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A study of defectless and vs-defectless extensions of valued fields

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Abstract. The phenomenon of defectless extensions is a classical notion in the framework of valued fields and valued vector spaces in valuation theory. The aim of this paper is to study various results regarding this concept and its applications.

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1. Introduction and preliminaries

In valuation theory, the notion of defectlessness plays an important role in several applications; specially, it is helpful to have equivalent characterizations because it makes the tight connection between valued fields, their invariants, value groups and residue fields. Moreover, it is one of the important tools of valuation theory which is used extensively in generalizing some results or making counterexamples. In this paper, we discuss the role that this concept plays in obtaining some important results in valuation theory. We compare the results of various research papers related to this concept, and state their findings in a logical and historical sequence. Moreover, we will attempt to state briefly some of applications of these results in two contexts of valued fields and valued vector spaces. In fact, this study includes two sections. In the first section, we deal with defectless extensions of valued fields, and in the second section, we treat defectless extensions in the context of valued vector spaces.

Let us first recall some basic notions. We refer the reader to [15, 16, 30, 33] for the elementary definitions and the basic facts of valuation theory.

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Let v be a valuation on a field K. The value group and the valuation ring of v denoted by $G(K)$ and $\mathcal{O}_K$, respectively. $\mathcal{O}_K$ is a local ring with the maximal ideal M_K, and $\mathcal{O}_K/M_K$ is the residue field of v denoted by $R(K)$. For any β in $\mathcal{O}_K$, β^* will denote its v-residue, i.e., the image of β under the residue map from the valuation ring of v onto its residue field.

Let K'/K be an extension of fields. A valued field (K', v') is called an extension of a valued field (K, v) if $v'|_K = v$. This statement is denoted by $(K', v')/(K, v)$ or $(K, v) \subseteq (K', v')$. We may denote briefly it by K'/K when there is no chance of confusion.

For a valued field extension $(K', v')/(K, v)$, there is a natural embedding of the value group $G(K)$ in the value group $G(K')$, and a natural embedding of the residue field $R(K)$ in the residue field $R(K')$. If both embeddings are onto (which we just express by writing $G(K) = G(K')$ and $R(K) = R(K')$), then the extension $(K', v')/(K, v)$ is called immediate. A valued field is called maximal if it does not admit any nontrivial immediate extensions.

Take $(K', v')/(K, v)$ to be finite. Then $R(K')/R(K)$ is a finite extension, and the number $f(K'/K) = [R(K') : R(K)]$ is called the residue degree (inertia degree) of K'/K. Moreover, the quotient group $G(K')/G(K)$ is finite, and its index denoted by $e(K'/K)$ is called the ramification index of K'/K (see [16, Corollary 3.2.3]). It is known that

$$[K' : K] \geqslant [G(K') : G(K)][R(K') : R(K)].$$

In fact, if $v'_1, v'_2, \ldots, v'_m$ are the distinct extensions of the valuation v on K to the field K', the so-called Lemma of Ostrowski (see [33, Chapter VI, §12, Corollary to Theorem 25]) establishes that

$$[K' : K] = \sum_{i=1}^{m} p^{n_i}[G_i(K') : G(K)][R_i(K') : R(K)]; \tag{1}$$

where $G_i(K')$ and $R_i(K')$ are respectively the value group and the residue field of v'_i, and p denotes the characteristic exponent of $R(K)$ (that is, $p = \text{char}R(K)$ if it is positive, and $p = 1$ otherwise), and for each $i \in \{1, 2, \ldots m\}$, n_i is a non-negative integer. We now give some definitions using the identity stated in (1).

Definition 1.1 Let $(K', v')/(K, v)$ be as above. For each $i \in \{1, \ldots m\}$, the factor p^{n_i} in (1) is called the defect of the valued field extension $(K', v'_i)/(K, v)$. When $p^{n_i} = 1$ for all $i \in \{1, \ldots m\}$, we say that K' is a defectless field extension of (K, v). Otherwise we call it a defect extension.

The multiplicativity of the degree of filed extensions, the ramification indexes and inertia degrees obviously implies that the defect is multiplicative. Consequently, every subextension of a finite defectless extension is again defectless. We usually center our study of defectless extensions to the particular case where $m = 1$, that is, where the valuation v extends uniquely to K'. Recall that a valued field (K, v) is called henselian if v has a unique extension to every algebraic extension of K. This holds if and only if (K, v) satisfies Hensel's Lemma (see [16, Theorem 4.1.3]), that is, if f is a polynomial with coefficients in the valuation ring $\mathcal{O}_K$ of (K, v) and there is $b \in \mathcal{O}_K$ such that $v(f(b)) > 0$ and $v(f'(b)) = 0$, then there exists $a \in \mathcal{O}_K$ such that $f(a) = 0$ and $v(b - a) > 0$.

