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Domination number of complements of functigraphs

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Abstract. Let $G = (V, E)$ be a simple graph. A subset $S \subseteq V(G)$ is a *dominating set* of G if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* of graph G, denoted by $\gamma(G)$, is the minimum size of a dominating set of vertices $V(G)$. Let G_1 and G_2 be two disjoint copies of graph G and $f : V(G_1) \to V(G_2)$ be a function. Then a *functigraph* G with function f is denoted by $C(G, f)$, its vertices and edges are $V(C(G, f)) = V(G_1) \cup V(G_2)$ and $E(C(G, f)) = E(G_1) \cup E(G_2) \cup \{vu | v \in V(G_1), u \in V(G_2), f(v) = u\}$, respectively. In this paper, we investigate domination number of complements of functigraphs. We show that for any connected graph G, $\gamma(\overline{C(G, f)}) \leqslant 3$. Also we provide conditions for the function f in some graphs such that $\gamma(\overline{C(G, f)}) = 3$. Finally, we prove if G is a bipartite graph or a connected $k-$ regular graph of order $n \geqslant 4$ for $k \in \{2, 3, 4\}$ and $G \notin \{K_3, K_4, K_5, H_1, H_2\}$, then $\gamma(\overline{C(G, f)}) = 2$.

Keywords: Domination, domination number, functigraph.

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1. Introduction

All graphs throughout this paper considered simple, finite and undirected. The *open neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices adjacent to v in G. The *closed neighborhood* of a vertex v in graph G is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) =| N_G(v) |$. We denote the *maximum* degree of

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G with $\Delta(G)$ and its *minimum degree* with $\delta(G)$. A vertex is called a *universal* vertex if it is adjacent to all of the vertices of the graph.

The *complement* of graph G is denoted by $\overline{G}$ and defined as a graph with vertex set $V(G)$ which $e \in E(\overline{G})$ if and only if $e \notin E(G)$. For any $S \subseteq V(G)$, the *induced subgraph* on S is denoted by $G[S]$.

A subset $S \subseteq V(G)$ is a dominating set of G if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum size of a dominating set of G.

The notations P_n, C_n, K_n, $K_{1,n}$, W_n and K_3^n are used for path, cycle, complete graph, star, wheel and friendship graph, respectively.

Let G_1 and G_2 be two disjoint copies of graph G and $f : V(G_1) \to V(G_2)$ be a function, where $V(G_1) = \{v_1, v_2, ..., v_n\}$ and $V(G_2) = \{u_1, u_2, ..., u_n\}$. Then a *functigraph* G with function f is denoted by $C(G, f)$, its vertices and edges are $V(C(G, f)) = V(G_1) \cup V(G_2)$ and $E(C(G, f)) = E(G_1) \cup E(G_2) \cup \{vu | v \in V(G_1), u \in V(G_2), f(v) = u\}$, respectively. For $u \in V(G_2)$, $f^{-1}(u) = \{v \in V(G_1) : f(v) = u\}$ and $R(f) = \{f(v) | v \in V(G_1)\}$. Also for each $0 \leqslant \ell \leqslant n$ we define $B_\ell = \{u \in V(G_2) \mid |f^{-1}(u)| = \ell\}$, where $n = |V(G)|$.

In recent years much attention drawn to the domination theory which is very interesting branch in graph theory. Recently, the concept of domination expanded to other parameters of domination such as $2-$rainbow domination, total domination, signed domination and Roman domination. For more details we refer reader to [1, 3, 4]. In 2012, Erol et al. studied the domination in functigraphs, see [2]. They proved that $\gamma(G) \leqslant \gamma(C(G, f)) \leqslant 2\gamma(G)$ and studied the domination number of $C(C_n, f)$.
In this paper, we study domination number of complements of functigraphs. We show that for any connected graph G, $\gamma(\overline{C(G, f)}) \leqslant 3$ and provide conditions for the function f such that $\gamma(\overline{C(G, f)}) = 3$. Finally, we prove if G is a bipartite graph or a connected $k-$ regular graph of order $n \geqslant 4$ for $k \in \{2, 3, 4\}$ and $G \notin \{K_3, K_4, K_5, H_1, H_2\}$, then $\gamma(\overline{C(G, f)}) = 2$.
The main results are the following.
Theorem A. Let graph G has a universal vertex. Then $\gamma(\overline{C(G, f)}) = 3$ if and only if:

(1) $\delta(G) \neq 1$,
(2) $B_1 = \emptyset$,
(3) For any $i \geqslant 2$ and any $u \in B_i$, $\delta(G_1[f^{-1}(u)]) \geqslant 1$,
(4) Every vertex in B_0 is adjacent to all of the vertices of B_i, for any $i \geqslant 2$,
(5) If $\{u, u'\} \subseteq \cup_{i \geqslant 2} B_i$ and u is not adjacent to u', then all of the vertices of $f^{-1}(u)$ are adjacent to each vertex of $f^{-1}(u')$.

