

## A certain studies on Nörlund summability of series

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**Abstract.** In this paper, we have obtained two theorems for Nörlund summability of Fourier series and their conjugate series under very general conditions. These two theorems are closely related to the great works of the analysts Patti [24], McFadden [15] and Siddiqui [25] but not the same.

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### 1. Introduction

The old hazy notion of convergence of infinite series was placed on the sound foundation with the appearance of Cauchy's monumental work "course d'analyse algebrigue" in 1821 and Abel's researches [1] on the binomial series in 1826. However, it was observed that there were certain non-convergent series which particularly in dynamical astronomy furnished nearly correct results. A theory of divergent series was formulated explicitly for the first time in 1890 when Cesàro [3] published a paper on the multiplication of series. Since the theory of series whose sequence of partial sums oscillates has been the center of creative activity for most of the leading mathematical analysts. After persistent efforts in which several celebrated mathematicians took part like Holder, Hausdorff, Riesz, Nörlund, etc. It was only in the closing decade of 19<sup>th</sup> century and the early years

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of the present century that satisfactory methods were devised to associate with them by process closely connected Cauchy concept of convergence certain values which may be called their sums in a reasonable way. These processes of associating generalized sums, known as methods of summability (Szász [26, 27] and Hardy [6] provide a natural generalization of the classical notion of convergence (Hobson [9], Titchmarsh [28]) and are thus responsible for bringing within the fold of applicability a wider class of erstwhile rejected series that used to be tabooed as divergent. The idea of convergence has been thus generalized, it was quite natural to study the possibilities of extending the notion of absolute convergence. As a matter of fact, just as the notion of convergence has led to the development of its extension under the general title of summability so also by analogy, the concept of absolute convergence led to the formulation of the various process of absolute summability [11]. As the ideas of ordinary and absolute convergent were instrumental to the development of ordinary and absolute summability respectively. Also, the notion of uniform convergence would have certainly insisted on the analysis of think of uniform summability. Hardy and Littlewood [7], for the first time in 1913, introduced the notion of “strong summability” of Fourier series (Fekete [4] in 1916 defined that a series  $\sum a_n$ ). This type is known as Cesàro summability of order 1 or  $(C, 1)$  summability. It is important to note that strong summability is weaker than absolute summability and strong than ordinary summability. Lorentz [12], for the first time in 1948, defined almost convergence of a bounded sequence  $\{S_n\}$  of an infinite series  $\sum a_n$ . It is easy to see that a convergent sequence is almost convergent and the limits are the same [14].

Mishra [16], Mishra et al. [17], Mishra et al. ([18, 19]) and Mishra [20] were the first mathematicians to use some problems on approximation of functions in Banach spaces, approximation of functions belonging to  $Lip(\xi(t), t)$  class by  $(N, p_n)(E, q)$  summability of conjugate series of Fourier series, on the trigonometric approximations of signals belonging to generalized weighted Lipschitz  $W'(L_r, \xi(t))$ ,  $(r \geq 1)$ – class by matrix  $(C^1.N_p)$  operator of conjugate series of its Fourier series, trigonometric approximation of periodic signals belonging to generalized weighted Lipschitz  $W'(L_r, \xi(t))$ ,  $(r \geq 1)$ -class by Nörlund-Euler  $(N, p_n)(E, q)$  operator of conjugate series of its Fourier series and trigonometric approximation of signals (functions) in  $L_p$ -norm respectively. The idea of almost convergence led to the formulation of almost summability methods.

## 2. Study of “T” and “ $\Phi$ ” Process

Some of the most familiar methods of summability and with which shall be concerned in the sequel are those that are known as method of Nörlund summability absolute summability, Cesàro summability, ordinary and absolute  $(\bar{N}, p_n)$  summability,  $(f, d_n)$  summability, almost matrix summability,  $(N, P_n)$  summability and strong summability. It may however be mentioned that all those methods can be derived from two basic general processes, which are T- Process and  $\Phi$ -process.