Definition 1.2 An infinite algebraic extension $(K', v')/(K, v)$ such that the valuation v admits a unique extension from K to K' is called defectless if every finite subextension of $(K', v')/(K, v)$ is defectless.

Definition 1.3 A valued field (K, v) is called a defectless valued field (or briefly a defectless field) if every finite extension of K is defectless.

Observe that by the Lemma of Ostrowski, any valued field (K, v) of residue characteristic zero is a defectless field. Moreover, the property of defectlessness is transitive:

Proposition 1.4 [22, Lemma 11.6] Let (K, v) be an arbitrary valued field, K'/K a finite extension and E/K a subextension of K'/K. Let $v_1, \ldots, v_m$ be all extensions of v from K to E. Then (K, v) is defectless in K' if and only if (K, v) is defectless in E and (E, v_i) is defectless in K' for $1 \leqslant i \leqslant m$.

Every finite valued field extension of a defectless field is again defectless. More precisely,

Proposition 1.5 [22, Lemma 11.9] If $v_1, \ldots, v_m$ are all extensions of the valuation v from K to a finite extension K', then (K, v) is a defectless field if and only if (K, v) is defectless in K' and (K', v_i) are defectless fields for all $i = 1, \ldots, m$.

2. Defectless extensions of valued fields

In this section, we study the concept of defectlessness in the context of valued fields. We see that how it has been extensively used to solve some problems, establishing classifications, and developing some results concerned with valued fields.

Throughout this section, unless otherwise stated, v is a valuation of a field K and $\bar{v}$ is a fixed extension of v to an algebraic closure $\overline{K}$ of K. For any overfield K' of K contained in $\overline{K}$, we will denote by $G(K')$ and $R(K')$ respectively the value group and the residue field of the valuation v' of K' obtained by restricting $\bar{v}$ to K'.

2.1 *Complete distinguished chains*

Since one of the applications of the concept of defectlessness has been to generalize some results in the context of complete discrete rank one valued fields (also named local fields) to valued fields of arbitrary rank, we start by considering a paper including the important results about local fields.

In [29], Popescu and Zaharescu defined some invariants associated to an irreducible polynomial to investigate the structure of irreducible polynomials. More precisely, they defined the notion of "lifting polynomial" relative to a residual transcendental extension of the local field (K, v) to the rational function field $K(x)$ over K in an variable x. There they prove that every lifting of an irreducible polynomial over a local field is also irreducible [29, Theorem 2.1]. This leads to some known criteria of irreducibility and also gives new criteria which generalized the usual Eisenstein irreducibility criterion [29, Proposition 2.2]. Khanduja and Saha generalized the notion of lifting polynomials over valuations of arbitrary rank in [20]. More precisely, they applied it to extend the irreducibility criterion presented in [29] to polynomials with coefficients from a valued field (K, v), where v is a valuation of any rank. In what follows, we give a description of lifting polynomials:

Definition 2.1 A pair (α, δ) in $\overline{K} \times G(\overline{K})$ is said to be minimal (with respect to (K, v)) if whenever $\beta \in \overline{K}$ satisfies $\bar{v}(\alpha - \beta) \geqslant \delta$, then $[K(\alpha) : K] \leqslant [K(\beta) : K]$.

If $f(x)$ is a fixed nonzero polynomial in $K[x]$, then using the Euclidean algorithm, each $F(x) \in K[x]$ can be uniquely represented as a finite sum $\sum_{i \geqslant 0} F_i(x)f(x)^i$, where for any i, the polynomial $F_i(x)$ is either 0 or has degree less than that of $f(x)$. The above representation will be referred to as the f-expansion of $F(x)$.

For a pair $(\alpha, \delta) \in \overline{K} \times G(\overline{K})$, the valuation $\overline{w}_{\alpha,\delta}$ of $\overline{K}(x)$ defined on $\overline{K}[x]$ by

$$\overline{w}_{\alpha,\delta}\left(\sum_i c_i(x-\alpha)^i\right) = \min_i\{\bar{v}(c_i) + i\delta\},\ c_i \in \overline{K}, \tag{2}$$

will be referred to as the valuation defined by the pair (α, δ). If $f(x)$ is the minimal polynomial of α over K of degree n with $w_{\alpha,\delta}(f(x)) = \lambda$ and e is the smallest positive integer such that $e\lambda \in \bar{v}(K(\alpha))$, say $e\lambda = \bar{v}(h(\alpha))$, $h(x) \in K[x]$, $\deg h(x) < n$, then the $w_{\alpha,\delta}$-residue $\left(\frac{f(x)^e}{h(x)}\right)^*$ of $\left(\frac{f(x)^e}{h(x)}\right)$ is transcendental over $R(K(\alpha))$ and the residue field of $w_{\alpha,\delta}$ is canonically isomorphic to $R(K(\alpha))\left(\left(\frac{f(x)^e}{h(x)}\right)^*\right)$ (see [5, Theorem 2.1]).

