert space and $\xi \in X$. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq X$ is said to be a 2-frame associated to ξ if there exist positive constants A and B such that

$$A \|f, \xi\|^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i | \xi \rangle|^2 \leqslant B \|f, \xi\|^2 \quad \forall f \in X.$$

Theorem 1.14 [1] Let L_ξ denote the 1-dimensional linear subspace of X generated by a fixed $\xi \in X$. Let M_ξ be the algebraic complement of L_ξ. Define $\langle x, y \rangle_\xi = \langle x, y | \xi \rangle$ on X. This semi-inner product induces an inner product on the quotient space X/L_ξ which is given by

$$\langle x + L_\xi, y + L_\xi \rangle_\xi = \langle x, y \rangle_\xi = \langle x, y | \xi \rangle \quad \forall x, y \in X.$$

By identifying X/L_ξ with M_ξ in an obvious way, we obtain an inner product on M_ξ. Define $\|x\|_\xi = \sqrt{\langle x, x \rangle_\xi} \quad (x \in M_\xi)$. Then $(M_\xi, \| \cdot \|_\xi)$ is a norm space. Let X_ξ be the completion of the inner product space M_ξ.

Theorem 1.15 [1] Let $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-Hilbert space and $\xi \in X$. Then a sequence $\{f_i\}_{i=1}^{\infty}$ in X is a 2-frame associated to ξ with bounds $A \& B$ if and only if it is a frame for the Hilbert space X_ξ with bounds $A \& B$.

2. Frame and it's properties in n-Hilbert space

In this section, we introduce the notion of frame in n-Hilbert space and then we discuss its several properties.

Proposition 2.1 Let $(X, \langle\cdot,\cdot|\cdot,\cdots,\cdot\rangle)$ be a n-inner product space and $x_1, \cdots, x_n$ be elements in X. Then

$$\|x_1, x_2, \cdots, x_n\| = \sup\{|\langle x_1, y | x_2, \cdots, x_n\rangle| : y \in X, \|y, x_2, \cdots, x_n\| = 1\}.$$

Proof. Now,

$$\begin{aligned}
\|x_1, x_2, \cdots, x_n\| &= \left\langle x_1, \frac{x_1}{\|x_1, x_2, \cdots, x_n\|} | x_2, \cdots, x_n \right\rangle \\
&\leqslant \sup\{|\langle x_1, y | x_2, \cdots, x_n\rangle| : y \in X \text{ and } \|y, x_2, \cdots, x_n\| = 1\} \\
&\quad \left[\text{where } y = \frac{x_1}{\|x_1, x_2, \cdots, x_n\|}\right] \\
&\leqslant \sup\{\|x_1, x_2, \cdots, x_n\| \, \|y, x_2, \cdots, x_n\| : \|y, x_2, \cdots, x_n\| = 1\} \\
&\quad \text{[by Cauchy-Schwarz inequality]} \\
&= \|x_1, x_2, \cdots, x_n\|.
\end{aligned}$$

This completes the proof. ■

Let L_F denote the linear subspace of X spanned by the non-empty finite set $F = \{a_2, a_3, \cdots, a_n\}$, where $a_2, a_3, \cdots, a_n$ are fixed elements in X. Then the quotient space X/L_F is a normed linear space with respect to the norm, $\|x + L_F\|_F = \|x, a_2, \cdots, a_n\|$ for every $x \in X$. Let M_F be the algebraic complement of L_F, then $X = L_F \oplus M_F$. Define

$$\langle x, y\rangle_F = \langle x, y | a_2, \cdots, a_n\rangle \quad \text{on} \quad X.$$

Then $\langle\cdot,\cdot\rangle_F$ is a semi-inner product on X and this semi-inner product induces an inner product on the quotient space X/L_F which is given by

$$\langle x + L_F, y + L_F\rangle_F = \langle x, y\rangle_F = \langle x, y | a_2, \cdots, a_n\rangle \quad \forall x, y \in X.$$

By identifying X/L_F with M_F in an obvious way, we obtain an inner product on M_F. Now, for every $x \in M_F$, we define $\|x\|_F = \sqrt{\langle x, x\rangle_F}$ and it can be easily verify that $(M_F, \|\cdot\|_F)$ is a norm space. Let X_F be the completion of the inner product space M_F.

For the remaining part of this paper, $(X, \langle\cdot,\cdot|\cdot,\cdots,\cdot\rangle)$ is consider to be a n-Hilbert space. I_F will denote the identity operator on X_F and $\mathcal{B}(X_F)$ denote the space of all bounded linear operator on X_F.