Theorem B. Let $n \geqslant 6$ and G be a $(n-2)$-regular graph of order n. Then $\gamma(\overline{C(G, f)}) = 3$ if and only if:

1) $B_1 = \emptyset$.
2) If $u \in B_0$ and $u' \notin N_{G_2}(u)$, then $u' \in B_0$.
3) For each $x \in \cup_{i \geqslant 2} B_i$; $\delta(G_1[f^{-1}(x)]) \geqslant 1$.

Theorem C. Let G be a connected k-regular graph of order $n \geqslant 4$, which is not isomorphic to K_3, K_4, K_5, H_1 and H_2. If $k \in \{2, 3, 4\}$, then $\gamma(\overline{C(G, f)}) = 2$.

2. Preliminary

For investigating the domination number of complements of functigraphs, the following Lemmas are useful.

Lemma 2.1 For any connected graph G, $\gamma(\overline{C(G,f)}) \leqslant 3$.

Proof. Let $v_i \in V(G_1)$ and $u_j \in V(G_2) \setminus \{f(v_i)\}$. Then v_i dominates all of the vertices $V(G_2) \setminus \{f(v_i)\}$, u_j dominates all of the vertices $V(G_1) \setminus S_j$ and $f(v_i)$ dominates all of the vertices $V(G_1) \setminus S_i$ in $\overline{C(G,f)}$, where $S_j = f^{-1}(u_j)$ and $S_i = f^{-1}(f(v_i))$. Since $S_i \cap S_j = \emptyset$, so $\{v_i, f(v_i), u_j\}$ is a dominating set of $\overline{C(G,f)}$. Hence $\gamma(\overline{C(G,f)}) \leqslant 3$. ■

Lemma 2.2 Let G be a graph of order n. Then $\gamma(\overline{C(G,f)}) = 1$ if and only if there is an isolated vertex x in G such that $x \notin R(f)$.

Proof. If $x \in V(G)$ is an isolated vertex and $x \notin R(f)$, then x is an isolated vertex in $C(G,f)$. So x is a universal vertex in $\overline{C(G,f)}$. Thus $\{x\}$ is a dominating set of $\overline{C(G,f)}$ and $\gamma(\overline{C(G,f)}) = 1$.
Conversely, let $\gamma(\overline{C(G,f)}) = 1$ and $\{x\}$ be a dominating set of $\overline{C(G,f)}$. Then x is an isolated vertex in $C(G,f)$. Hence x is an isolated vertex in G and $x \notin R(f)$. ■

Lemma 2.3 Let G be a graph of order n with $\delta(G) \geqslant 1$. If $B_0 = \emptyset$ or $B_1 \neq \emptyset$, then $\gamma(\overline{C(G,f)}) = 2$.

Proof. If $B_0 = \emptyset$, then $B_1 = \{u_1, u_2, ..., u_n\}$. It is easy to see that for every $1 \leqslant i \leqslant n$, $\{v_i, f(v_i)\}$ is a dominating set of $\overline{C(G,f)}$. So $\gamma(\overline{C(G,f)}) \leqslant 2$. Since G does not have any isolated vertex, by Lemma 2.2, we have $\gamma(\overline{C(G,f)}) = 2$.
If $B_1 \neq \emptyset$ and $u \in B_1$, then we can see that $\{u, f^{-1}(u)\}$ is a dominating set of $\overline{C(G,f)}$. Hence $\gamma(\overline{C(G,f)}) \leqslant 2$. Since G does not have any isolated vertex, by Lemma 2.2, we have $\gamma(\overline{C(G,f)}) = 2$. ■

By Lemma 2.3, if f is a bijective function, then $\gamma(\overline{C(G,f)}) = 2$. So, in the following lemmas and theorems, f is not a bijective function.