Definition 2.2 For a (K, v)-minimal pair (α, δ), let $f(x)$, n, λ, and e be as above. A monic polynomial $F(x)$ belonging to $K[x]$ is said to be a lifting of a monic polynomial $Q(Y)$ belonging to $R(K(\alpha))[Y]$ having degree $m \geqslant 1$ with respect to (α, δ) if there exists $h(x) \in K[x]$ of degree less than n such that

(i) $\deg F(x) = emn$,
(ii) $w_{\alpha,\delta}(F(x)) = mw_{\alpha,\delta}(h(x)) = em\lambda$,
(iii) the $w_{\alpha,\delta}$-residue of $F(x)/h(x)^m$ is $Q((f^e/h)^*)$.

The concept of lifting of polynomials is now one of the most important tools of valuation theory in studying the properties of irreducible polynomials with coefficients in a valued field (K, v) (see for example [3, 9, 10, 13]).

In [29], it was also defined the notions of "distinguished pairs" and "complete distinguished chains" (also called saturated distinguished chains) which lead to a sequence of invariants of an irreducible polynomial. These invariants are characteristic, i.e., by using them, we may describe the set of irreducible polynomials over a local field [29, Theorem 4.6]. As an application, these invariants may be used to understand sufficiently well the extension of the natural valuation of a local field K to the field given by the considered polynomial. The notion of distinguished pairs originates from an invariant $\delta_K(\theta)$ referred to as the main invariant of an algebraic element θ over K. $\delta_K(\theta)$ was defined for algebraic elements $\theta \in \overline{K} \setminus K$ when (K, v) is a complete discrete rank one valued field [29]. By the main invariant of an algebraic element θ is the supremum of the set $M(\theta, K)$ defined by

$$M(\theta, K) = \{\bar{v}(\theta - \xi) |\ \xi \in \overline{K},\ [K(\xi) : K] < [K(\theta) : K]\},$$

where, for the sake of definition of supremum, $G(\overline{K})$ may be viewed as a subset of its Dedekind order completion. Popescu and Zaharescu proved that if (K, v) is a local field, then $M(\theta, K)$ has an upper bound in $G(\overline{K})$, and moreover, $\delta_K(\theta) \in M(\theta, K)$ for each $\theta \in \overline{K} \setminus K$ [28, p. 74]. However, there are instances when $\delta_K(\theta) \in G(\overline{K})$ but fails to belong to $M(\theta, K)$ (see [2, Example 2.1]). This arose a question of how to characterize those valued fields (K, v) for which to each $\theta \in \overline{K} \setminus K$, there corresponds $\xi \in \overline{K}$ satisfying $[K(\xi) : K] < [K(\theta) : K]$ and $\delta_K(\theta) = \bar{v}(\theta - \xi)$.

Aghigh and Khanduja solved this problem by using defectless extensions over henselian valued fields as follows:

Theorem 2.3 [2, Theorem 1.1] Let v be a henselian valuation of a field K and $\overline{K}$ be the algebraic closure of K with valuation $\bar{v}$. The following two statements are equivalent:

(i) To each $\alpha \in \overline{K} \setminus K$, there corresponds $\beta \in \overline{K}$ with $[K(\beta) : K] < [K(\alpha) : K]$ such that $\delta_K(\alpha) = \bar{v}(\alpha - \beta)$.
(ii) For each $\theta \in \overline{K}$, $K(\theta)/K$ is a defectless extension with respect to the valuation obtained by restricting $\bar{v}$.

When $M(\theta, K)$ has a maximum element for $\theta \in \overline{K} \setminus K$, we may choose an element $\xi \in \overline{K}$ of smallest degree over K such that $\bar{v}(\theta - \xi) = \delta_K(\theta)$. This forms a pair (θ, ξ) which is referred to as distinguished pairs:

Definition 2.4 A pair (θ, ξ) of elements of $\overline{K}$ is called a distinguished pair if the following three conditions are satisfied:

(i) $\bar{v}(\theta - \xi) = \delta_K(\theta)$,
(ii) $[K(\theta) : K] > [K(\xi) : K]$,
(iii) if γ belonging to $\overline{K}$ has degree less than that of ξ, then $\bar{v}(\theta - \gamma) < \bar{v}(\theta - \xi)$.

Distinguished pairs give rise to distinguished chains in a natural manner:

Definition 2.5 A chain $\theta = \theta_0, \theta_1, \ldots, \theta_r$ of elements of $\overline{K}$ will be called a complete distinguished chain for θ if (θ_i, θ_{i+1}) is a distinguished pair for $0 \leqslant i \leqslant r - 1$ and $\theta_r \in K$.

Popescu and Zaharescu [29] proved the existence of a complete distinguished chain for each $\theta \in \overline{K} \setminus K$ in case (K, v) is a local field, and also gave some invariants associated to a chain for θ, and Ota [27] gave a method to determine these invariants in this case. Aghigh and Khanduja again used the concept of defectlessness to generalize these results to henselian valued fields of arbitrary rank. They proved:

Theorem 2.6 [1, Theorem 1.2] Let (K, v) and $(\overline{K}, \bar{v})$ be as in Theorem 2.3. An element $\theta \in \overline{K} \setminus K$ has a complete distinguished chain if and only if $K(\theta)/K$ is defectless.

