Journal of
Linear and Topological Algebra
Vol. 10, *No.* 02, 2021, 79- 91

Some properties of Moore−Penrose inverse of weighted composition operators

M. Sohrabi[a,*]

[a]*Department of Mathematics, Lorestan University, Khorramabad, Iran.*

Received 1 June 2020; Revised 16 April 2021; Accepted 24 April 2021.
Communicated by Ghasem Soleimani Rad

Abstract. In this paper, we give an explicit formula for the Moore-Penrose inverse of W, denoted by $W^\dagger$, on $L^2(\Sigma)$. As an application, we give a characterization for some operator classes that are weaker than p-hyponormal with $W^\dagger$. Moreover, we give specific examples illustrating these classes.

Keywords: Moore-Penrose inverse, weighted composition operator, polar decomposition, partial isometry, hyponormal operator.

2010 AMS Subject Classification: 47B20, 47B38.

1. Introduction and preliminaries

Let (X, Σ, μ) be a sigma finite measure space and let $\varphi : X \to X$ be a measurable transformation such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ \varphi^{-1}/d\mu$ is finite valued or equivalently $(X, \varphi^{-1}(\Sigma), \mu)$ is sigma finite. We use the notation $L^2(\varphi^{-1}(\Sigma))$ for $L^2(X, \varphi^{-1}(\Sigma), \mu_{|\varphi^{-1}(\Sigma)})$ and henceforth we write μ in place of $\mu_{|\varphi^{-1}(\Sigma)}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote that the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. For a finite valued function $u \in L^0(\Sigma)$, the weighted composition operator W on $L^2(\Sigma)$ induced by φ and u is given by $W = M_u \circ C_\varphi$ where M_u is a multiplication operator and C_φ is a composition operator on $L^2(\Sigma)$ defined by $M_u f = uf$ and $C_\varphi f = f \circ \varphi$, respectively.

*Corresponding author.
E-mail address: sohrabi.mo@lu.ac.ir (M. Sohrabi).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

Let $\mathcal{A} = \varphi^{-1}(\Sigma)$, $0 \leqslant u \in L^0(\Sigma)$. If $\varphi^{-1}(\Sigma) \subseteq \Sigma$, there exists an operator $E := E^{\varphi^{-1}(\Sigma)} : L^p(\Sigma) \to L^p(\mathcal{A})$ which is called conditional expectation operator. $\mathcal{D}(E)$, the domain of E, contains the set of all non-negative measurable functions and each $f \in L^p(\Sigma)$ with $1 \leqslant p \leqslant \infty$, which satisfies

$$\int f d\mu = \int_A E(f) d\mu, \qquad A \in \varphi^{-1}(\Sigma).$$

Recall that $E : L^2(\Sigma) \to L^2(\mathcal{A})$ is a surjective, positive and contractive orthogonal projection. For more details on the properties of E see [16, 22, 23]. Since by the change of variable formula,

$$\int_X f \circ \varphi d\mu = \int_X h f d\mu, \qquad f \in L^1(\Sigma),$$

then $\|Wf\|_2 = \|\sqrt{hE(|u|^2) \circ \varphi^{-1}} f\|_2$. Put $J = hE(|u|^2) \circ \varphi^{-1}$. It follows that W is bounded on $L^2(\Sigma)$ if and only if $J \in L^\infty(\Sigma)$ (see [17] and also [8] for a discussion of $E(\cdot) \circ \varphi^{-1}$ when φ is not invertible).

Composition operators as an extension of shift operators are a good tool for separating weak hyponormal classes. Classic seminormal (weighted) composition operators have been extensively studied by Harrington and Whitley [15], Lambert [17, 22], Singh [24], Campbell [6–8] and Stochel [11]. In [4, 5] some weak hyponormal classes of composition operators are studied. In those works, examples were given which show that composition operators can be used to separate each partial normality class from quasinormal through w-hyponormal. But in some cases composition operators can not be separated some of these classes. Hence, it is better that we consider the weighted case of composition operators. In [10, 18], the authors generalized the work done in [4] and have obtained some characterizations of related p-hyponormal weighted composition operators as separately. In [18] some examples were presented to illustrate that weighted composition operators lie between those classes. We then give specific examples illustrating these classes.

Given a complex separable Hilbert space H, let $B(H)$ denote the linear space of all bounded linear operators on H. $N(T)$ and $R(T)$ denote the null-space and range of an operator T, respectively. Let $T = U|T|$ be the polar decomposition of T. Associated with $T \in B(H)$ there is a useful related operator $\widetilde{T} = |T|^{1/2} U |T|^{1/2}$, called the Aluthge transform of T. Let $CR(H)$ be the set of all bounded linear operators on H with closed range. For $T \in CR(H)$, the Moore-Penrose inverse of T, denoted by $T^\dagger$, is the unique operator $T^\dagger \in CR(H)$ that satisfies the following:

$$TT^\dagger T = T, \quad T^\dagger T T^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.$$

We recall that $T^\dagger$ exists if and only if $T \in CR(H)$. The Moore-Penrose inverse is designed as a measure for the invertibility of an operator. If $T = U|T|$ is invertible, then $T^{-1} = T^\dagger$, U is unitary and so $|T| = (T^*T)^{1/2}$ is invertible. It is a classical fact that the polar decomposition of T^* is $U^*|T^*|$. It is easy to check that $U^*|T^*|^\dagger$ and $|T^\dagger|^{\frac{1}{2}} U^* |T^\dagger|^{\frac{1}{2}}$ are the polar decomposition and Aluthge transform of $T^\dagger$, respectively. For other important properties of $T^\dagger$ see [1–3, 9, 12, 13, 20].