Lemma 2.4 Let G be a graph and $\{u_i, u_j\} \subseteq V(G_2)$. Then

1) if u_i is not adjacent to u_j, $u_i \in R(f)$ and $u_j \notin R(f)$, then $\gamma(\overline{C(G,f)}) \leqslant 2$.
2) if $N_{G_2}(u_i) \cap N_{G_2}(u_j) = \emptyset$, then $\gamma(\overline{C(G,f)}) \leqslant 2$.

Proof.

1) Let $v_\ell \in V(G_1)$ and $f(v_\ell) = u_i$. Then v_ℓ dominates all of the vertices $V(G_2) \setminus \{u_i\}$ and u_j dominates all of the vertices $V(G_1) \cup \{u_i\}$ in $\overline{C(G,f)}$. So $\{v_\ell, u_j\}$ is a dominating set of $\overline{C(G,f)}$. Thus $\gamma(\overline{C(G,f)}) \leqslant 2$.
2) Let $N_{G_2}(u_i) = N_i$, $N_{G_2}(u_j) = N_j$, $f^{-1}(u_i) = S_i$ and $f^{-1}(u_j) = S_j$. Then u_i dominates all of the vertices $(V(G_1) \setminus S_i) \cup (V(G_2) \setminus N_i)$ and u_j dominates all of the vertices $(V(G_1) \setminus S_j) \cup (V(G_2) \setminus N_j)$. So $\{u_i, u_j\}$ is a dominating set of $\overline{C(G,f)}$. Therefore $\gamma(\overline{C(G,f)}) \leqslant 2$.

■

Lemma 2.5 Let $G \cong H_1$ and $R(f) = \{x_2, x_4, x_5\}$. If $\delta(G_1[f^{-1}(x_i)]) \geqslant 1$, for $i \in \{2, 4, 5\}$, then $\gamma(\overline{C(G,f)}) = 3$.

Proof. Let $\{a, b\}$ be a dominating set of $\overline{C(G,f)}$. Since $\overline{H_1}$ is a disconnected graph with two component of C_4 and K_3, so $\{a, b\} \nsubseteq V(G_i)$, $i \in \{1, 2\}$. Hence we may assume that $a \in V(G_1)$ and $b \in V(G_2)$. We know that $f(a)$ is not dominated by a in $\overline{C(G,f)}$. So $b \in \{x_2, x_4, x_5\}$. Since $\delta(G_1[f^{-1}(x_i)]) \geqslant 1$, there is at least one vertex in $f^{-1}(b)$ that is not dominated by a and b, which is a contradiction. So $\gamma(\overline{C(G,f)}) = 3$.

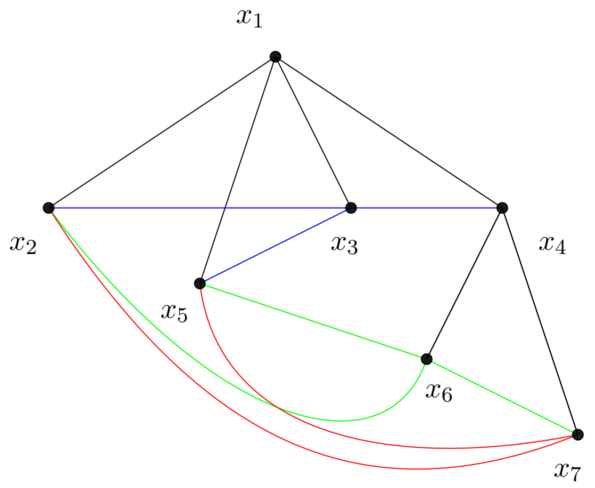

Figure 1: H_1

■

Lemma 2.6 Let G be a graph with $\delta(G) \geqslant 1$ and x a vertex of G such that the induced subgraph on $N_G(x)$ has at least an isolated vertex. Then $\gamma(\overline{C(G,f)}) = 2$.

Proof. Let $u_i \in V(G_2)$ be corresponding to vertex $x \in V(G)$. Then all of the vertices $(V(G_2) \setminus N_i) \cup (V(G_1) \setminus S_i)$ are dominated by u_i, where $S_i = f^{-1}(u_i)$ and $N_i = N_{G_2}(u_i)$. Let u_j be an isolated vertex in $G_2[N_{G_2}(u_i)]$. Then all of the vertices N_i and S_i are dominated by u_j. So $\{u_i, u_j\}$ is a dominating set of $\overline{C(G,f)}$ and $\gamma(\overline{C(G,f)}) \leqslant 2$. By Lemma 2.2, $\gamma(\overline{C(G,f)}) = 2$. ■

3. The proof of our main results

The main results are proven in this section.