They also showed that complete distinguished chains for an element $\theta \in \overline{K} \setminus K$ give rise to several invariants associated with θ which satisfy some fundamental relations (see [1, Theorems 1.4 and 1.5]):

Theorem 2.7 Let (K, v) be a henselian valued field, and $\bar{v}$ be the unique extension of v to a fixed algebraic closure $\overline{K}$ of K. If $\theta = \theta_0, \theta_1, \ldots, \theta_r$ and $\theta = \eta_0, \eta_1, \ldots, \eta_s$ are two complete distinguished chains for $\theta \in \overline{K} \setminus K$, then $r = s$ and $[K(\theta_i) : K] = [K(\eta_i) : K]$ for $1 \leqslant i \leqslant s$.

Theorem 2.8 With (K, v) and $(\overline{K}, \bar{v})$ as above, let $\theta = \theta_0, \theta_1, \ldots, \theta_s$ and $\theta = \eta_0, \eta_1, \ldots, \eta_s$ be two complete distinguished chains for an element $\theta \in \overline{K} \setminus K$. If $f_i(x)$ and $g_i(x)$ denote respectively the minimal polynomials of θ_i and η_i over K, then the following hold for $1 \leqslant i \leqslant s$:

(i) $G(K(\theta_i)) = G(K(\eta_i))$,
(ii) $R(K(\theta_i)) = R(K(\eta_i))$,
(iii) $\bar{v}(\theta_{i-1} - \theta_i) = \bar{v}(\eta_{i-1} - \eta_i)$,
(iv) $\bar{v}(f_i(\theta_{i-1})) = \bar{v}(g_i(\eta_{i-1}))$.

Theorems 2.7 and 2.8 show that the invariants associated with θ happen to be the same for all K-conjugates of θ and hence are invariants of the minimal polynomial of θ over K. Consequently, complete distinguished chains has been used to obtain results about irreducible polynomials over valued fields. There have been described some methods of constructing complete distinguished chains for algebraic elements over valued fields (see [4, 31] for example).

2.2 *Defectless polynomials*

Continuing to examine the importance of the notion of defectlessness, it is worthwhile to study defectless polynomials and tame polynomials.

Definition 2.9 A polynomial $h(x) \in K[x]$ is called defectless over a valued field (K, v) if it has a root α such that $\deg h$ is the product of the ramification index and the residue degree of some extension of v to $K[\alpha]$.

Definition 2.10 A polynomial $h(x) \in K[x]$ is called tame if it is defectless and admits a root α such that the characteristic of $R(K)$ does not divide the ramification index of the field extension $K[\alpha]/K$, and the residue class field extension of $K[\alpha]/K$ is separable.

In 2009, Ron Brown [12] introduced a strict system of polynomial extensions over a valued field (K, v) as follows:

We denote by $\mathbb{Q}G(K)$ a fixed divisible hull of the value group $G(K)$ of v. By an extension w of v to $K[x]$, we mean a mapping

$$w : K[x] \to \mathbb{Q}G(K) \cup \{\infty\}$$

satisfying $w(f + g) \geqslant \min\{w(f), w(g)\},\ w(fg) = w(f) + w(g)$ for all $f, g \in K[x]$, with $w^{-1}(\infty)$ not necessarily the zero ideal. If $w^{-1}(\infty)$ is a nonzero (prime) ideal I, then w gives rise to a valuation w_I of the field $K[x]/I$. We shall denote by K_w the residue field of w_I and by $\tau_w : K[x] \to K_w \cup \{\infty\}$ the associated place.

Definition 2.11 Suppose that $n \geqslant 0$. A strict system of polynomial extensions over (K, v) of length $n + 1$ is a finite sequence $(g_0, w_0, \gamma_0), \ldots,$
$(g_{n+1}, w_{n+1}, \gamma_{n+1})$, where each w_i is an extension of v to $K[x]$ and $\gamma_i \in \mathbb{Q}G(K) \cup \{-\infty\}$ such that the following properties are satisfied:
(A) $g_0 = x - a,\ a \in K,\ \gamma_0 = -\infty,\ w_0(h) = v(h(a))$ for every $h \in K[x]$; and for $0 \leqslant i \leqslant n$:
(B) $\deg g_{i+1} > \deg g_i,\ \deg g_i$ divides $\deg g_{i+1}$;
(C) $\gamma_{i+1} = w_i(g_{i+1})$;
(D) $w_{i+1}(g_{i+1}) = \infty$;
(E) The g_i-expansion of g_{i+1} given by $g_{i+1} = g_i^{d_i} + \sum\limits_{r<d_i} A_r g_i^r$ ($\deg A_r < \deg g_i$) satisfies $\frac{w_i(A_r)}{d_i - r} \geqslant \frac{w_i(A_0)}{d_i} > \gamma_i$ for all $r < d_i$;
(F) If e_i is the least positive integer such that $e_i w_i(A_0) \in d_i w_i(K[x])$ and $l_i = d_i/e_i$, then the polynomial

$$Y^{l_i} + \sum_{r<l_i} \tau_{w_i}(s^{l_i - r} A_{e_i r}) Y^r$$

is irreducible over K_{w_i} for all s in $K[x]$ with $w_i(A_0 s^{l_i}) = 0$.