2. Main results

Let $CR(H)$ be the set of all bounded linear operators on H with closed range. Also, $W \in CR(L^2(\Sigma))$ if and only if J is bounded away from zero on $\sigma(J)$ (see [17]). In the following we give an explicit formula for the Moore-Penrose inverse of W. In addition, to avoid tedious calculations, we investigate some characterizations of weak p-hyponormal and $A(p)$ classes of Moore-Penrose inverse of weighted composition operators on $L^2(\Sigma)$. We give specific examples illustrating these classes. From now on, we assume that W has closed range.

Theorem 2.1 Let $W \in CR(L^2(\Sigma))$. Then $W^\dagger = M_{\frac{\chi_{\sigma(J)}}{J}} W^*$ and $(W^*)^\dagger = M_{\frac{\chi_{\sigma(J\circ\varphi)}}{J\circ\varphi}} W$.

Proof. Since $W \in CR(L^2(\Sigma))$ has closed range. Put $\Lambda = M_{\frac{\chi_{\sigma(J)}}{J}} W^*$. Then $\Lambda \in CR(L^2(\Sigma))$, because $\frac{\chi_{\sigma(J)}}{J}$ is bounded away from zero on X. Since for each $f \in L^2(\Sigma)$, $W^* f = hE(uf) \circ \varphi^{-1}$, then we have

$$\begin{aligned} W\Lambda W f &= u(\Lambda W f) \circ \varphi \\ &= u(\frac{\chi_{\sigma(J)}}{J} hE(u^2) \circ \varphi^{-1} f) \circ \varphi \\ &= u(\frac{\chi_{\sigma(J)}}{J} hE(u^2)\varphi^{-1} f) \circ \varphi \\ &= u\chi_{\sigma(J\circ\varphi)} f \circ \varphi. \end{aligned}$$

Since, $u \geqslant 0$ and $\sigma(h \circ \varphi) = X$, $\sigma(J \circ \varphi)) = \sigma(h \circ \varphi E(u^2)) = \sigma(E(u^2)) \supseteq \sigma(u)$, and so

$$\begin{aligned} W\Lambda W f &= (u\chi_{\sigma(u)})\chi_{\sigma(E(u^2))} f \circ \varphi \\ &= (u\chi_{\sigma(u)}) f \circ \varphi = W f. \end{aligned}$$

Also,

$$\begin{aligned} \Lambda W \Lambda f &= \chi_{\frac{\sigma(J)}{J}} hE(uW\Lambda f) \circ \varphi^{-1} \\ &= \chi_{\frac{\sigma(J)}{J}} hE(u^2(\Lambda f) \circ \varphi) \circ \varphi^{-1} \\ &= \chi_{\frac{\sigma(J)}{J}} h(E(u^2)(\Lambda f) \circ \varphi) \circ \varphi^{-1} \\ &= \chi_{\frac{\sigma(J)}{J}} (hE(u^2) \circ \varphi^{-1})\Lambda f \\ &= \chi_{\sigma(J)} \Lambda f = \Lambda f. \end{aligned}$$

Similar computations show that

$$W\Lambda = M_{\frac{u\chi_{\sigma(E(u))}}{E(u^2)}} EM_u = (W\Lambda)^*$$

and $\Lambda W = M_{\chi_{\sigma(J)}} = (\Lambda W)^*$. This completes the proof. ∎

Corollary 2.2 Let $C_\varphi \in CR(L^2(\Sigma))$. Then ${C_\varphi}^\dagger = M_{\frac{\chi_{\sigma(h)}}{h}} {C_\varphi}^*$.

Theorem 2.3 Let $W \in CR(L^2(\Sigma))$. Then $\|W^\dagger\| = \|\frac{\chi_{\sigma(J)}}{\sqrt{J}}\|_\infty$.

Proof. We have $W^\dagger = M_{\frac{\chi_{\sigma(J)}}{J}} W^*$. Put $\omega := \frac{\chi_{\sigma(J)}}{J}$, hence $W^\dagger = M_\omega W^*$, and so $(W^\dagger)^* = WM_\omega$. Then for each $f \in L^2(\Sigma)$, we get that

$$\begin{aligned}\infty > \|(W^\dagger)^*\|^2 = |WM_\omega f\|^2 &= \int_X |u^2(f\circ\varphi)(\omega\circ\varphi)|^2 d\mu \\ &= \int_X E(u^2)|(f.\omega)\circ\varphi|^2 d\mu = \int_X (hE(u^2)\circ\varphi^{-1})|f.\omega|^2 d\mu \\ &= \int_X |\omega\sqrt{J}f|^2 d\mu = \|M_{\omega\sqrt{J}}f\|^2.\end{aligned}$$

Thus,

$$\|W^\dagger\| = \|M_{\omega\sqrt{J}}\| = \|\omega\sqrt{J}\|_\infty = \|\frac{\chi_{\sigma(J)}}{\sqrt{J}}\|_\infty.$$