Definition 2.2 A sequence $\{f_i\}_{i=1}^{\infty}$ in X is said to be a frame associated to $(a_2, \cdots, a_n)$ for X if there exist constants $0 < A \leqslant B < \infty$ such that

$$A \|f, a_2, \cdots, a_n\|^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n\rangle|^2 \leqslant B \|f, a_2, \cdots, a_n\|^2$$

for all $f \in X$. The infimum of all such B is called the optimal upper frame bound and supremum of all such A is called the optimal lower frame bound. If the sequence $\{f_i\}_{i=1}^{\infty}$ satisfies only

$$\sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \leqslant B \, \|f, a_2, \cdots, a_n\|^2 \quad \forall f \in X,$$

is called a Bessel sequence associated to $(a_2, \cdots, a_n)$ in X with bound B.

Theorem 2.3 Let $(X, \langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle)$ be a n-Hilbert space. Then $\{f_i\}_{i=1}^{\infty} \subseteq X$ is a frame associated to $(a_2, \cdots, a_n)$ with bounds $A \& B$ if and only if it is a frame for the Hilbert space X_F with bounds $A \& B$.

Proof. This theorem is an extension of the Theorem 1.15 and the proof of this theorem directly follows from that of the Theorem 1.15. ■

Theorem 2.4 Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence associated to $(a_2, \cdots, a_n)$ in X with bound B. Then the operator given by

$$T_F : l^2(\mathbb{N}) \to X_F, T_F \left(\{c_i\}_{i=1}^{\infty} \right) = \sum_{i=1}^{\infty} c_i f_i$$

is well-defined and bounded. Furthermore, the adjoint operator of T_F is given by

$$T_F^* : X_F \to l^2(\mathbb{N}), T_F^*(f) = \{\langle f, f_i | a_2, \cdots, a_n \rangle\}_{i=1}^{\infty} \quad \forall f \in X_F.$$

Proof. Let $\{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$. Then

$$\begin{aligned}
\left\| \sum_{i=1}^{l} c_i f_i - \sum_{i=1}^{k} c_i f_i \right\|_F^2 &= \left\| \sum_{i=k+1}^{l} c_i f_i, a_2, \cdots, a_n \right\|^2 \\
&= \sup_{y \in X} \left\{ \left| \left\langle \sum_{i=k+1}^{l} c_i f_i, y | a_2, \cdots, a_n \right\rangle \right|^2 : \|y, a_2, \cdots, a_n\| = 1 \right\} \\
&= \sup_{y \in X} \left\{ \left| \sum_{i=k+1}^{l} c_i \langle f_i, y | a_2, \cdots, a_n \rangle \right|^2 : \|y, a_2, \cdots, a_n\| = 1 \right\} \\
&\leqslant \sum_{i=k+1}^{l} |c_i|^2 \sup_{y \in X} \left\{ \sum_{i=k+1}^{l} |\langle f_i, y | a_2, \cdots, a_n \rangle|^2 : \|y, a_2, \cdots, a_n\| = 1 \right\} \\
&\quad \text{[using Cauchy-Schwarz inequality]} \\
&\leqslant B \sum_{i=k+1}^{l} |c_i|^2 \\
&\quad \text{[since } \{f_i\}_{i=1}^{\infty} \text{ is a Bessel sequence associated to } (a_2, \cdots, a_n)\text{]}.
\end{aligned}$$

This implies that $\sum\limits_{i=1}^{\infty} c_i f_i$ is convergent in X_F if $\{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$. Using the continuity of

n-norm function, we get

$$\left\|\sum_{i=1}^{\infty} c_i f_i\right\|_F^2 \leqslant B \sum_{i=1}^{\infty} |c_i|^2 \Rightarrow \|T_F \{c_i\}_{i=1}^{\infty}\|_F \leqslant \sqrt{B}\, \|\{c_i\}_{i=1}^{\infty}\|_{l^2}.$$

The above calculation shows that T_F is well-defined and bounded. To find the expression for T_F^*, let $f \in X_F$ and $\{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$. Then

$$\langle f, T_F \{c_i\}_{i=1}^{\infty} | a_2, \cdots, a_n\rangle = \left\langle f, \sum_{i=1}^{\infty} c_i f_i | a_2, \cdots, a_n \right\rangle = \sum_{i=1}^{\infty} \overline{c_i} \langle f, f_i | a_2, \cdots, a_n\rangle.$$