A T-methods are based upon the formation of an auxiliary sequence  $t_n$ , defined by the sequence-to-sequence transformation such that  $t_n = \sum a_{n,k} S_k$ ,  $n = 0, 1, 2, 3$ , where  $a_{n,k}$  being the elements of  $n^{th}$  and  $k^{th}$  column of Toeplitz matrix method [10] ( $T = a_{n,k}$ ) and  $S_k$  the  $k^{th}$  partial sum of infinite series  $\sum a_n$ . The  $\Phi$ -methods are based upon the formation of functional transformation  $t(x)$  defined by sequence to - functional transformation  $t(x) = \sum \Phi_n(x) S(y)$  or by function to - function transform  $t(x) = \int_0^\infty \Phi(x, y) S(y) dy$ , where  $x$  is constant parameter and the function  $\Phi(x, y)$  is defined over a suitable interval of  $x$  and  $y$ . The series  $\sum a_n$  or the sequence  $\{S_n\}$  is said to be summable to a finite limits S by T-method or  $\Phi$ -method according as the sequence  $\{t_n\}$  or the function  $t(x)$  tends

to  $S$  as  $n$  tends to infinity depending upon the method. We considered infinite matrices  $A = (a_{n,k})$  and corresponding matrix transforms and summability, methods (compare [13, 30]). A sequence  $\{S_k\}$  is said to be  $\bar{A}$ -summable to the value  $\sigma$ , if all sums

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk} S_k, \quad n = 0, 1, \dots \tag{1}$$

exist and converge to  $\sigma$  for  $k \rightarrow \infty$ .

The sequence  $\{S_k\}$  is strongly  $A$ -summable or  $\bar{A}$ -summable to the value, if all sums

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk} |S_k - \sigma|, \quad n = 0, 1, \dots \tag{2}$$

exist and converge to zero. Strong summability is usually considered only for positive  $A$  (i.e. for  $a_{nk} \geq 0$ ). In this case the limit  $\sigma$  is uniquely determined [2, 5]. The method of summability considered under Nörlund summability was first introduced Woroni [29] in 1902, it is customary to associate with the name of Nörlund [23] who independently introduced this method in 1919. In 1948, Siddiqui [25] for the first time to introduce the notion of harmonic summability of Fourier series.

Let  $t_n = \sum_{r=0}^n \frac{p_{n-r} S_r}{P_n}$ , ( $p_n \neq 0$ ) or

$$t_n = P_n^{-1} \sum_{r=0}^n p_r S_{n-r}. \tag{3}$$

If  $t_n \rightarrow S$  as  $n \rightarrow \infty$ , we write  $\sum_{n=0}^{\infty} a_n = S(N, p_n)$  or  $S_n \rightarrow S(N, p_n)$ .

The conditions of regularity of the method of summability  $(N, p_n)$  defined by (3) is

$$\lim_{n \rightarrow \infty} \frac{p_n}{P_n} = 0. \tag{4}$$

Strong Cesàro summability  $[C, \alpha, q]$  ( $\alpha > 0, q > 0$ ) of series  $\sum a_n$  is defined by Hyslop [10, 22], which is extended to strong Nörlund summability  $[N, p_n, q]$ ,  $q \geq 1$  by the Mittal and Kumar [21], Mittal, Singh and Mishra [22]. This definition can be further extended to summability  $[N, p_n^\alpha, q]$ ,  $\alpha \geq 1, q \geq 1$ , by taking  $p_n^\alpha (= \sum_{r=0}^n E_{n-r}^{\alpha-1} p_r)$  for  $p_n$ , where  $T_n^\alpha$  is obtained from [21, 22] replacing  $p_n$  by  $p_n^\alpha$ .

### 3. Preliminaries

Let  $\{p_n\}$  be a sequence of non-zero constants real or complex with  $p_n$  as its non-vanishing  $n^{th}$  partial sum and let  $\{S_n\}$  be the sequence of partial sum of a given infinite series  $\sum a_n$ . Following the lines of Fekete [4], Hardy and Littlewood [8], if

$$\sum_{m=0}^n p_m |S_{n-m} - S| = o(p_n), \quad \text{as } n \rightarrow \infty, \tag{5}$$

then the sequences  $\{S_n\}$  or the series  $\sum a_n$  is said to be strongly summable  $(N, p_n)$  to the fixed finite sum  $S$ .