■

***Example* 2.4** Let $w := \{m_n\}_{n=1}^\infty$ be a sequence of positive real numbers. Consider the space $\ell^2(w) = L^2(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $2^{\mathbb{N}}$ is the power set of natural numbers and μ is a measure on $2^{\mathbb{N}}$ defined by $\mu(\{n\}) = m_n$. Let $u = \{u(j)\}_{j=1}^\infty$ be a sequence of non-negative real numbers. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a non-singular measurable transformation. Direct computation shows that (see [22])

$$J(k) = \frac{1}{m_k} \sum_{j\in\varphi^{-1}(k)} (u(j))^2 m_j.$$

Thus $\sigma(J)^c = \{k \in \mathbb{N} : \varphi^{-1}(k) = \emptyset \text{ or } u(\varphi^{-1}(k)) = \{0\}\}$. It follows that $\|W^\dagger\| = \frac{1}{\mu}$, where

$$\mu := \inf_{k\in\sigma(J)} \frac{1}{m_k} \sum_{j\in\varphi^{-1}(k)} (u(j))^2 m_j > 0.$$

In particular, if $h \in \ell^\infty(w)$ and bounded away from zero on $\sigma(h)$, then $\|C_\varphi^\dagger\| = \frac{1}{\lambda}$, where $\lambda = \inf\{\frac{1}{m_k}\sum_{j\in\varphi^{-1}(k)} m_j : \varphi^{-1}(k) \neq \emptyset\}$.

Now, let B and C be bounded and positive operators on H such that $BC = CB$. Put $A = BC$. Then by using of the functional calculus we obtain $A^p = B^pC^p$, for each $p > 0$. In particular, take $B = M_\nu$ and $C = M_\omega EM_\omega$, where $0 \leqslant \nu \in L^0(\mathcal{A})$ and $0 \leqslant \omega \in L^0(\Sigma)$. Then we can obtain from direct computations that $C^p = M_{\omega E(\omega^2)^{p-1}} EM_\omega$. Consequently, we have the following lemma.

Lemma 2.5 [19] Let $0 \leqslant \nu \in L^0(\mathcal{A})$, $0 \leqslant \omega \in L^0(\Sigma)$ and let $A := M_{\nu\omega}EM_\omega \in B(L^2(\Sigma))$. Then for each $p \in (0,\infty)$, $A^p = M_{\nu^p\omega E(\omega^2)^{p-1}}EM_\omega$.

Let $p > 0$, in the following two theorems, to avoid tedious calculations, we investigate only p-hyponormal and p-quasihyponormal classes of Moore-Penrose inverse of weighted composition operators. Note that an operator $T \in B(H)$ is p-hyponormal if and only if $(T^*T)^p \geqslant (TT^*)^p$, for $0 < p \leqslant 1$[1]. If $p = 1$, T is called hyponormal, and T is p-quasihyponormal if and only if $T^*(T^*T)^pT \geqslant T^*(TT^*)^pT$. An operator T is said to be p-paranormal if $\||T|^pU|T|^px\| \geqslant \||T|^px\|^2$ and operator T is absolute-p-paranormal operator if $\||T|^pTx\| \geqslant \|Tx\|^{p+1}$, for every unit vector $x \in H$. From now on, we assume that $K := hE(u) \circ \varphi^{-1}$.

Theorem 2.6 [5] W is p-hyponormal if and only if $\sigma(u) \subseteq \sigma(J)$ and

$$(h^p \circ \varphi)E(\frac{u^2(E(u^2))^{p-1}\chi_{\sigma(J)}}{J^p}) \leqslant 1.$$

Theorem 2.7 Let $W \in CR(L^2(\Sigma))$. Then $W^\dagger$ is p-hyponormal if and only if

$$\frac{(E(u))^2}{(J \circ \varphi)^p E(u^2)} \geqslant \frac{1}{J^p} \quad \text{on } \sigma(J \circ \varphi).$$

Proof. Let $f \in L^2(\Sigma)$. By direct computations and Lemma 2.5, we get that

$$(W^{\dagger *} W^\dagger)^p f = |W^\dagger|^{2p} f = \frac{\chi_{\sigma(J\circ\varphi)}}{(J \circ \varphi)^p E(u^2)} uE(uf),$$
$$(W^\dagger W^{\dagger *})^p f = |W^{*\dagger}|^{2p} f = \frac{1}{J^p} f.$$

Then $W^\dagger$ is p-hyponormal if and only if

$$\langle \frac{\chi_{\sigma(J\circ\varphi)}}{(J \circ \varphi)^p E(u^2)} uE(uf) - \frac{1}{J^p} f, f\rangle \geqslant 0. \tag{1}$$

Put $f = \chi_{\varphi^{-1}B}$ with $\mu(\varphi^{-1}B) < \infty$. Hence, (1) holds if and only if

$$\int_{\varphi^{-1}B} \{\frac{\chi_{\sigma(J\circ\varphi)}}{(J \circ \varphi)^p E(u^2)} uE(u) - \frac{1}{J^p}\} d\mu \geqslant 0.$$

Equivalently,

$$\int_B \{\frac{\chi_{\sigma(J)}}{J^p E(u^2) \circ \varphi^{-1}} (E(u) \circ \varphi^{-1})^2 - \frac{1}{J^p \circ \varphi^{-1}}\} h d\mu \geqslant 0.$$

But, this is equivalent to

$$\frac{\chi_{\sigma(J)}}{J^p E(u^2) \circ \varphi^{-1}} (E(u) \circ \varphi^{-1})^2 - \frac{1}{J^p \circ \varphi^{-1}} \geqslant 0,$$

on $\sigma(J)$. Equivalently, by composing with φ, we get that

$$\frac{(E(u))^2}{(J \circ \varphi)^p E(u^2)} \geqslant \frac{1}{J^p},$$

on $\sigma(J \circ \varphi)$. Then $W^\dagger$ is p-hyponormal if and only if

$$\frac{(E(u))^2}{(J \circ \varphi)^p E(u^2)} \geqslant \frac{1}{J^p},$$

on $\sigma(J\circ\varphi)$. ■

Corollary 2.8 Let $C_\varphi \in CR(L^2(\Sigma))$. Then ${C_\varphi}^\dagger$ is p-hyponormal if and only if $(h\circ\varphi)^p \leqslant h^p$.