The convergence of the series $\sum_{i=1}^{\infty} \overline{c_i} \langle f, f_i | a_2, \cdots, a_n\rangle$ for all $\{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$ implies that $\{\langle f, f_i | a_2, \cdots, a_n\rangle\}_{i=1}^{\infty} \in l^2(\mathbb{N})$, (see page 145 of [10]). Therefore,

$$\langle f, T_F \{c_i\}_{i=1}^{\infty}\rangle_F = \langle \{\langle f, f_i | a_2, \cdots, a_n\rangle\}_{i=1}^{\infty}, \{c_i\}_{i=1}^{\infty}\rangle_{l^2(\mathbb{N})}$$

and hence $T_F^*(f) = \{\langle f, f_i | a_2, \cdots, a_n\rangle\}_{i=1}^{\infty}$ for all $f \in X_F$. This completes the proof. ■

Remark 1 *The operator T_F, defined in Theorem 2.4, is called pre-frame operator and the adjoint operator of T_F is called analysis operator for $\{f_i\}_{i=1}^{\infty}$.*

Definition 2.5 Let $\{f_i\}_{i=1}^{\infty}$ be a frame associated to $(a_2, \cdots, a_n)$ for X. Then the operator $S_F : X_F \to X_F$ defined by $S_F(f) = \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n\rangle f_i$ for all $f \in X_F$ is called the frame operator for $\{f_i\}_{i=1}^{\infty}$. It can be easily verify that

$$\langle S_F f, f | a_2, \cdots, a_n\rangle = \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n\rangle|^2 \quad \forall f \in X_F. \tag{2}$$

Theorem 2.6 Let $\{f_i\}_{i=1}^{\infty}$ be a frame associated to $(a_2, \cdots, a_n)$ for X with bounds $A \& B$. Then the corresponding frame operator S_F is bounded, invertible, self-adjoint and positive.

Proof. For each $f \in X_F$, we have

$$\begin{aligned}
\|S_F f\|_F^2 &= \|S_F f, a_2, \cdots, a_n\|^2 \\
&= \sup\left\{ |\langle S_F f, g | a_2, \cdots, a_n\rangle|^2 : \|g, a_2, \cdots, a_n\| = 1 \right\} \\
&= \sup\left\{ \left| \left\langle \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n\rangle f_i, g | a_2, \cdots, a_n \right\rangle \right|^2 : \|g, a_2, \cdots, a_n\| = 1 \right\} \\
&\leqslant \sup\left\{ \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n\rangle|^2 \sum_{i=1}^{\infty} |\langle g, f_i | a_2, \cdots, a_n\rangle|^2 : \|g, a_2, \cdots, a_n\| = 1 \right\} \\
&\quad [\text{ using Cauchy-Schwarz inequality}] \\
&\leqslant B\, \|f, a_2, \cdots, a_n\|^2\, B \; [\text{since } \{f_i\}_{i=1}^{\infty} \text{ is a frame associated to } (a_2, \cdots, a_n)] \\
&= B^2\, \|f, a_2, \cdots, a_n\|^2 = B^2 \|f\|_F^2.
\end{aligned}$$

This shows that S_F is bounded. Since $S_F = T_F T_F^*$, it is easy to verify that S_F is self-adjoint. By (2), frame inequality of Definition 2.2 can be written as

$$A\langle f, f|a_2,\cdots,a_n\rangle \leqslant \langle S_F f, f|a_2,\cdots,a_n\rangle \leqslant B\langle f, f|a_2,\cdots,a_n\rangle$$

and therefore according to the Theorem 1.2, we can write $AI_F \leqslant S_F \leqslant BI_F$. Thus, S_F is positive and consequently it is invertible. ■

Remark 2 *In Theorem 2.6, it is proved that $AI_F \leqslant S_F \leqslant BI_F$. Since S_F^{-1} commutes with both S_F and I_F, multiplying in the inequality, $AI_F \leqslant S_F \leqslant BI_F$ by S_F^{-1}, we get $B^{-1}I_F \leqslant S_F^{-1} \leqslant A^{-1}I_F$.*

Theorem 2.7 Let $\{f_i\}_{i=1}^{\infty}$ be a frame associated to $(a_2,\cdots,a_n)$ for X with frame bounds A, B and S_F be the corresponding frame operator. Then $\left\{S_F^{-1}f_i\right\}_{i=1}^{\infty}$ is also a frame associated to $(a_2,\cdots,a_n)$ for X with bounds B^{-1} and A^{-1}.