Theorem 3.1 Let G be a bipartite graph and $\delta(G) \geqslant 1$. Then $\gamma(\overline{C(G,f)}) = 2$.

Proof. Let $V(G_2) = X \cup Y$. If $B_0 = \emptyset$, then by Lemma 2.3, $\gamma(\overline{C(G,f)}) = 2$. Let $B_0 \neq \emptyset$ and $u \in B_0$. If $u \in X$, then u dominates all of the vertices $V(G_1) \cup X$ and a vertex $u_i \in Y$ dominates all of the vertices Y in $\overline{C(G,f)}$. So $\{u, u_i\}$ is a dominating set of $\overline{C(G,f)}$. By Lemmas 2.2 and 2.4, $\gamma(\overline{C(G,f)}) = 2$. ■

Corollary 3.2 If G is a tree, then $\gamma(\overline{C(G,f)}) = 2$.

Proof. By Theorem 3.1, $\gamma(\overline{C(G,f)}) = 2$. ■

Theorem 3.3 Let graph G has a universal vertex. Then $\gamma(\overline{C(G,f)}) = 3$ if and only if:

(1) $\delta(G) \neq 1$,
(2) $B_1 = \emptyset$,
(3) For any $i \geqslant 2$ and any $u \in B_i$, $\delta(G_1[f^{-1}(u)]) \geqslant 1$,
(4) Every vertex in B_0 is adjacent to all of the vertices of B_i, for any $i \geqslant 2$,
(5) If $\{u, u'\} \subseteq \cup_{i \geqslant 2} B_i$ and u is not adjacent to u', then all of the vertices of $f^{-1}(u)$ are adjacent to each vertex of $f^{-1}(u')$.

Proof. Let $\gamma(\overline{C(G,f)}) = 3$ and w be a universal vertex of G_2.
1) Let $\delta(G) = 1$, $u_i \in V(G_2)$ and $deg_{G_2}(u_i) = 1$. Then u_i dominates all of the vertices

$(V(G_2)\setminus\{w\})\cup(V(G_1)\setminus f^{-1}(u_i))$ and w dominates all of the vertices $V(G_1)\setminus f^{-1}(w)$. So $\{u_i, w\}$ is a dominating set of $\overline{C(G,f)}$. Hence $\gamma(\overline{C(G,f)}) \leqslant 2$, which is a contradiction.
2) Let $B_1 \neq \emptyset$. Then by Lemma 2.3, we have $\gamma(\overline{C(G,f)}) = 2$, which is a contradiction.
3) If there exists an $i \geqslant 2$ and a $u \in B_i$ such that $G_1[f^{-1}(u)]$ has an isolated vertex v, Then v dominates all of the vertices $(V(G_2) \setminus \{u\}) \cup f^{-1}(u)$ and u dominates all of the vertices $V(G_1) \setminus f^{-1}(u)$. Hence $\{v, u\}$ is a dominating set of $\overline{C(G,f)}$. Hence, $\gamma(\overline{C(G,f)}) \leqslant 2$, which is not true.
4) If there exists a $u_0 \in B_0$ that is not adjacent to $u \in B_i$ for some $i \geqslant 2$, then u_0 dominates all of the vertices $V(G_1)\cup\{u\}$ and v_k dominates all of the vertices $V(G_2)\setminus\{u\}$, where $f(v_k) = u$. Hence $\{u_0, v_k\}$ is a dominating set of $\overline{C(G,f)}$. Therefore $\gamma(\overline{C(G,f)}) \leqslant 2$, which is a contradiction to the fact $\gamma(\overline{C(G,f)}) = 3$.
5) If $\{u, u'\} \subseteq \cup_{i\geqslant 2}B_i$, u is not adjacent to u' and choose $v \in f^{-1}(u)$ such that v is not adjacent to any vertex of $f^{-1}(u')$, then v dominates all of the vertices $(V(G_2) \setminus \{u\}) \cup f^{-1}(u')$. Also all of the vertices $(V(G_1) \setminus f^{-1}(u')) \cup \{u\}$ are dominated by u'. Hence $\{v, u'\}$ is a dominating set of $\overline{C(G,f)}$ and so $\gamma(\overline{C(G,f)}) \leqslant 2$, which is impossible.
Conversely, on the contrary let $\gamma(\overline{C(G,f)}) = 2$ and $D = \{a, b\}$ be a dominating set of $\overline{C(G,f)}$. We need only consider 3 cases:
Case 1: Let $D = \{a, b\} \subseteq V(G_1)$. If a and b are universal vertices of G, then by (1), $G \ncong P_2$ and so there is a $v_k \in V(G_1) \setminus \{a, b\}$ such that it is not dominated by D in $\overline{C(G,f)}$. If a is a universal vertex and b is not a universal vertex, then by (1), there is a $v_k \neq a$ such that it is adjacent to b. So D does not dominate v_k. If a and b are not universal vertices, then universal vertices of G_1 are not dominated by D in $\overline{C(G,f)}$, which is a contradiction.
Case 2: Let $D = \{a, b\} \subseteq V(G_2)$. Similarly, $D = \{a, b\} \subseteq V(G_2)$ leads to a contradiction.
Case 3: Now let $a \in V(G_1)$ and $b \in V(G_2)$. Then all of the vertices $V(G_2) \setminus f(a)$ are dominated by a in $\overline{C(G,f)}$. If $f(a) = b$, then since $\{a, b\}$ is a dominating set of $\overline{C(G,f)}$, so all of the vertices $f^{-1}(b)$ are dominated by a. By (2), $|f^{-1}(b)| \geqslant 2$ and a must be an isolated vertex of $G_1[f^{-1}(b)]$, which contradicts to (3). Let $f(a) \neq b$. Since $\{a, b\}$ is a dominating set of $\overline{C(G,f)}$, b is not adjacent to $f(a)$ in G_2. Since $B_1 = \emptyset$, by (4), $b \notin B_0$. Hence, $|f^{-1}(b)| \geqslant 2$. Therefore, a is not adjacent to any vertices of $f^{-1}(b)$, which contradicts to (5). This completes the proof. ■