Let  $f(t)$  be a  $2\pi$ -periodic and Lebesgue integral function of  $t$  in  $(-\pi, \pi)$ . Then the Fourier series corresponding to the function  $f(t)$  is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \quad (6)$$

and  $\sum_{n=0}^{\infty} A_n(t)$  and its conjugate series is given by

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t). \quad (7)$$

Let us write with a fixed point,

$$\Phi(t) = f(x+t) + f(x-t) - 2f(x), \quad \psi(t) = f(x+t) - f(x-t),$$

and

$$K_n(t) = \frac{1}{2\pi} \sum_{m=0}^n \left\{ \frac{p_m}{\sin \frac{t}{2}} \epsilon_m \sin \left( n - m + \frac{1}{2} \right) t \right\},$$

$$\bar{K}_n(t) = -\frac{1}{2\pi} \sum_{m=0}^n \left\{ \frac{p_m}{\sin \frac{t}{2}} \epsilon_m \cos \left( n - m + \frac{1}{2} \right) t \right\},$$

and  $p\left(\frac{1}{t}\right) = p_\tau$  and  $P\left(\frac{1}{t}\right) = P_\tau$ , where  $\tau$  denote the internal part of  $\frac{1}{t}$ .

The summability of Fourier series by ordinary Nörlund method has been studied by various analysts. In this direction, the following result due Pati [24] is worth stating.

**Theorem 3.1** Let  $(N, p_n)$  be a regular Nörlund method defined by a real, non-negative, monotonic non-increasing sequence of coefficients  $p_n$  such that  $p_n \rightarrow \infty$  and  $\log n = o(P_n)$  as  $n \rightarrow \infty$ . If

$$\int_0^t |\phi(u)| du = o\left(\frac{t}{P_\tau}\right) \quad \text{as } t \rightarrow +0, \quad (8)$$

then the Fourier series of  $f(t)$  is summable  $(N, p_n)$  to the sum  $f(x)$  at the point  $t = x$ .

The object of this article is to improve the result of the above theorem under very general condition by establishing the following theorem.

**Theorem 3.2** Let  $\{p_n\}$  be a non-negative, monotonic non increasing sequence of constants with  $\{\epsilon_n\}$  a suitable sequence of constants  $\pm 1$  such that

$$\sum_{m=0}^n p_m \epsilon_m = o\left(P_n\right), \quad (9)$$

where  $P_n$  is non-vanishing  $n^{th}$  partial sum of the sequence of constants  $\{p_n\}$ , and  $p_n$  tending to  $\infty$  as  $n \rightarrow \infty$ . Also, let  $\mu(t)$  and  $\nu(t)$  be two positive functions of  $t$  such that  $\mu(t)$ ,  $\nu(t)$  and  $t \frac{\mu(t)}{\nu(t)}$  increases monotonically with  $t$  and

$$\mu(n)P_n = o[\nu(P_n)] \quad \text{as } n \rightarrow \infty. \tag{10}$$

If

$$\Phi(t) = \int_0^t |\Phi(u)|du = o \left[ \frac{\mu(\frac{1}{t}) \cdot p_\tau}{\nu(p_\tau)} \right] \text{ as } t \rightarrow +0, \tag{11}$$

then

$$\sum_{m=0}^n p_m |\sigma_{n-m}(x) - f(x)| = o(P_n) \text{ as } n \rightarrow \infty \tag{12}$$

at point  $t = x$ , where  $\sigma_n(x)$  is the  $n^{th}$  partial sum of the Fourier series (6) of the function  $f(t)$  at  $t = x$ .

**Theorem 3.3** Let  $\{p_n\}$  and  $\{\epsilon_n\}$  be the same as in Theorem 3.2 satisfying (9). Again, let  $\mu(t)$  and  $\nu(t)$  are same satisfying the condition (10), and  $\bar{\sigma}_n(x)$  be the  $n^{th}$  partial sum of the conjugate series (7) at  $t = x$ . If

$$\Psi(t) = \int_0^t |\psi(u)|du = o \left[ \frac{\mu(\frac{1}{t}) p_\tau}{\nu(P_\tau)} \right] \text{ as } t \rightarrow 0, \tag{13}$$

then

$$\sum_{m=0}^n p_m |\bar{\sigma}_{n-m}(x) - \bar{f}(x)| = o(P_n) \text{ as } n \rightarrow \infty \tag{14}$$

at every point  $x$ , where  $\bar{f}(x) = \frac{1}{2\pi} \int_0^\pi \Psi(t) \cos \frac{1}{2} t dt$  at every point this integral exists.