Proof. By Theorem 2.6, $S_F^{-1} : X_F \to X_F$ is self-adjoint. Now, for each $f \in X_F$, since $\{f_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2,\cdots,a_n)$, we have

$$\begin{aligned}\sum_{i=1}^{\infty}\left|\left\langle f, S_F^{-1}f_i|a_2,\cdots,a_n\right\rangle\right|^2 &= \sum_{i=1}^{\infty}\left|\left\langle \left(S_F^{-1}\right)^* f, f_i|a_2,\cdots,a_n\right\rangle\right|^2 \\ &= \sum_{i=1}^{\infty}\left|\left\langle S_F^{-1}f, f_i|a_2,\cdots,a_n\right\rangle\right|^2 \\ &\leqslant B\left\|S_F^{-1}f, a_2,\cdots,a_n\right\|^2 \\ &\leqslant B\left\|S_F^{-1}\right\|^2\|f, a_2,\cdots,a_n\|^2.\end{aligned}$$

This shows that $\left\{S_F^{-1}f_i\right\}_{i=1}^{\infty}$ is a Bessel sequence associated to $(a_2,\cdots,a_n)$ in X. Also, for each $f \in X_F$, we have

$$\begin{aligned}\sum_{i=1}^{\infty}\left\langle f, S_F^{-1}f_i|a_2,\cdots,a_n\right\rangle S_F^{-1}f_i &= S_F^{-1}\left(\sum_{i=1}^{\infty}\left\langle S_F^{-1}f, f_i|a_2,\cdots,a_n\right\rangle f_i\right) \\ &= S_F^{-1}\left(S_F\left(S_F^{-1}f\right)\right) = S_F^{-1}f. \qquad (3)\end{aligned}$$

This shows that S_F^{-1} is the frame operator for $\left\{S_F^{-1}f_i\right\}_{i=1}^{\infty}$. Now, for each $f \in X_F$, using the inequality $B^{-1}I_F \leqslant S_F^{-1} \leqslant A^{-1}I_F$, we get

$$B^{-1}\|f, a_2,\cdots,a_n\|^2 \leqslant \left\langle S_F^{-1}f, f|a_2,\cdots,a_n\right\rangle \leqslant A^{-1}\|f, a_2,\cdots,a_n\|^2.$$

Now, using (3),

$$\begin{aligned}\left\langle S_F^{-1}f, f|a_2,\cdots,a_n\right\rangle &= \left\langle \sum_{i=1}^{\infty}\left\langle f, S_F^{-1}f_i|a_2,\cdots,a_n\right\rangle S_F^{-1}f_i, f|a_2,\cdots,a_n\right\rangle \\ &= \sum_{i=1}^{\infty}\left\langle f, S_F^{-1}f_i|a_2,\cdots,a_n\right\rangle\left\langle S_F^{-1}f_i, f|a_2,\cdots,a_n\right\rangle \\ &= \sum_{i=1}^{\infty}\left|\left\langle f, S_F^{-1}f_i|a_2,\cdots,a_n\right\rangle\right|^2.\end{aligned}$$

Therefore, for each $f \in X_F$,

$$B^{-1}\|f, a_2, \cdots, a_n\|^2 \leqslant \sum_{i=1}^{\infty} \left|\left\langle f, S_F^{-1} f_i | a_2, \cdots, a_n \right\rangle\right|^2 \leqslant A^{-1}\|f, a_2, \cdots, a_n\|^2.$$

Hence, by Theorem 2.3, $\left\{S_F^{-1} f_i\right\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X with bounds B^{-1} and A^{-1}. ■

Remark 3 *From the Theorem 2.7, we can conclude that if $\{f_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ with optimal frame bounds A and B, then B^{-1} and A^{-1} are also optimal frame bounds for $\left\{S_F^{-1} f_i\right\}_{i=1}^{\infty}$. The frame $\left\{S_F^{-1} f_i\right\}_{i=1}^{\infty}$ is called the canonical dual frame associated to $(a_2, \cdots, a_n)$ of $\{f_i\}_{i=1}^{\infty}$.*

Proposition 2.8 Let $\{f_i\}_{i=1}^{\infty}$ be a frame associated to $(a_2, \cdots, a_n)$ for X and S_F be the corresponding frame operator. Then, for every $f \in X_F$,

$$f = \sum_{i=1}^{\infty} \left\langle f, S_F^{-1} f_i | a_2, \cdots, a_n \right\rangle f_i \text{ and } f = \sum_{i=1}^{\infty} \left\langle f, f_i | a_2, \cdots, a_n \right\rangle S_F^{-1} f_i,$$

provided both the series converges unconditionally for all $f \in X_F$.