Lemma 2.9 [21] Let α and β be nonnegative and measurable functions. Then, for every $f \in L^2(\Sigma)$,

$$\int_X \alpha|f|^2 d\mu \geqslant \int_X |E(\beta f)|^2 d\mu$$

if and only if $\sigma(\beta) \subseteq \sigma(\alpha)$ and $E(\frac{\beta^2}{\alpha}\chi_{\sigma(\alpha)}) \leqslant 1$.

Theorem 2.10 Let $W \in CR(L^2(\Sigma))$. Then $(W^*)^\dagger$ is p-hyponormal if and only if $\sigma(u) \subseteq \sigma(J)$ and

$$E(\frac{u^2}{J^p}\chi_{\sigma(J)}) \leqslant (J^p\circ\varphi)E(u^2).$$

Proof. Let $f \in L^2(\Sigma)$. Since $(W^*)^\dagger = (W^\dagger)^*$, then we get that

$$\langle(({W^*}^\dagger)^*({W^*}^\dagger))^p f, f\rangle = \langle((W^\dagger)(W^\dagger)^*)^p f, f\rangle = \int_X \frac{\chi_{\sigma(J)}}{J^p}|f|^2 d\mu,$$

and

$$\begin{aligned}\langle(({W^*}^\dagger)({W^*}^\dagger)^*)^p f, f\rangle = \langle((W^\dagger)^*(W^\dagger))^p f, f\rangle &= \int_X \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^{2p}} u(h^p\circ\varphi)(E(u^2))^{p-1}E(uf)\bar{f}d\mu \\ &= \int_X |E(\frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^p}(h^{\frac{p}{2}}\circ\varphi)(E(u^2))^{\frac{p-1}{2}}uf)|^2 d\mu.\end{aligned}$$

Put $\alpha = \frac{\chi_{\sigma(J)}}{J^p}$ and $\beta = \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^p}(h^{\frac{p}{2}}\circ\varphi)(E(u^2))^{\frac{p-1}{2}}u = \frac{u\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^{\frac{p}{2}}(E(u^2))^{\frac{1}{2}}}$. Then $\sigma(\alpha) = \sigma(J)$ and $\sigma(\beta) = \sigma(u)$. Now, the desired conclusion follows from Lemma 2.9. ■

Let $B(H)$ be the algebra of all bounded linear operators on the infinite dimensional complex Hilbert space H. Let $T = U|T|$ be the polar decomposition for $T \in B(H)$, where U is a partial isometry and $|T| = (T^*T)^{1/2}$. In the following we concentrate on the polar decomposition of $W^\dagger$ and $(W^*)^\dagger$.

Proposition 2.11 Let $W \in CR(L^2(\Sigma))$. Then $W^\dagger = U^*|W^\dagger|$ is the polar decomposition of $W^\dagger$, such that

$$\begin{aligned}|W^\dagger|(f) &= \frac{\chi_{\sigma(J\circ\varphi)}}{\sqrt{J\circ\varphi}(E(u^2))}uE(uf), \\ U^*(f) &= (\frac{\chi_{\sigma(J)}}{J})^{\frac{1}{2}}W^*f.\end{aligned}$$

Proof. Let $f \in L^2(\Sigma)$. Then $({W^\dagger}^* W^\dagger)(f) = \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^2}u(h\circ\varphi)E(uf)$. Now $|W^\dagger|$ follows from Lemma 2.5. Moreover, by direct computations we have $U^* = M_{\frac{\chi_{(J)}}{\sqrt{J}}}W^*$. Moreover, it is easy to check that $U^*|W^\dagger| = W^\dagger$, $U^*UU^* = U^*$ and $\mathcal{N}(U^*) = \mathcal{N}(W^*) = \mathcal{N}(W^\dagger)$. This completes the proof. ■

Similar to above result we obtain the following proposition.

Proposition 2.12 Let $W \in CR(L^2(\Sigma))$. Then $(W^*)^\dagger = U|(W^*)^\dagger|$ is the polar decomposition of $(W^*)^\dagger$ such that

$$|(W^*)^\dagger|(f) = \frac{1}{\sqrt{J}}(f),$$
$$U(f) = \frac{\chi_{\sigma(J\circ\varphi)}}{\sqrt{J\circ\varphi}} Wf.$$

***Example* 2.13** Let $X = (0,1)$, equipped with the Lebesgue measure μ on the Lebesgue measurable subsets. Set $u(x) = \sqrt{x}$ and let $\varphi : X \to X$ is defined by

$$\varphi(x) = \begin{cases} 2x & 0 < x < \frac{1}{2}, \\ 2-2x & \frac{1}{2} \leqslant x < 1. \end{cases}$$

Direct computation (see [25]) shows that for each $f \in L^2(\Sigma)$,

$$E(f)\circ\varphi^{-1}(x) = \frac{1}{2}\left(f(\frac{x}{2}) + f(1-\frac{x}{2})\right).$$