For the proof of discussed theorems, we need the following lemmas.

**Lemma 3.4** If the sequences  $\{p_n\}$  and  $\{\epsilon_n\}$  are considered in Theorem 3.2 satisfying the condition (9), then for  $0 \leq a < b \leq \infty$ ,  $0 \leq t \leq \infty$  and for any  $n$ ,  $|\sum_{m=a}^b p_m \epsilon_m e^{i(n-m)t}| = o(P_\tau)$ .

It is immediate consequence of a well-known result due to McFadden [15].

**Lemma 3.5** Let the sequences  $\{p_n\}$  and  $\{\epsilon_n\}$  are considered in Theorem 3.2 satisfying the condition (9). Then, for  $0 \leq t \leq \frac{1}{n}$ ,

$$|K_n(t)| = o(nP_n) \tag{15}$$

and

$$\left| \frac{1}{2\pi} \sum_{m=0}^n p_m \epsilon_m \frac{\cos \frac{t}{2} - \cos(n-m+\frac{1}{2})}{\sin \frac{1}{2}t} \right| = o(nP_n) \text{ as } n \rightarrow \infty. \tag{16}$$

**Proof.** For  $0 \leq t \leq \frac{1}{n}$ , we have

$$\begin{aligned} |K_n(t)| &= \left| \frac{1}{2\pi} \sum_{m=0}^n p_m \epsilon_m \frac{\sin(n-m+\frac{1}{2})t}{\sin\frac{t}{2}} \right| \\ &= o \left\{ \sum_{m=0}^n p_m \epsilon_m (2n-2m+1) \frac{|\sin\frac{1}{2}t|}{|\sin\frac{1}{2}t|} \right\} = o(nP_n) \quad (\text{using (9)}) \end{aligned}$$

and

$$\begin{aligned} \left| \frac{1}{2\pi} \sum_{m=0}^n p_m \epsilon_m \frac{\cos\frac{t}{2} - \cos(n-m+\frac{1}{2})t}{\sin\frac{t}{2}} \right| &\leq \frac{1}{2\pi} \sum_{m=0}^n p_m \epsilon_m \sum_{k=0}^{n-m} 2|\sin kt| \\ &= o \left[ \sum_{m=0}^n p_m \epsilon_m (n-m) \right] \text{ as } n \rightarrow \infty \\ &= o(nP_n) \text{ as } n \rightarrow \infty. \end{aligned}$$

■

**Lemma 3.6** With the sequences  $\{p_n\}$  and  $\{\epsilon_n\}$  as above satisfying (9), and for  $\frac{1}{n} \leq t \leq \delta < \pi$ , we have

$$|K_n(t)| = o \left[ \frac{P_\tau}{t} \right] \quad (17)$$

$$|\bar{K}_n(t)| = o \left[ \frac{P_\tau}{t} \right] \text{ as } n \rightarrow \infty. \quad (18)$$

**Proof.** Here, for  $\frac{1}{n} \leq t \leq \delta < \pi$ , we have

$$\begin{aligned} |K_n(t)| &= \frac{1}{2\pi} \left| \sum_{m=0}^n p_m \epsilon_m \frac{\sin(n-m+\frac{1}{2})t}{\sin\frac{t}{2}} \right| \\ &= \frac{1}{2\pi|\sin\frac{t}{2}|} \cdot \left| I_m \sum_{m=0}^n p_m \epsilon_m e^{i(n-m+\frac{1}{2})t} \right| \\ &= \frac{1}{2\pi|\sin\frac{t}{2}|} \left| I_m \left[ e^{\frac{it}{2}} \sum_{m=0}^n p_m \epsilon_m e^{i(n-m)t} \right] \right| \\ &= o \left[ \frac{P_\tau}{t} \right] \text{ as } n \rightarrow \infty \quad (\text{by Lemma 3.4}). \end{aligned}$$