Proof. Let $f \in X_F$. Then

$$\begin{aligned} f = S_F S_F^{-1} f &= S_F \left(\sum_{i=1}^{\infty} \left\langle f, S_F^{-1} f_i | a_2, \cdots, a_n \right\rangle S_F^{-1} f_i\right) \text{[using (3)]} \\ &= \sum_{i=1}^{\infty} \left\langle f, S_F^{-1} f_i | a_2, \cdots, a_n \right\rangle S_F \left(S_F^{-1} f_i\right) \\ &= \sum_{i=1}^{\infty} \left\langle f, S_F^{-1} f_i | a_2, \cdots, a_n \right\rangle f_i. \end{aligned}$$

Since $\left\{\left\langle f, S_F^{-1} f_i | a_2, \cdots, a_n \right\rangle\right\}_{i=1}^{\infty} \in l^2(\mathbb{N})$ and $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$ in X, the above series converges unconditionally. On the other hand,

$$f = S_F^{-1} S_F f = S_F^{-1} \left(\sum_{i=1}^{\infty} \left\langle f, f_i | a_2, \cdots, a_n \right\rangle f_i\right) = \sum_{i=1}^{\infty} \left\langle f, f_i | a_2, \cdots, a_n \right\rangle S_F^{-1} f_i$$

for all $f \in X_F$. This completes the proof. ■

Definition 2.9 A sequence $\{f_i\}_{i=1}^{\infty}$ in X is said to be a tight frame associated to $(a_2, \cdots, a_n)$ for X with bound A if for all $f \in X$,

$$\sum_{i=1}^{\infty} \left| \left\langle f, f_i | a_2, \cdots, a_n \right\rangle \right|^2 = A \left\|f, a_2, \cdots, a_n\right\|^2. \tag{4}$$

If $A = 1$ then it is called normalized tight frame associated to $(a_2, \cdots, a_n)$. From (4), we have

$$\sum_{i=1}^{\infty} \left|\left\langle f, \frac{1}{\sqrt{A}} f_i | a_2, \cdots, a_n \right\rangle\right|^2 = \left\|f, a_2, \cdots, a_n\right\|^2.$$

Therefore, if $\{f_i\}_{i=1}^{\infty}$ is a tight frame associated to $(a_2, \cdots, a_n)$ with bound A, then family $\left\{\frac{1}{\sqrt{A}} f_i\right\}_{i=1}^{\infty}$ is a normalized tight frame associated to $(a_2, \cdots, a_n)$. According to

Theorem 2.3, $\{f_i\}_{i=1}^{\infty}$ is a tight frame associated to $(a_2, \cdots, a_n)$ for X with bound A if and only if it is a tight frame for X_F with bound A.

Proposition 2.10 Let $\{f_i\}_{i=1}^{\infty}$ be a tight frame associated to $(a_2, \cdots, a_n)$ for X with frame bound A. Then, for every $f \in X_F$,

$$f = \frac{1}{A} \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n \rangle f_i.$$

Proof. Since $\{f_i\}_{i=1}^{\infty}$ is a tight frame associated to $(a_2, \cdots, a_n)$ for X with bound A, we have

$$\sum_{i=1}^{\infty} | \langle f, f_i | a_2, \cdots, a_n \rangle |^2 = A \, \|f, a_2, \cdots, a_n\|^2 \quad \forall f \in X_F.$$

Let S_F be the corresponding frame operator for $\{f_i\}_{i=1}^{\infty}$. Then, by (2),

$$\langle S_F f, f | a_2, \cdots, a_n \rangle = \sum_{k=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 = A \, \|f, a_2, \cdots, a_n\|^2 = \langle Af, f | a_2, \cdots, a_n \rangle$$

$$\Rightarrow \langle (S_F - AI_F) f, f | a_2, \cdots, a_n \rangle = 0 \quad \forall f \in X_F \Rightarrow S_F = AI_F.$$

Therefore, for each $f \in X_F$, we get

$$Af = S_F(f) = \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n \rangle f_i \Rightarrow f = \frac{1}{A} \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n \rangle f_i.$$