Corollary 3.4 Let $n \geqslant 3$ and $G \cong K_n$. Then $\gamma(\overline{C(G,f)}) = 3$ if and only if $B_1 = \emptyset$

Corollary 3.5 Let $n \geqslant 5$ and $G \cong W_n$. Then $\gamma(\overline{C(G,f)}) = 3$ if and only if $R(f) = \{w\}$, where w is a universal vertex of W_n.

Proof. Let $\gamma(\overline{C(G,f)}) = 3$. Then by Theorem 3.3, $B_1 = \emptyset$ and so $B_0 \neq \emptyset$. Assume that $u_i \in B_0$. By (4) in Theorem 3.3, $R(f) \subseteq \{w, u_{i-1}, u_{i+1}\}$. Hence, $\{u_{i-2}, u_{i+2}\} \subseteq B_0$ and by (4) in Theorem 3.3, $R(f) \subseteq \{w, u_{i+1}, u_{i+3}\}$ and $R(f) \subseteq \{w, u_{i-1}, u_{i-3}\}$. Thus $R(f) = \{w\}$. Conversely, let $R(f) = \{w\}$. Then, by Theorem 3.3, $\gamma(\overline{C(G,f)}) = 3$. ■

Corollary 3.6 Let $m \geqslant 2$ and $G \cong K_3^m$. Then $\gamma(\overline{C(G,f)}) = 3$ if and only if $R(f) = \{w\}$, where w is a universal vertex of K_3^m.

Proof. Let vertices of $i-$th triangle of G be $\{w, u_{i1}, u_{i2}\}$ and $\gamma(\overline{C(G,f)}) = 3$. Then by Theorem 3.3, $B_1 = \emptyset$ and so $B_0 \neq \emptyset$. Suppose $u_{i1} \in B_0$. By (4) in Theorem 3.3, $R(f) \subseteq \{w, u_{i2}\}$. So for every $j \neq i$, $u_{j1} \in B_0$ and by (4) in Theorem 3.3, $R(f) \subseteq \{w, u_{j2}\}$. Therefore $R(f) = \{w\}$. Conversely, let $R(f) = \{w\}$. By Theorem 3.3, $\gamma(\overline{C(G,f)}) = 3$. ■

Theorem 3.7 Let $n \geqslant 6$ and G be an $(n-2)$-regular graph of order n. Then $\gamma(\overline{C(G,f)}) = 3$ if and only if:

1) $B_1 = \emptyset$.
2) If $u \in B_0$ and $u' \notin N_{G_2}(u)$, then $u' \in B_0$.
3) For each $x \in \cup_{i \geqslant 2} B_i$; $\delta(G_1[f^{-1}(x)]) \geqslant 1$.