It follows that $J(x) = \frac{1}{2}\left(u^2(\frac{x}{2}) + u^2(1-\frac{x}{2})\right) = \frac{1}{2}$, $J\circ\varphi = E(u^2) = \frac{1}{2}$. Hence we get that

$$W^\dagger(f) = \sqrt{\frac{x}{2}} f(\frac{x}{2}) + \sqrt{1-\frac{x}{2}} f(1-\frac{x}{2});$$
$$|W^\dagger|(f) = \sqrt{2x}\{\sqrt{x} f(x) + \sqrt{1-x} f(1-x)\};$$
$$U^*(f) = \sqrt{x} f(\frac{x}{2}) + \sqrt{2-x} f(1-\frac{x}{2});$$
$$(W^\dagger)^*(f) = \begin{cases} 2\sqrt{x} f(2x) & 0 < x < \frac{1}{2}, \\ 2\sqrt{x} f(2-2x) & \frac{1}{2} \leqslant x < 1; \end{cases}$$
$$|(W^*)^\dagger|(f) = \sqrt{2}(f);$$
$$U(f) = \begin{cases} \sqrt{2x} f(2x) & 0 < x < \frac{1}{2}, \\ \sqrt{2x} f(2-2x) & \frac{1}{2} \leqslant x < 1. \end{cases}.$$

Theorem 2.14 [14] Let $W \in L^2(\Sigma)$. Then W is p-quasihyponormal if and only if

$$E(u^2 J^p) \geqslant (h^p \circ \varphi)(E(u^2))^{p+1}.$$

Theorem 2.15 Let $W \in CR(L^2(\Sigma))$. Then $W^\dagger$ is p-quasihyponormal if and only if

$$\frac{(E(u))^2}{E(u^2)} E(\frac{K}{J}) \geqslant (\frac{J\circ\varphi}{J})^p \frac{K}{J}, \quad \text{on } \sigma(J).$$

Proof. Let $f \in L^2(\Sigma)$. Direct computations show that

$$(W^\dagger)^*((W^\dagger)^*W^\dagger)^p W^\dagger f = \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)(J\circ\varphi^2)^p E(u^2)\circ\varphi} u(u\circ\varphi)E(u\frac{\chi_{\sigma(J)}}{J}hE(uf)\circ\varphi^{-1})\circ\varphi := af,$$

$$(W^\dagger)^*(W^\dagger(W^\dagger)^*)^p W^\dagger f = \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^{p+2}} u(h\circ\varphi)E(uf) := bf.$$

Then $W^\dagger$ is p-quasihyponormal if and only if

$$\langle af - bf, f\rangle \geqslant 0 \tag{2}$$

for each $\lambda \in (0,\infty)$. Put $f = \chi_{\varphi^{-1}B}$ with $\mu(\varphi^{-1}B) < \infty$. Hence, (2) holds if and only if

$$\int_{\varphi^{-1}B} \{\frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)(J\circ\varphi^2)^p E(u^2)\circ\varphi} u(u\circ\varphi)E(u\frac{\chi_{\sigma(J)}}{J}hE(u)\circ\varphi^{-1})\circ\varphi$$
$$- \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^{p+2}} u(h\circ\varphi)E(u)\} d\mu \geqslant 0.$$

Equivalently,

$$\int_B \{\frac{\chi_{\sigma(J\circ\varphi)}}{J(J\circ\varphi)^p E(u^2)}(E(u))(E(u)\circ\varphi^{-1})E(u\frac{\chi_{\sigma(J)}}{J}hE(u)\circ\varphi^{-1})$$
$$- \frac{\chi_{\sigma(J)}}{J^{p+2}} hE(u)^2\circ\varphi^{-1}\} h d\mu \geqslant 0.$$

This is equivalent to

$$\frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^p E(u^2)}(E(u))^2(E(u)\circ\varphi^{-1})E(\frac{K}{J}) - \frac{\chi_{\sigma(J)}}{J^{p+1}} hE(u)^2\circ\varphi^{-1} \geqslant 0,$$

on $\sigma(J)$. Then $W^\dagger$ is p-quasihyponormal if and only if

$$\frac{(E(u))^2}{E(u^2)}E(\frac{K}{J}) \geqslant (\frac{J\circ\varphi}{J})^p \frac{K}{J}, \quad \text{on } \sigma(J).$$

■

Corollary 2.16 Let $C_\varphi \in CR(L^2(\Sigma))$. Then ${C_\varphi}^\dagger$ is p-quasihyponormal if and only if $h \geqslant (h\circ\varphi)$ on $\sigma(h)$.

Theorem 2.17 Let $W \in CR(L^2(\Sigma))$. Then $(W^*)^\dagger$ is p-quasihyponormal if and only if

$$(h^p\circ\varphi)(E(u^2))^{p-1}E(\frac{u^2}{J^p}) \geqslant 1, \quad \text{on } \sigma(J).$$

Proof. Let $f \in L^2(\Sigma)$. It is easy to check that

$$W^\dagger(W^\dagger(W^\dagger)^*))^p(W^\dagger)^* f = \frac{\chi_{\sigma(J)}}{J^2} hE(\frac{u^2}{J^p})\circ\varphi^{-1} f,$$

$$(W^\dagger)((W^\dagger)^*)W^\dagger)^p(W^\dagger)^* f = \frac{\chi_{\sigma(J)}}{J^{p+1}} f.$$

Thus, $(W^*)^\dagger$ is p-quasihyponormal if and only if

$$\frac{hE(\frac{u^2}{J^p})\circ\varphi^{-1}}{J^2} \geqslant \frac{1}{J^{p+1}}$$

on $\sigma(J)$. Equivalently, $(h^p\circ\varphi)(E(u^2))^{p-1}E(\frac{u^2}{J^p}) \geqslant 1$ on $\sigma(J)$. ■

For $k>0$, an operator $T\in B(H)$ belongs to class $A(p)$ if $(T^*|T|^{2p}T)^{\frac{1}{p+1}} \geqslant |T|^2$ ([26]).