Similarly, we obtain  $|\bar{K}_n(t)| = o \left[ \frac{P_\tau}{t} \right] \text{ as } n \rightarrow \infty.$

■

**Lemma 3.7** If  $\frac{1}{n} \leq t \leq \delta < \pi$ , then

$$|\bar{K}_n(t)| = \left| \frac{-1}{2\pi} \sum_{m=0}^n \left\{ \frac{P_m}{\sin \frac{t}{2}} \epsilon_m \cos(n - m + \frac{1}{2})t \right\} \right| = o \left[ \frac{p_\tau}{p_n(t)} \right].$$

**Proof.** The proof is similar to that of Lemma 3.6. ■

Now, we prove Theorems 3.2 and 3.3.

**Proof of Theorem 3.2.** Let  $\sigma_n$  denotes the  $n^{th}$  partial sum of the Fourier series (6), then we have

$$\sigma_n(x) - f(x) = \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\sin(n + \frac{1}{2})t}{\sin \frac{t}{2}} dt.$$

Hence,

$$\begin{aligned} \sum_{m=0}^n p_m \left[ \sigma_{n-m}(x) - f(x) \right] &= \sum_{m=0}^n p_m \epsilon_m \left[ \sigma_{n-m}(x) - f(x) \right] \\ &= \int_0^\pi \phi(t) \left\{ \sum_{m=0}^n p_m \epsilon_m \frac{\sin(n - m + \frac{1}{2})t}{\sin \frac{t}{2}} \right\} dt \\ &= \int_0^\pi \phi(t) K_n(t) dt \\ &= \left\{ \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^\delta + \int_\delta^\pi \right\} \phi(t) K_n(t) dt \quad (0 < \delta < \pi) \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{19}$$

where  $\{\epsilon_n\}$  is a suitable sequence of constants  $\pm 1$  such that  $p_m \epsilon_n \{\sigma_{n-m}(x) - f(x)\} \geq 0$  for all  $m$ . Now,

$$\begin{aligned} |I_1| &\leq \int_0^{\frac{1}{n}} |\phi(t)| |K_n(t)| dt \\ &= o \left[ nP_n \right] \int_0^{\frac{1}{n}} |\phi(t)| dt \\ &= o \left[ nP_n \right] \cdot o \left[ \frac{p_n \cdot \mu_n}{\nu(P_n)} \right] \quad (\text{by (8)}) \\ &= o \left[ P_n \right] \text{ as } n \rightarrow \infty, \quad (\text{by (10)}) \end{aligned} \tag{20}$$

because  $np_n \leq P_n$ , considering,  $I_2$ , we have

$$\begin{aligned}
 |I_2| &\leq \int_{\frac{1}{n}}^{\delta} |\phi(t)| |K_n(t)| dt \\
 &= o \left[ \int_{\frac{1}{n}}^{\delta} |\phi(t)| \frac{p_{\tau}}{t} dt \right] \quad (\text{by (17)}) \\
 &= o \left[ \Phi(t) \frac{p_{\tau}}{t} \right]_{\frac{1}{n}}^{\delta} + o \left[ \int_{1/n}^{\delta} \Phi(t) \frac{p_{\tau}}{t^2} dt \right] + o \left[ \int_{\frac{1}{n}}^{\delta} \Phi(t) \frac{1}{t} d(p_{\tau}) \right] \\
 &= I_{2.1} + I_{2.2} + I_{2.3}.
 \end{aligned} \tag{21}$$

Now,

$$I_{2.1} = o \left[ \Phi(t) \frac{p_{\tau}}{t} \right]_{\frac{1}{n}}^{\delta} = o \left[ p_n \frac{\mu(n) \cdot np_n}{\nu(P_n)} \right] = o(P_n) \text{ as } n \rightarrow \infty. \tag{22}$$

Again,

$$I_{2.2} = o \left[ \int_{\frac{1}{n}}^{\delta} \Phi(t) \frac{p_{\tau}}{t^2} dt \right] = o \left[ \sum_{m=1}^{n-1} \int_m^{m+1} \Phi \left( \frac{1}{\nu} \right) p_{[\nu]} dv \right].$$