Proof. Let $\gamma(\overline{C(G,f)}) = 2$ and $D = \{a, b\}$ be a dominating set of $\overline{C(G,f)}$. Since $\overline{G} \cong \bigcup P_2$, so $\{a,b\} \nsubseteq G_i$ for $i \in \{1,2\}$. Without loss of generality, let $a \in V(G_1)$ and $b \in V(G_2)$. If $f(a) = b$, then vertex a dominates all of the vertices $V(G_2) \setminus \{b\}$ and vertex b dominates all of the vertices $V(G_1) \setminus f^{-1}(b)$ in $\overline{C(G,f)}$. Since $\{a,b\}$ is a dominating set of $\overline{C(G,f)}$, so vertex a dominates $f^{-1}(b)$. Thus a is an isolated vertex of $G_1[f^{-1}(b)]$, which is a contradiction. Let $f(a) \neq b$. Since $f(a)$ is not dominated by a in $\overline{C(G,f)}$, So $f(a)$ is dominated by b. Hence $b \notin N_{G_2}(f(a))$. If $b \notin B_0$, then since $B_1 = \emptyset$, so $|f^{-1}(b)| \geqslant 2$. It is clear that the vertices of $f^{-1}(b)$ are not dominated by b. Thus the vertices of $f^{-1}(b)$ are dominated by a and so they are not adjacent to a. This is impossible. So $b \in B_0$, this is contradicts to (2). Therefore $\gamma(\overline{C(G,f)} = 3$.
Conversely, on the contrary if $B_1 \neq \emptyset$, then by Lemma 2.3, $\gamma(\overline{C(G,f)}) = 2$. This is a contradiction to the fact that $\gamma(\overline{C(G,f)} = 3$.
Assume that there are u and u', $u' \notin N_{G_2}(u)$, $u \in B_0$ and $u' \notin B_0$. If $v_k \in V(G_1)$ and $f(v_k) = u'$, then $\{v_k, u\}$ is a dominating set of $\overline{C(G,f)}$. Hence $\gamma(\overline{C(G,f)}) \leqslant 2$, which is impossible.
Finally, let $u_i \in V(G_2)$ such that $G_1[f^{-1}(u_i)]$ has an isolated vertex v_k. Then $\{v_k, u_i\}$ is a dominating set of $\overline{C(G,f)}$ and so $\gamma(\overline{C(G,f)}) \leqslant 2$, which is impossible. This completes the proof. ■

Lemma 3.8 Let $G \cong H_2$. Then $\gamma(\overline{C(G,f)}) = 3$ if and only if $|R(f)| = 2$, $G_2[R(f)] = \emptyset$ and $\delta(G_1[f^{-1}(x)] \geqslant 1$ for every $x \in R(f)$.

Proof. If $|R(f)| = 2$, $G_2[R(f)] = \emptyset$ and $\delta(G_1[f^{-1}(x)] \geqslant 1$ for every $x \in R(f)$, then by Theorem 3.7, $\gamma(\overline{C(G,f)}) = 3$. Conversely, let $\gamma(\overline{C(G,f)}) = 3$. Then by Theorem 3.7 (1), $B_1 = \emptyset$. So $|R(f)| \neq 4$. If $|R(f)| \in \{1,3\}$, then there is an $u_j \in R(f)$ such that u_j is not adjacent to u_i, where $u_i \notin R(f)$. By Theorem 3.7 (2), $u_j \in B_0$ that is not true. So $|R(f)| = 2$. Let $R(f) = \{a,b\}$ and $x \in B_0$. Then by Theorem 3.7 (2), $\{a,b\} \subseteq N_{G_2}(x)$. Since $deg_{G_2}(a) = deg_{G_2}(b) = 4$, so a is not adjacent to b. Thus $G_2[R(f)] = \emptyset$. By (3) in Theorem 3.7, $\delta(G_1[f^{-1}(a)]) \geqslant 1$ and $\delta(G_1[f^{-1}(b)]) \geqslant 1$. This completes the proof.