Notation : We let $\mathcal{P}(K)$ denote the set of all polynomials h with $h = g_{n+1}$ for some strict system g as in Definition 2.11.

The class $\mathcal{P}(K)$ of polynomials first arose as the class of “key polynomials” in MacLane’s seminal construction [23], in the case that v is discrete rank one, of the extensions of v to a rank one valuation on the polynomial ring $K[x]$. His construction gave rise to a vast generalization of many known irreducibility criteria (see the list in [25, p. 232]). In work related to MacLane's, Brown studied the class $\mathcal{P}(K)$ of polynomials in full generality; he showed that for maximal fields the extensions of v to $K[x]$ were bijective with a class of objects defined purely in terms of the residue class field and value group of v (certain "signatures"); a corresponding computation was given for the irreducible polynomials. In an effort to make a part of this material more accessible, he introduced the notion of a "strict system of polynomial extensions" (Definition 2.11) by adapting the notion of an "extension of a polynomial" [11, Definition (5.5)and Sec. 7]. Another approach to the study of extensions of v to $K(x)$, particularly for local fields, was developed by N. Popescu and several collaborators (see for example [29]), and then generalized and strikingly applied by Khanduja and her collaborators by using the notion of defectless extensions (see for example [1, 8, 32]). In [12] and [13], Brown and Merzel established strong connections between strict systems of polynomial extensions and complete distinguished chains of polynomials, and used these connections to make applications to both. We here explain some of their results:

As one of the most important results of Brown's work [12], it may be worth mentioning some results showing in case that (K, v) is a maximal field, then $\mathcal{P}(K)$ is the set of all monic nonlinear irreducible polynomials over K; if (K, v) is discrete rank one, then $\mathcal{P}(K)$ is the set of monic polynomials over K which are irreducible over the completion of (K, v), and if (K, v) is maximally complete ([30, Definition 9, p. 36]), then $\mathcal{P}(K)$ is the set of all monic irreducible polynomials over K. Moreover, the following theorem shows that if (K, v) is henselian, then every complete distinguished chain of polynomials in $K[x]$ in the sense of [1, 29] is a strict system of polynomial extensions.

Theorem 2.12 [13, Theorem 9.1] Suppose that $(g_{n+1}, \ldots, g_0)$ is a complete distinguished chain over a henselian valued field (K, v). Let $\gamma_0 = -\infty$. For each $0 \leqslant i \leqslant n+1$, let w_i denote the unique extension of v to $K[x]$ with $w_i(g_i) = \infty$ and for each $0 \leqslant i \leqslant n$ set $\gamma_{i+1} = w_i(g_{i+1})$. Then the sequence $g = ((g_i, w_i, \gamma_i))_{i \leqslant n+1}$ is a strict system of polynomial extensions over (K, v).

An very important application of Theorem 2.12 shows that if (K, v) is henselian, then $\mathcal{P}(K)$ consists precisely of the monic nonlinear defectless polynomials over (K, v) [13, Theorem 9.3]:

Theorem 2.13 If (K, v) is henselian, then $\mathcal{P}(K)$ consists exactly of the monic nonlinear defectless polynomials over (K, v).

Brown and Merzel [13] also studied the rich sequences of invariants of defectless polynomials which arise naturally in both approaches of strict systems of polynomial extensions and complete distinguished chains of polynomials; these sequences are often essentially identical. They particularly dealt with tame polynomials in $K[x]$. For such polynomials which are a special case of defectless polynomials (see Definition 2.10), we can present the following main result:

Theorem 2.14 [13, Theoem 2.3] Let h be a tame polynomial in $K[x]$ and that α is a root of h in the algebraic closure $\overline{K}$. Suppose that $h \in \mathcal{P}(K)$ with $h = g_{n+1}$ for some strict system g as in Definition 2.11. Then for each integer r with $0 \leqslant r \leqslant n$, the set

$$\mathcal{S}_r = \{\beta \in \overline{K} \ : \ \bar{v}(\alpha - \beta) = m_r, \ h(\beta) = 0\}$$

has $(d_r - 1) \prod_{i=r+1}^{n} d_i$ elements.