Theorem 2.18 [18] Let $W\in L^2(\Sigma)$. Then is belongs to class $A(p)$ if and only if

$$E(u^2J^p) \geqslant (h^p\circ\varphi)(E(u^2))^{p+1}.$$

Theorem 2.19 Let $W\in CR(L^2(\Sigma))$. Then $W^\dagger$ is belongs to class $A(p)$ if and only if

$$(\frac{E(u^2)}{E(u^2)\circ\varphi})^{p+1}E(\frac{K}{J})\circ\varphi \geqslant (\frac{E(u)}{E(u)\circ\varphi})^2, \quad \text{on } \sigma(J).$$

Proof. Let $f\in L^2(\Sigma)$. By easy calculations we get that

$$(W^\dagger)^*|W^\dagger|^{2p}W^\dagger f = \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)(J\circ\varphi^2)^p(E(u^2)\circ\varphi)}(u\circ\varphi)E(u\frac{\chi_{\sigma(J)}}{J}W^*f),$$
$$|W^\dagger|^2 f = \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^2}u(h\circ\varphi)E(uf).$$

Then by similar to the proof of Theorem 2.15, the proof is complete. ■

Lemma 2.20 Let $T\in B(H)$ and let $U|T|$ be its polar decomposition. Suppose p is a positive real number. Then we have the following assentations:
(i) [26] T is p-paranormal if and only if for each $\lambda>0$,

$$|T^*|^p|T|^{2p}|T^*|^p - 2\lambda|T^*|^{2p} + \lambda^2 \geqslant 0.$$

(ii) [26] T is absolute-p-paranormal if and only if for each $\lambda>0$,

$$|T^*||T|^{2p}|T^*| - (p+1)\lambda^p|T^*|^2 + p\lambda^{p+r} \geqslant 0.$$

Theorem 2.21 Let $W\in CR(L^2(\Sigma))$. Then The following statements are hold.
(i) $W^\dagger$ is p-paranormal if and only if

$$\frac{E(u)}{E(u^2)}E(\frac{u}{\sqrt{J^p}}) \geqslant (\frac{J\circ\varphi}{J\sqrt{J}})^p, \quad \text{on } \sigma(J).$$

(ii) $W^\dagger$ is absolute-p-paranormal if and only if

$$\sqrt{J}\frac{E(u)}{E(u^2)}E(\frac{u}{\sqrt{J}}) \geqslant (\frac{J\circ\varphi}{J})^p, \quad \text{on } \sigma(J).$$

Proof. (i) Let $f \in L^2(\Sigma)$. It is easy to check that

$$|W^{\dagger}|^{2p} f = \frac{\chi_{\sigma(J\circ\varphi)}}{(J\circ\varphi)^p E(u^2)} uE(uf),$$
$$|W^{\dagger}|^{p} f = \frac{\chi_{\sigma(J\circ\varphi)}}{\sqrt{(J\circ\varphi)^p} E(u^2)} uE(uf),$$
$$|W^{\dagger *}|^{2p} f = \frac{1}{J^p} f.$$

It follows that

$$|W^{\dagger *}|^{p}|W^{\dagger}|^{2p}|W^{\dagger *}|^{p} f = \frac{\chi_{\sigma(J\circ\varphi)}}{\sqrt{J^p}(J\circ\varphi)^p E(u^2)} uE(\frac{uf}{\sqrt{J^p}}).$$

Now, by Lemma 2.20 (i), $W^{\dagger}$ is p-paranormal if and only if

$$\langle \frac{\chi_{\sigma(J)}}{\sqrt{J^p}(J\circ\varphi)^p E(u^2)} uE(\frac{uf}{\sqrt{J^p}}) - 2\lambda \frac{\chi_{\sigma(J)}}{J^p} f + \lambda^2, f\rangle \geqslant 0, \tag{3}$$

for each $\lambda \in (0,\infty)$. Set $f = \chi_{\varphi^{-1}B}$ with $\mu(\varphi^{-1}B) < \infty$. Hence, (3) holds if and only if

$$\int_{\varphi^{-1}B} \{\frac{\chi_{\sigma(J)}}{\sqrt{J^p}(J\circ\varphi)^p E(u^2)} uE(\frac{u\chi_B\circ\varphi}{\sqrt{J^p}}) - 2\lambda \frac{\chi_{\sigma(J)}}{J^p}(\chi_B\circ\varphi) + \lambda^2\} d\mu \geqslant 0.$$

Equivalently,

$$\int_B \{\frac{\chi_{\sigma(J)} E(u)\circ\varphi^{-1}}{\sqrt{J^p}\circ\varphi^{-1} J^p E(u^2)\circ\varphi^{-1}} E(\frac{u}{\sqrt{J^p}})\circ\varphi^{-1} - 2\lambda \frac{\chi_{\sigma(J)}}{J^p\circ\varphi^{-1}} + \lambda^2\} h d\mu \geqslant 0.$$