■

Theorem 2.11 Let $\{f_i\}_{i=1}^{\infty}$ be a frame associated to $(a_2, \cdots, a_n)$ for X and S_F be the corresponding frame operator. Then $\left\{ S_F^{-1/2} f_i \right\}_{i=1}^{\infty}$ is a normalized tight frame associated to $(a_2, \cdots, a_n)$ and furthermore, for each $f \in X_F$,

$$f = \sum_{i=1}^{\infty} \left\langle f, S_F^{-1/2} f_i | a_2, \cdots, a_n \right\rangle S_F^{-1/2} f_i.$$

Proof. By Theorem 1.4, a unique positive square root $S_F^{-1/2}$ of S_F^{-1} exists, which is self-adjoint and commutes with S_F. Therefore, each $f \in X_F$ can be written as

$$\begin{aligned} f &= S_F^{-1/2} S_F^{-1/2} (S_F f) = S_F^{-1/2} S_F \left(S_F^{-1/2} f \right) \\ &= S_F^{-1/2} \left(\sum_{i=1}^{\infty} \left\langle S_F^{-1/2} f, f_i | a_2, \cdots, a_n \right\rangle f_i \right) \\ &= \sum_{i=1}^{\infty} \left\langle S_F^{-1/2} f, f_i | a_2, \cdots, a_n \right\rangle S_F^{-1/2} f_i \\ &= \sum_{i=1}^{\infty} \left\langle f, S_F^{-1/2} f_i | a_2, \cdots, a_n \right\rangle S_F^{-1/2} f_i. \end{aligned}$$

Now, for each $f \in X_F$, we have

$$\begin{aligned}\|f, a_2, \cdots, a_n\|^2 &= \langle f, f | a_2, \cdots, a_n \rangle \\ &= \left\langle \sum_{i=1}^{\infty} \left\langle f, S_F^{-1/2} f_i | a_2, \cdots, a_n \right\rangle S_F^{-1/2} f_i, f | a_2, \cdots, a_n \right\rangle \\ &= \sum_{i=1}^{\infty} \left\langle f, S_F^{-1/2} f_i | a_2, \cdots, a_n \right\rangle \left\langle S_F^{-1/2} f_i, f | a_2, \cdots, a_n \right\rangle \\ &= \sum_{i=1}^{\infty} \left| \left\langle f, S_F^{-1/2} f_i | a_2, \cdots, a_n \right\rangle \right|^2 .\end{aligned}$$

Hence, $\left\{ S_F^{-1/2} f_i \right\}_{i=1}^{\infty}$ is a normalized tight frame associated to $(a_2, \cdots, a_n)$ for X. ■

3. Frame and operator in n-Hilbert space

In this section, we establish that an image of frame associated to $(a_2, \cdots, a_n)$ becomes a frame associated to $(a_2, \cdots, a_n)$ under a bounded linear operator if and only if the bounded linear operator have to be invertible. In general, a Bessel sequence does not form a frame in n-Hilbert space. We give some sufficient condition for Bessel sequence associated to $(a_2, \cdots, a_n)$ to be a frame associated to $(a_2, \cdots, a_n)$ in n-Hilbert space.

Theorem 3.1 Let $\{f_i\}_{i=1}^{\infty}$ be a frame associated to $(a_2, \cdots, a_n)$ for X with bounds A and B, S_F be the corresponding frame operator and $U : X_F \to X_F$ be a bounded linear operator. Then $\{U f_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X if and only if U is invertible on X_F.

Proof. Suppose $U : X_F \to X_F$ is invertible. Then, for each $f \in X_F$,

$$\|f, a_2, \cdots, a_n\|^2 = \left\| (U^{-1})^* U^* f, a_2, \cdots, a_n \right\|^2 \leqslant \left\| U^{-1} \right\|^2 \|U^* f, a_2, \cdots, a_n\|^2 . \quad (5)$$

Since $\{f_i\}_{i=1}^{\infty}$is a frame associated to $(a_2, \cdots, a_n)$ for each $f \in X_F$,

$$\begin{aligned}\sum_{i=1}^{\infty} |\langle f, U f_i | a_2, \cdots, a_n \rangle|^2 &= \sum_{i=1}^{\infty} |\langle U^* f, f_i | a_2, \cdots, a_n \rangle|^2 \\ &\geqslant A \|U^* f, a_2, \cdots, a_n\|^2 \\ &\geqslant A \left\| U^{-1} \right\|^{-2} \|f, a_2, \cdots, a_n\|^2 \text{ [by (5)]}.\end{aligned}$$

On the other hand, for each $f \in X_F$, we have

$$\sum_{i=1}^{\infty} |\langle f, U f_i | a_2, \cdots, a_n \rangle|^2 = \sum_{i=1}^{\infty} |\langle U^* f, f_i | a_2, \cdots, a_n \rangle|^2 \leqslant B \|U\|^2 \|f, a_2, \cdots, a_n\|^2 .$$

Hence, $\{U f_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X.