Theorem 2.14 can be regarded as a description of the Newton polygon of $h(x+\alpha)$ (see Remark 3.1 of [13]). In fact, it implies that

$$|\{\alpha\}| + \sum_{r=0}^{n} |\mathcal{S}_r| = d_0 \cdots d_n = \deg h,$$

where $|A|$ denotes the number of elements in any finite set A. Since the m_r are all distinct, the sets $\mathcal{S}_r$ are pairwise disjoint and hence Theorem 2.14 also implies that

$$\{\beta \in \overline{K} \ : \ h(\beta) = 0\} = \{\alpha\} \cup \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_n.$$

As β ranges over the roots of h in $\overline{K}$ the differences $\beta - \alpha$ are exactly the roots of $h(x+\alpha)$, and therefore the values $\bar{v}(\beta - \alpha)$ are exactly the slopes of the Newton polygon of $h(x+\alpha)$, regarded as a polynomial over the valued field $(\overline{K}, \bar{v})$.

Let h be an element of $\mathcal{P}(K)$ and that $h = g_{n+1}$ where g is a strict system as in Definition 2.11, and let $\alpha \in \overline{K}$ denote a root of h. We remark some familiar invariants associated to h as follows:

Definition 2.15 The Krasner constant of h, denoted $\omega_K(h)$, is the maximum of the set

$$\{\bar{v}(\alpha - \alpha') \ : \ \alpha \neq \alpha' \in \overline{K} \text{ and } h(\alpha') = 0\}.$$

This set is independent of the choice of α, so $\omega_K(h)$-often called the Krasner constant of α-is an invariant of h.

Definition 2.16 The minimum of the above set is called the diameter of h and denoted by $\Omega_K(h)$.

Note that $\Omega_K(h)$ was studied by Ax [6] and later by Khanduja [18], and was called the diameter of h in [13].

Definition 2.17 The maximum of the following set is called the separant of h:

$$\{\bar{v}(h'(\alpha)(\alpha - \alpha')) \ : \ \alpha \neq \alpha' \in \overline{K}, h(\alpha') = 0\}.$$

One of the applications of Theorem 2.14 is to calculate the invariants of a tame polynomial $h \in \mathcal{P}(K)$ such as simple formulas for the Krasner constant and the separant of h as well as the diameter of h (see Sec. 4 of [13]). We remark that a Krasner constant of a tame irreducible polynomial can be regarded as the Krasner constant of any of its roots.

In Sections 5 and 6 of [13], there are important results about roots of tame polynomials over the valued field (K, v). For example, a result is given showing that a sufficiently good approximation in an extension field L of K to a root of a defectless polynomial h over K guarantees the existence of an exact root of h in L. Also in the tame case, a (best possible) result is given describing when a polynomial is sufficiently close to a defectless polynomial so as to guarantee that the roots of the two polynomials generate the same extension fields.

In [19], Khanduja and Khassa completed the work begun by Brown and Merzel toward establishing a one-to-one correspondence between strict systems of polynomial extensions and conjugacy classes of complete distinguished chains over henselian valued fields (K, v). In fact, in the case henselian valued field (K, v), with the help of this equivalence, they determined explicitly (see [19, Theorem 1.3 and Corollary 1.4]) the best possible constant λ_h associated to any defectless polynomial $h(x)$ over a henselian valued field (K, v) satisfying the property that whenever $\bar{v}(h(\beta)) > \lambda_h$, $\beta \in \overline{K}$, then some root of $h(x)$ comes sufficiently close to β; in the particular case when $h(x)$ is a tame polynomial, then the above result implies that $K(\beta)$ contains a root of $h(x)$ which yields a result of Brown proved in [12]. Recall that a finite defectless extension of henselian valued fields $(K', v')/(K, v)$ is said to be tamely ramified if the residue field of v' is a separable extension of the residue field of v and the ramification index of $(K', v')/(K, v)$ is not divisible by the characteristic of the residue field of v.

Theorem 2.18 Let $K(\theta)$ be a defectless extension of a henselian valued field (K, v) and $g(x)$ be the minimal polynomial of θ over K. If $\theta = \theta_0, \theta_1, \ldots, \theta_n$ is a complete distinguished chain for θ, then given any β in $\overline{K}$ with $\bar{v}(g(\beta)) > \bar{v}(g(\theta_1))$, there exists a K-conjugate θ' of θ such that $\bar{v}(\theta' - \beta) > \delta_K(\theta)$. Moreover, the constant $\lambda_g = \bar{v}(g(\theta_1))$ depends only on $g(x)$ and is the least element λ of $G(\overline{K})$ such that for any β in $\overline{K}$ with $\bar{v}(g(\beta)) > \lambda$, there exists a K-conjugate θ' of θ satisfying $\bar{v}(\theta' - \beta) > \delta_K(\theta)$.

Corollary 2.19 Let the hypothesis be as in Theorem 2.18. Assume in addition that either $K(\theta)$ or $K(\beta)$ is a tamely ramified extension of (K, v). Then $K(\beta)$ contains a root of $g(x)$.