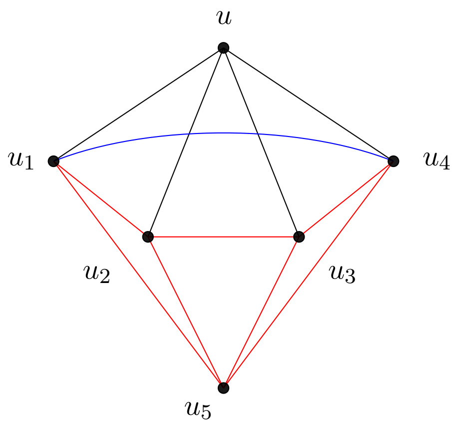

Figure 2: H_2

■

Theorem 3.9 Let G be a connected k-regular graph of order $n \geqslant 4$, which is not isomorphic to K_3, K_4, K_5, H_1 and H_2. If $k \in \{2, 3, 4\}$, then $\gamma(\overline{C(G, f)}) = 2$.

Proof. Let $k = 2$ and $v \in V(G)$. Then since $n \geqslant 4$, induced subgraph on $N_G(v)$ has an isolated vertex. By Lemma 2.6, $\gamma(\overline{C(G, f)}) = 2$.

Let $k = 3$, $a \in V(G)$ and $N_G(a) = \{x, y, z\}$. If $G[N_G(a)]$ has an isolated vertex, then by Lemma 2.6, $\gamma(\overline{C(G, f)}) = 2$.
If $G[N_G(a)]$ has no isolated vertex, then since $G \ncong K_4$, we have $G[N_G(a)] \cong P_3$. (*See Figure* 3) Since G is a 3−regular graph, there is a $t \in V(G) \setminus \{x, y\}$ such that $t \in N_G(z)$. It is easy to see that z is an isolated vertex of $G[N_G(t)]$. By Lemma 2.6, $\gamma(\overline{C(G, f)}) = 2$.

Let $k = 4$. If $B_0 = \emptyset$ or $B_1 \neq \emptyset$, then by Lemma 2.3, $\gamma(\overline{C(G, f)}) = 2$. Let $B_0 \neq \emptyset$, $u \in B_0$ and $N_{G_2}(u) = \{u_1, u_2, u_3, u_4\}$. If $R(f) \nsubseteq N_{G_2}(u)$ and $B_1 = \emptyset$, then there is an $u_i \in V(G_2)$ such that $u_i \notin N_{G_2}(u)$ and $|f^{-1}(u_i)| \geqslant 2$. Suppose, $v_k \in V(G_1)$ and $f(v_k) = u_i$. Then all of the vertices $V(G_1) \cup (V(G_2) \setminus \{u_1, u_2, u_3, u_4\})$ are dominated by vertex u and the vertices u_1, u_2, u_3 and u_4 are dominated by v_k in $\overline{C(G, f)}$. So $\{u, v_k\}$ is a dominating set of $\overline{C(G, f)}$. Thus $\gamma(\overline{C(G, f)}) \leqslant 2$. Therefore $\gamma(\overline{C(G, f)}) = 2$ by Lemma 2.2.
If $R(f) \subseteq N_{G_2}(u)$, we have three following cases:
Case 1: Let induced subgraph on $N_{G_2}[u] = \{u, u_1, u_2, u_3, u_4\}$ has a vertex of degree 1. Then by Lemma 2.6, $\gamma(\overline{C(G, f)}) = 2$.
Case 2: Let $\delta(G_2[N_{G_2}[u]]) \geqslant 2$ and $G_2[N_{G_2}[u]]$ has a vertex of degree 2. Without loss of generality, let $deg_{G_2[N_{G_2}[u]]}(u_4) = 2$ and u_4 is adjacent to u_3. Also let u_4 be adjacent to u_5 and u_6. (*See Figure* 4) If $N_{G_2}(u) = N_{G_2}(u_5) = N_{G_2}(u_6)$, then since $\delta(G_2[N_{G_2}[u]]) \geqslant 2$, u_1 is adjacent to u_2 and $G \cong H_1$. (*See Figure* 1) u_5 or u_6 is not adjacent to at least one of the vertices $N_{G_2}(u) \setminus \{u_4\}$.
If $|R(f)| = 4$, then by Lemmas 2.2 and 2.4, $\gamma(\overline{C(G, f)}) = 2$.
Let $|R(f)| = 3$. If $u_4 \notin R(f)$, then $u_i \in R(f)$, for $i \in \{1, 2\}$. Since u_1 and u_2 are not adjacent to u_4, by Lemmas 2.2 and 2.4, $\gamma(\overline{C(G, f)}) = 2$. Assume that $u_4 \in R(f)$. If $u_1 \notin R(f)$ or $u_2 \notin R(f)$, then since u_4 is not adjacent to u_1 and u_2, by Lemmas 2.2 and 2.4, $\gamma(\overline{C(G, f)}) = 2$.
Let $\{u_1, u_2, u_4\} = R(f)$. If $u_3 \notin N_{G_2}(u_1)$ or $u_3 \notin N_{G_2}(u_2)$, then by Lemmas 2.2 and 2.4, $\gamma(\overline{C(G, f)}) = 2$. Let $u_3 \in N_{G_2}(u_1) \cap N_{G_2}(u_2)$. Since $G \ncong H_1$, (*See Figure* 1) so there is a vertex $x \in V(G_2) \setminus R(f)$ such that x is not adjacent to u_4. Therefore by Lemmas 2.2 and 2.4, $\gamma(\overline{C(G, f)}) = 2$.
Let $|R(f)| = 2$. If $u_4 \in R(f)$, then u_1 or u_2 is not in $R(f)$. If $u_4 \notin R(f)$, then u_1 or u_2 is in $R(f)$. However, by Lemmas 2.2 and 2.4, $\gamma(\overline{C(G, f)}) = 2$.
Finally, let $|R(f)| = 1$. If $R(f) \subseteq \{u_1, u_2, u_4\}$, then by Lemmas 2.2 and 2.4, $\gamma(\overline{C(G, f)}) = 2$. Let $R(f) = \{u_3\}$. If u_1 and u_2 are adjacent to u_3, then u_5 is not adjacent to u_3. So by Lemmas 2.2 and 2.4, $\gamma(\overline{C(G, f)}) = 2$. If u_1 or u_2 is not adjacent to u_3, then by Lemmas 2.2 and 2.4, $\gamma(\overline{C(G, f)}) = 2$.
Case 3: Let $\delta(G_2[N_{G_2}[u]]) \geqslant 3$. Since $G \ncong K_5$, we may assume that there is a vertex $u_5 \in V(G_2)$ such that $u_5 \notin N_{G_2}(u)$ and $u_5 \in N_{G_2}(u_4)$. This involves no loss of generality (*See Figure* 5). If $N_{G_2}(u_5) = N_{G_2}(u)$, then $G \cong H_2$ (*See Figure* 2), which is impossible. So u_5 is adjacent to vertex u_6, where $u_6 \in V(G_2) \setminus \{u, u_1, u_2, u_3, u_4\}$. Since G is a 4-regular graph and $R(f) \subseteq N_{G_2}(u)$, for each $y \in R(f)$ if $y \in N_{G_2}(u_5)$, then $y \notin N_{G_2}(u_6)$ or if $y \in N_{G_2}(u_6)$, then $y \notin N_{G_2}(u_5)$. By Lemmas 2.2 and 2.4, $\gamma(\overline{C(G, f)}) = 2$.