Conversely, suppose that $\{U f_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X. Now, for each $f \in X_F$, we have

$$\sum_{i=1}^{\infty} \langle f, Uf_i | a_2, \cdots, a_n \rangle Uf_i = U \left(\sum_{i=1}^{\infty} \langle U^* f, f_i | a_2, \cdots, a_n \rangle \right) = US_F U^* f.$$

This shows that US_FU^* is the corresponding frame operator for $\{Uf_i\}_{i=1}^{\infty}$. By Theorem 2.6, US_FU^* is invertible and hence $U : X_F \to X_F$ is invertible. ■

Corollary 3.2 Let $\{f_i\}_{i=1}^{\infty}$ be a frame associated to $(a_2, \cdots, a_n)$ for X and $U : X_F \to X_F$ be a bounded linear operator. Then $\{f_i + Uf_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X if and only if $I + U$ is invertible on X_F.

Proof. For each $f \in X_F$, we can write

$$\sum_{i=1}^{\infty} |\langle f, f_i + Uf_i | a_2, \cdots, a_n \rangle|^2 = \sum_{i=1}^{\infty} |\langle (I+U)^* f, f_i | a_2, \cdots, a_n \rangle|^2 .$$

Thus, $\{f_i + Uf_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X if and only if $\{f_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X. By Theorem 3.1, the family $\{f_i + Uf_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X if and only if $I + U$ is invertible on X_F. ■

Remark 4 *Furthermore, for each $f \in X_F$, we have*

$$\begin{aligned}
\sum_{i=1}^{\infty} \langle f, f_i + Uf_i | a_2, \cdots, a_n \rangle (f_i + Uf_i) &= \sum_{i=1}^{\infty} \langle f, (I+U)f_i | a_2, \cdots, a_n \rangle (I+U)f_i \\
&= (I+U) \sum_{i=1}^{\infty} \langle (I+U)^* f, f_i | a_2, \cdots, a_n \rangle f_i \\
&= (I+U) S_F (I+U)^* f.
\end{aligned}$$

This implies that $(I+U)S_F(I+U)^*$ is the corresponding frame operator for the frame $\{f_i + Uf_i\}_{i=1}^{\infty}$.

Corollary 3.3 Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be two Bessel sequences associated to $(a_2, \cdots, a_n)$ in X with pre-frame operators T_F and T'_F, respectively. Then, for $L_1, L_2 \in \mathcal{B}(X_F)$, the sequence $\{L_1 f_i + L_2 g_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X if and only if $\left[T_F^* L_1^* + (T'_F)^* L_2^*\right]$ is an invertible on X_F.

Proof. Since T_F and T'_F are pre-frame operators for $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$,

$$T_F^*(f) = \{\langle f, f_i | a_2, \cdots, a_n \rangle\}_{i=1}^{\infty}, \text{ and } \left(T'_F\right)^*(f) = \{\langle f, g_i | a_2, \cdots, a_n \rangle\}_{i=1}^{\infty} \quad \forall f \in X_F.$$

By Theorem 3.1, $\{L_1 f_i + L_2 g_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X if and only if its analysis operator $T : X_F \to l^2(\mathbb{N})$ defined by

$$T(f) = \{\langle f, L_1 f_i + L_2 g_i | a_2, \cdots, a_n \rangle\}_{i=1}^{\infty} \quad \forall f \in X_F,$$

is invertible on X_F. Now, for each $f \in X_F$,

$$\begin{aligned}
T(f) &= \{\langle f, L_1 f_i + L_2 g_i | a_2, \cdots, a_n \rangle\}_{i=1}^{\infty} \\
&= \{\langle f, L_1 f_i | a_2, \cdots, a_n \rangle + \langle f, L_2 g_i | a_2, \cdots, a_n \rangle\}_{i=1}^{\infty} \\
&= \{\langle L_1^* f, f_i | a_2, \cdots, a_n \rangle\}_{i=1}^{\infty} + \{\langle L_2^* f, g_i | a_2, \cdots, a_n \rangle\}_{i=1}^{\infty} \\
&= \left[T_F^* L_1^* + \left(T'_F\right)^* L_2^*\right] f.
\end{aligned}$$