This is equivalent to

$$\frac{\chi_{\sigma(J)} E(u)\circ\varphi^{-1}}{\sqrt{J^p}\circ\varphi^{-1} J^p E(u^2)\circ\varphi^{-1}} E(\frac{u}{\sqrt{J^p}})\circ\varphi^{-1} - 2\lambda \frac{\chi_{\sigma(J)}}{J^p\circ\varphi^{-1}} + \lambda^2 \geqslant 0.$$

Put

$$\frac{E(u)\circ\varphi^{-1}}{\sqrt{J^p}\circ\varphi^{-1} J^p E(u^2)\circ\varphi^{-1}} E(\frac{u}{\sqrt{J^p}})\circ\varphi^{-1} := a$$

and

$$b := \frac{1}{J^p\circ\varphi^{-1}}$$

Then, $W^{\dagger}$ is p-paranormal if and only if

$$D(\lambda) := a - 2b\lambda + \lambda^2 \geqslant 0, \qquad \lambda \in (0,\infty).$$

Since $\min_{\lambda\in(0,\infty)} D(\lambda) = D(b)$, it follows that

$$\begin{aligned}
D(b) \geqslant 0 &\Longleftrightarrow a \geqslant b^2 \\
&\Longleftrightarrow \frac{E(u)\circ\varphi^{-1}}{\sqrt{J^p}\circ\varphi^{-1}J^pE(u^2)\circ\varphi^{-1}}E(\frac{u}{\sqrt{J^p}})\circ\varphi^{-1} \geqslant (\frac{1}{J^p\circ\varphi^{-1}})^2 \\
&\Longleftrightarrow \frac{E(u)}{\sqrt{J^p}(J^p\circ\varphi)E(u^2)}E(\frac{u}{\sqrt{J^p}}) \geqslant \frac{1}{J^{2p}}, \\
&\Longleftrightarrow \frac{E(u)}{E(u^2)}E(\frac{u}{\sqrt{J^p}}) \geqslant (\frac{J\circ\varphi}{J\sqrt{J}})^p, \quad \text{on } \sigma(J)
\end{aligned}$$

(ii) The proof is similar to part (i). ∎

***Example* 2.22** Let $X = (0,1)$, equipped with the Lebesgue measure $d\mu = dx$ on the Lebesgue measurable subsets of X and let $\varphi : X \to X$ be a non-singular measurable transformation defined by and

$$\varphi(x) = \begin{cases} 2x & 0 < x \leqslant \frac{1}{2}, \\ 2-2x & \frac{1}{2} \leqslant x < 1. \end{cases}$$

Then for each $f \in L^2(\Sigma)$ and $x \in X$ we have

$$\begin{aligned}
h(x) &= \left| \frac{d}{dx}(\frac{x}{2})\right| + \left|\frac{d}{dx}(\frac{2-x}{2})\right| = 1; \\
(Ef)(x) &= \frac{f(x)+f(1-x)}{2}; \\
(E(f)\circ\varphi^{-1})(x) &= \frac{1}{2}(f(\frac{x}{2}) + f(1-\frac{x}{2})).
\end{aligned}$$

Put $u(x) = \sqrt{x}$. Direct computation show that

$$\begin{aligned}
E(u) &= \frac{\sqrt{x}+\sqrt{1-x}}{2}; \\
E(u)\circ\varphi^{-1} &= \frac{\sqrt{\frac{x}{2}}+\sqrt{1-\frac{x}{2}}}{2}; \\
E(u^2) &= \frac{1}{2}; \\
E(u^2)\circ\varphi^{-1} &= \frac{1}{2}; \\
J &= \frac{1}{2}; \\
J\circ\varphi &= \frac{1}{2}.
\end{aligned}$$

Then by Theorem 2.20, $W^\dagger$ is p-hyponormal if and only if $x = \frac{1}{2}$. Also, by Theorem

2.15, $W^\dagger$ is p-quasihyponormal if and only if

$$(1+2\sqrt{x(1-x)})\{\sqrt{(\sqrt{\frac{x}{2}}+\sqrt{1-\frac{x}{2}})}+\sqrt{1-(\sqrt{\frac{x}{2}}+\sqrt{1-\frac{x}{2}})}\}\geqslant 2(\sqrt{\frac{x}{2}}+\sqrt{1-\frac{x}{2}}). \tag{4}$$

In particular, if $p>0$, $x=\frac{1}{2}$ then $W^\dagger$ is p-hyponormal, but (4) is not holds on $X=(0,1)$. So $W^\dagger$ is not p-quasihyponormal, for every $p>0$. Also $W^\dagger$ is p-paranormal if and only if

$$(\sqrt{x}-\sqrt{1-x})(\sqrt{2^p x}-\sqrt{1-2^p x})\geqslant 2\sqrt{2^p}.$$

But, the above inequality is not holds on $X=(0,1)$, then $W^\dagger$ is not p-paranormal, and also $W^\dagger$ is absolute-p-paranormal if and only if $\sqrt{2}(\sqrt{x}-\sqrt{1-x})(\sqrt{2x}-\sqrt{1-2x})\geqslant 1$. But this relationship is hold on the $(0.5,1)$, then $W^\dagger$ is absolute-p-paranormal on $(0.5,1)$.