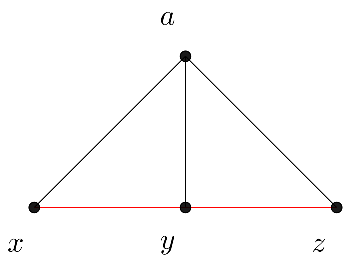

Figure 3

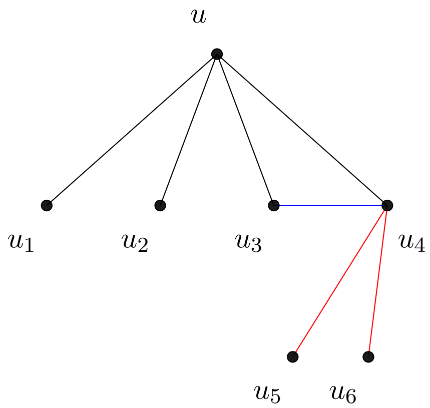

Figure 4

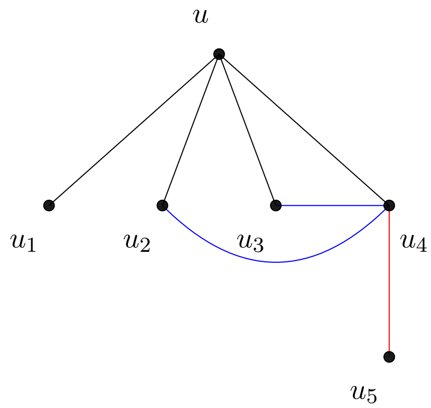

Figure 5

■

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