In the end of this section, we note that recently Moraes De Oliveira and Nart have presented constructive methods to produce approximations to defectless prime polynomials [26]. They have used key polynomials for inductive valuations that are the important tools of valuation theory introduced by MacLane in [23, 24].

3. Vector-space defectless extensions of valued fields

In this section we consider the valued field extension $(K, v) \subseteq (K', v')$ as $(K, v) \subseteq (K', v)$ (or briefly $(K'/K, v)$) i.e., v is a valuation on K' and K is equipped with the restriction of v to K, also the notation res for the residue map corresponding to the valuation v. We will work over a valued field extension $(K'/K, v)$ unless otherwise stated.

As usual in the field theory, we may view K' as a K-vector space. Even more, we may view (K', v) as a valued K-vector space. In this section, we examine the notion of defectlessness in the context of valued vector spaces. Let us start with recalling some concepts on the vector pace defectless (or briefly vs-defectless) extensions.

Let $W \subseteq V$ be K-vector spaces with $V \subseteq K'$. The valuation and the residue map induce respectively a totally ordered subset $vV := v(V) \setminus \{\infty\}$ of $G(K')$ and a $R(K)$-vector subspace $Vv := \mathrm{res}(\mathcal{O}_V)$ of $R(K')$ where $\mathcal{O}_V := \{a \in V \mid v(a) \geqslant 0\}$. We say that the K-vector space extension $W \subseteq V$ is finite if $\dim_K V/W$ is finite.

Definition 3.1 A subset $B \subseteq V \setminus \{0\}$ is (K, v)-valuation independent over W if for every finite K-linear combination $\sum_{i=1}^{n} c_i b_i$ of (pairwise distinct) elements $b_i \in B$ and every $a \in W$, we have that

$$v\left(\sum_{i=1}^{n} c_i b_i + a\right) = \min_{1 \leqslant i \leqslant n} \{v(c_i b_i), v(a)\}.$$

Remark 1 *Note that if B is (K, v)-valuation independent over W, then it is K-linearly independent over W. Indeed, for any $a \in W$ and a finite K-linear combination $b :=$*

$\sum_{i=1}^{n} c_i b_i$ *with* $b_i \in B$ *and* $c_i \in K$ *such that* $b + a = 0$ *we have that*

$$\infty = v(0) = v\left(\sum_{i=1}^{n} c_i b_i + a\right) = \min_{1 \leqslant i \leqslant n} \{v(c_i b_i), v(a)\},$$

which imposes that $a = 0$ *and all* $c_i = 0$.

Definition 3.2 Given a (K, v)-valuation independent set $B \subseteq V$, if $V = \mathrm{Span}k_K(B) \oplus W$, B is called a (K, v)-valuation basis of V over W. The set B is (K, v)-valuation independent (resp. a (K, v)-valuation basis of V) if it is (K, v)-valuation independent over $W = \{0\}$ (resp. a (K, v)-valuation basis of V over $\{0\}$). It is called (K, v)-valuation dependent over W if it is not (K, v)-valuation independent over W.

Note that when the valued field (K, v) in consideration is clear from the context, we will often omit (K, v) and simply say K-valuation independent, K-valuation basis, etc.

Definition 3.3 The extension $(K'/K, v)$ is called vs-defectless if every finitely generated K-vector subspace of K' has a K-valuation basis.

Note that the previous definition is due to Baur [7], who originally called such extensions separated extensions. Unfortunately, the choice of terminology conflicts with standard vocabulary from other areas of mathematics which have a strong connection to valuation theory (in particular, algebraic geometry and model theory). The term "vs-defectless" chosen in this article was coined during the eighties by Roquette's group in Heidelberg. Green, Matignon and Pop in [17] introduced a vector space defect for valued function fields in one variable; it is trivial if and only if the function field is a vs-defectless extension.

In view of the fact that valued fields are ordinary valued vector spaces, we have the following transitivity of vs-defectless extensions:

Proposition 3.4 [22, Lemma 6.5] Let $(K'/K, v)$ be an extension of valued fields and E/K an arbitrary subextension of K'/K. If $(K'/K, v)$ is a vs-defectless extension, then so is $(E/K, v)$. Conversely, if $(K'/E, v)$ and $(E/K, v)$ are vs-defectless extensions, then so is $(K'/K, v)$.

Algebraic vs-defectless extensions may be characterized as follows:

Proposition 3.5 [22, Lemma 6.6] An algebraic extension $(K'/K, v)$ of valued fields is vs-defectless if and only if every finite subextension $(E/K, v)$ of $(K'/K, v)$ is vs-defectless.