Therefore, $\{L_1 f_i + L_2 g_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X if and only if $\left[T_F^* L_1^* + (T_F')^* L_2^*\right]$ is an invertible on X_F. ■

Theorem 3.4 A sequence $\{f_i\}_{i=1}^{\infty}$ in X is a frame associated to $(a_2, \cdots, a_n)$ for X if and only if $T_F : \{c_i\}_{i=1}^{\infty} \to \sum\limits_{i=1}^{\infty} c_i f_i$ is well-defined mapping of $l^2(\mathbb{N})$ onto X_F.

Proof. First we suppose that $\{f_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X. Then, by Theorem 2.4, T_F is well-defined bounded linear operator from $l^2(\mathbb{N})$ onto X_F. Also, by Theorem 2.6, the frame operator $S_F = T_F T_F^*$ is surjective and hence T_F is surjective.

Conversely, suppose that T_F is well-defined mapping of $l^2(\mathbb{N})$ onto X_F. By Theorem 1.6, T_F is bounded and that $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$. So, $T_F^* f = \{\langle f, f_i | a_2, \cdots, a_n \rangle\}_{i=1}^{\infty}$. Since T_F is surjective, by Theorem 1.1, there exists an operator $T_F^{\dagger} : X_F \to l^2(\mathbb{N})$ such that $T_F T_F^{\dagger} = I_F$. This implies that $\left(T_F^{\dagger}\right)^* T_F^* = I_F$. Then, for each $f \in X_F$, we get

$$\|f, a_2, \cdots, a_n\|^2 \leqslant \left\|\left(T_F^{\dagger}\right)^*\right\|^2 \|T_F^* f, a_2, \cdots, a_n\|^2 \leqslant \left\|T_F^{\dagger}\right\|^2 \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2$$

$$\Rightarrow \frac{1}{\left\|T_F^{\dagger}\right\|^2} \|f, a_2, \cdots, a_n\|^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2.$$

Therefore, $\{f_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X. ■

Theorem 3.5 Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be two Bessel sequences associated to $(a_2, \cdots, a_n)$ in X with bounds $C$and$D$, respectively. Suppose that T_F and T_F' be their pre-frame operators such that $T_F (T_F')^* = I_F$. Then $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are frames associated to $(a_2, \cdots, a_n)$ for X.

Proof. Since T_F and T_F' are pre-frame operators for $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$, respectively. For each $f \in X_F$, we have

$$\|T_F^* f\|_F^2 = \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \quad \text{and} \quad \left\|(T_F')^* f\right\|_F^2 = \sum_{i=1}^{\infty} |\langle f, g_i | a_2, \cdots, a_n \rangle|^2 .$$

Now, for each $f \in X_F$, we have

$$\begin{aligned}
\|f, a_2, \cdots, a_n\|^4 &= [\langle f, f | a_2, \cdots, a_n \rangle]^2 \\
&= \left[\left\langle T_F \left(T_F'\right)^* f, f | a_2, \cdots, a_n \right\rangle\right]^2 [\text{since} T_F \left(T_F'\right)^* = I_F] \\
&= \left[\left\langle \left(T_F'\right)^* f, T_F^* f | a_2, \cdots, a_n \right\rangle\right]^2 \\
&\leqslant \|T_F^* f\|_F^2 \left\|\left(T_F'\right)^* f\right\|_F^2 [\text{by Cauchy-Schwarz inequality}] \\
&= \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \sum_{i=1}^{\infty} |\langle f, g_i | a_2, \cdots, a_n \rangle|^2 \\
&\leqslant \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 D \|f, a_2, \cdots, a_n\|^2
\end{aligned}$$

$$\Rightarrow \frac{1}{D}\|f, a_2, \cdots, a_n\|^2 \leqslant \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n\rangle|^2 .$$

Hence, $\{f_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X. Similarly, it can be shown that $\{g_i\}_{i=1}^{\infty}$ is a frames associated to $(a_2, \cdots, a_n)$ for X with the lower bound $\dfrac{1}{C}$. ■

4. Conclusion

In this paper, in the setting of n-Hilbert space, we give the idea of frame and tight frame and establish some characterizations of them. Yet it remains to establish another few important concepts of frame theory like, perturbation, stability etc. in the setting of n-Hilbert space.

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