***Example* 2.23** Let $X=(1,\infty)$, equipped with the Lebesgue measure $d\mu$ on the Lebesgue measurable subsets. The transformation φ and the weighted function $u(x)$ are given by $\varphi(x)=\sqrt{x}$ and $u(x)=\frac{1}{\sqrt{1+x}}$. Then $h(x)=2x$, $E=I$, $J(x)=\frac{2x}{1+x^2}$, $h\circ\varphi(x)=2\sqrt{x}$, $J\circ\varphi(x)=\frac{2\sqrt{x}}{1+x}$. So, by using the above theorems p-hyponormality, p-quasihyponormality and Belonging to class $A(p)$ for W is equivalent to $J\geqslant J\circ\varphi$. Therefor, W dose not lie in the above classes. Also, p-hyponormality of $W^\dagger$ is equivalent to $J\geqslant J\circ\varphi$, then $W^\dagger$ is not p-hyponormal. Moreover, p-quasihyponormality of $W^\dagger$ is equivalent to $J\geqslant J\circ\varphi$ and also $W^\dagger$ is a class $A(p)$ operator if and only if $(\frac{1+\sqrt{x}}{1+x})^p\sqrt{1+x^2}\geqslant 1$. In particular, if $p=1$, then $W^\dagger$ is not p-quasihyponormal but it is in the class of $A(p)$, while for very large values of p, $W^\dagger$ is not in none of the classes of $A(p)$ and p-quasihyponormal.

Acknowledgments

The author would like to express his deep gratitude to the referee(s) for his/her careful reading of the paper and helpful comments which improved the presentation of it.

References

[1] A. Aluthge, On p-hyponormal operators for $0<p<1$, Integr. Equ. Oper. Theory. 13 (1990), 307-315.
[2] M. L. Arias, G. Corach, M. C. Gonzalez, Generalized inverse and Douglas equations, Proc. Amer. Math. Soc. 136 (2008), 3177-3183.
[3] A. Ben-Israel, T. N. E. Greville, Generalized Inverses: Theory and Applications, Springer-Verlag, New York, 2003.
[4] C. Burnap, I. Jung, Composition operators with weak hyponormalities, J. Math. Anal. Appl. 337 (2008), 686-694.
[5] C. Burnap, I. Jung, A. Lambert, Separating partial normality classes with composition operators, J. Operator Theory. 53 (2005), 381-397.
[6] J. Campbell, M. Embry-Wardrop, R. Fleming, Normal and quasinormal weighted composition operators, Glasgow Math. J. 33 (1991), 275-279.
[7] J. Campbell, W. Hornor, Localising and seminormal composition operators on L^2, Proc. Roy. Soc. Edinburgh Sect. A. 124 (1994), 301-316.
[8] J. Campbell, J. Jamison, On some classes of weighted composition operators, Glasgow Math. J. 32 (1990), 87-94.
[9] S. R. Caradus, Generalized Inverse and Operator Theory, Queens Papers in Pure and Applied Mathematics, 50, Queens University, Kingston, Ont, 1978.
[10] M. Cho, T. Yamazaki, Characterizations of p-hyponormal and weak hyponormal weighted composition operators, Acta Sci. Math. 76 (2010), 173-181.
[11] A. Daniluk, J. Stochel, Seminormal composition operators induced by affine transformations, Hokkaido Math. J. 26 (1997), 377-404.

[12] D. S. Djordjević, Further results on the reverse order law for generalized inverses, Siam J. Matrix Anal. Appl. 29 (4) (2007), 1242-1246.
[13] D. S. Djordjević, N. C. Dincić, Reverse order law for the Moore-Penrose inverse, J. Math. Anal. Appl. 361 (2010), 252-261.
[14] H. Emamalipour, M. R. Jabbarzadeh, M. Sohrabi Chegeni, Some weak p-hyponormal classes of weighted composition operators, Filomat. 31 (9) (2017), 2643-2656.
[15] D. Harrington, R. Whitley, Seminormal composition operators, J. Operator Theory. 11 (1984), 125-135.
[16] J. Herron, Weighted conditional expectation operators, Oper. Matrices. 5 (2011), 107-118.
[17] T. Hoover, A. Lambert, J. Queen, The Markov process determined by a weighted composition operator, Studia Math. LXXII (1982), 225-235.
[18] M. R. Jabbarzadeh, M. R. Azimi, Some weak hyponormal classes of weighted composition operators, Bull. Korean. Math. Soc. 47 (2010), 793-803.
[19] M. R. Jabbarzadeh, M. Sohrabi Chegeni, Moore-Penrose inverse of conditional type operators, Oper. Matrices, 11 (2017), 289-299.
[20] F. Kimura, Analysis of non-normal operators via Aluthge transformation, Integr. Equ. Oper. Theory. 50 (2004), 375-384.
[21] A. Lambert, Hyponormal composition operators, Bull. London Math. Soc. 18 (1986), 395-400.
[22] A. Lambert, Localising sets for sigma-algebras and related point transformations, Proc. Roy. Soc. Edinburgh Sect. A. 118 (1991), 111-118.
[23] M. M. Rao, Conditional Measure and Applications, Marcel Dekker, New York, 1993.
[24] R. K. Singh, J. S. Manhas, Composition Operators on Function Spaces, North Holland Math. Studies 179, Amsterdam, 1993.
[25] R. Whitley, Normal and quasinormal composition operators, Proc. Amer. Math. Soc. 70 (1978), 114-118.
[26] T. Yamazaki, M. Yanagida, A further generalization of paranormal operators, Sci. Math. 3 (1) (2000), 23-31.