The vs-defectless extensions are anti-immediate in the sense that if $(K'/K, v)$ is a vs-defectless extension and $b \in K' \setminus K$, then the set $v(b - K) = \{v(b - c) \mid c \in K\}$ has a maximum. They are anti-immediate also in the sense that they are linearly disjoint from all immediate extensions:

Proposition 3.6 [22, Lemma 6.8] Let $(\Omega/K, v)$ be an arbitrary valued field extension with $(K'/K, v)$ a vs-defectless and $(F/K, v)$ an immediate subextension. Then K'/K is linearly disjoint from F/K, the extension $(K'.F/K', v)$ is immediate and the valuation on $K'.F$ is uniquely determined by the valuations on K' and F. If $b_1, \ldots, b_n \in K'$ are K-valuation independent, then they are also F-valuation independent.

Corollary 3.7 If $(K'/K, v)$ is a vs-defectless and $(F/K, v)$ is an immediate extension, then the compositum of (K', v) and (F, v) is unique up to isomorphism of valued fields over K.

Corollary 3.8 Let $(K'/K, v)$ be a vs-defectless extension. If (K', v) is a maximal field, then so is (K, v).

In [21], Kovacsics, Kuhlmann and Rzepka have studied vs-defectless extensions of valued fields and settled two open questions regarding such extensions. They use the notion of standard K-valuation basis to give a characterization of defectless extensions as follows:

For a subset X of the field K' and $a \in K'$, we set

$$\mathrm{res}(X, a) := \{\mathrm{res}(a'/a) \mid a' \in X \text{ and } v(a') = v(a)\}.$$

Also, we let $\mathrm{Res}(X, a)$ denote the multiset $\{\mathrm{res}(a'/a) \mid a' \in X \text{ and } v(a') = v(a)\}$, that is, we allow repetition of elements. This distinction between $\mathrm{res}(X, a)$ and $\mathrm{Res}(X, a)$ will be particularly useful concerning linear independence, as it may well be the case that $\mathrm{res}(X, a)$ is a K-linearly independent set while $\mathrm{Res}(X, a)$ is not (for instance when $\mathrm{res}(X, a)$ contains a unique element which is repeated in $\mathrm{Res}(X, a)$).

By the notation above, we give an important example of a valuation independent set:

Lemma 3.9 [16, Lemma 3.2.2] Let $X \subseteq K'$ such that for any two elements in X, their image under the valuation belong to distinct cosets modulo $G(K)$. Let $Y \subseteq \mathcal{O}_{K'}$ be such that $\mathrm{Res}(Y, 1)$ is $R(K)$-linearly independent. Then the set $B := \{xy \mid x \in X, y \in Y\}$ is K-valuation independent.

Definition 3.10 Let X, Y and B be as in Lemma 3.9. If in addition $1 \in X$ and $1 \in Y$, then the set B is called a standard K-valuation independent set. When $\mathrm{Span}_K(B) = K'$ we say that B is a standard K-valuation basis of K'.

We now present the following characterization of finite vs-defectless extensions showing the relation between defectless and vs-defectless extensions [21, Proposition 3.4]:

Proposition 3.11 Assume that the extension $(K'/K, v)$ is finite. Then the following conditions are equivalent:

(i) $[K' : K] = [G(K') : G(K)][R(K') : R(K)]$,
(ii) $(K'/K, v)$ admits a standard K-valuation basis,
(iii) $(K'/K, v)$ admits a K-valuation basis,
(iv) $(K'/K, v)$ is a vs-defectless extension.

Let us remark that every K-valuation independent set in a valued field extension $(K'/K, v)$ can be transformed by multiplication with elements from K into a valuation independent set where every two elements have already equal value if their values belong to the same coset modulo $G(K)$. Moreover, note that condition (i) of the proposition above implies that the valuation v extends in a unique way from K to K' and that $(K'/K, v)$ is defectless:

Corollary 3.12 Assume that the extension K'/K is algebraic. Then $(K'/K, v)$ is vs-defectless if and only if v extends in a unique way from K to K' and $(K'/K, v)$ is defectless.

In [21], it is proved a main result about arbitrary vs-defectless extensions (not necessarily finite). In particular, it is studied the implications between the following properties of a valued field extension:

Theorem 3.13 [21, Theorem 3.7] Consider the following properties of a valued field extension $(K'/K, v)$:

(A) the extension $(K'/K, v)$ is vs-defectless,
(B) for every K-vector space $V \subseteq K'$ of finite dimension and every $a \in K'$, the set $\{v(a-x) \mid x \in V\}$ has a maximal element,
(C) K' is linearly disjoint over K from every immediate extension M of K in every common field extension.

Then (A)⇔(B)⇒(C).

In [14], Delon proved that for any valued field extension $(K'/K, v)$, (B)⇒(A)⇒(C). Delon's proof of (A)⇒(C) uses tools from the model theory of pairs of valued fields as studied by Baur in [7]. It remained open whether implications (A)⇒(B) and (C)⇒(A) hold in general. Kovacsics et al. in [21] answer both questions by showing that the former implication does hold in general, while the latter does not. Finally, an example of a valued field extension that does not satisfy the implication (C)⇒(A) is given in Proposition 3.9 of [21].

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