But

$$\int_m^{m+1} \Phi \left( \frac{1}{\nu} \right) p_{(\nu)} dv \leq I \left( \frac{1}{m} \right) p_m = o \left[ p_m \mu(m) \frac{p_m}{\nu(p_m)} \right] = o(p_m) \text{ as } m \rightarrow \infty.$$

So,

$$I_{2.2} = o \left( \sum_{m=0}^{n-1} p_m \right) = o(p_n) \text{ as } n \rightarrow \infty \tag{23}$$

and

$$\begin{aligned}
 I_{2.3} &= o \left[ \int_{\frac{1}{n}}^{\delta} \Phi(t) \frac{1}{t} dp_{\tau} \right] \\
 &= o \left[ \int_{\frac{1}{\delta}}^n \Phi \left( \frac{1}{\nu} \right) V dp_{\nu} \right] \\
 &= o \left[ \sum_{m=1}^{n-1} m p_m \Phi \left( \frac{1}{m} \right) \right] \\
 &= o \left[ \sum_{m=1}^{n-1} p_m \mu(m) \frac{p_m}{\nu(p_m)} \right] \\
 &= o \left[ \sum_{m=1}^{n-1} p_m \right] \\
 &= o(p_n) \text{ as } n \rightarrow \infty.
 \end{aligned} \tag{24}$$



Combining (21), (22), (23) and (24), we get

$$|I_2| = o(p_n) \text{ as } n \rightarrow \infty. \quad (25)$$

Lastly, by virtue of Riemann-Lebesgue theorem and regularity of method of summation, we have

$$|I_3| \leq \int_{\delta}^{\pi} |\phi(t)| |K_n(t)| dt = o \left[ \int_{\delta}^{\mu} |\phi(t)| \frac{p_{\tau}}{t} dt \right] = o(p_n) \text{ as } n \rightarrow \infty. \quad (26)$$

Hence, on collecting (19), (20), (25) and (26), we get the required result in (12). This completes the proof of the Theorem 3.2.

**Proof of Theorem 3.3.** Let  $\bar{\sigma}_n(x)$  the  $n^{\text{th}}$  partial sum of the series  $\sum B_n(x)$ . Then, we have

$$\bar{\sigma}_n(x) = \frac{1}{2\pi} \int_0^{\pi} \Psi(t) \frac{\cos \frac{t}{2} - \cos(n + \frac{1}{2})t}{\sin \frac{t}{2}} dt.$$

Hence the following (5),

$$\begin{aligned} \sum_{m=0}^n p_m |\bar{\sigma}_{n-m}(x) - \bar{f}(x)| &= \sum_{m=0}^n |\bar{f}(x) - \bar{\sigma}_{n-m}(x)| p_m \\ &= \sum_{m=0}^n p_m \epsilon_m \left\{ \bar{f}(x) - \bar{\sigma}_{n-m}(x) \right\} \\ &= \sum_{m=0}^n p_m \epsilon_m \left\{ \frac{1}{2\pi} \int_0^{\pi} \Psi(t) \cdot \frac{\cos(n - m + \frac{1}{2})t}{\sin \frac{t}{2}} dt \right\} \\ &= \int_0^{\pi} \Psi(t) \frac{1}{2\pi} \sum_{m=0}^n p_m \epsilon_m \frac{\cos(n - m + \frac{1}{2})t}{\sin \frac{t}{2}} dt \\ &= \int_0^{\pi} \Psi(t) \bar{K}_n(t) dt \\ &= \left\{ \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi} \right\} \Psi(t) \bar{K}_n(t) dt \quad (0 < \delta < \pi) \\ &= J_1 + J_2 + J_3, \end{aligned} \quad (27)$$

where  $p_m \epsilon_m \left\{ \bar{f}(x) - \bar{\sigma}_{n-m}(x) \right\} \geq 0$ .

Since the conjugate function exists, therefore

$$\frac{1}{2\pi} \int_{t=0}^{\frac{1}{n}} \Psi(t) \cos \frac{t}{2} dt = o(1) \text{ as } n \rightarrow \infty. \quad (28)$$

Now,

$$\begin{aligned}
|J_1| &\leq \int_{t=0}^{\frac{1}{n}} |\Psi(t)| \left| \bar{K}_n(t) \right| dt \\
&= \int_{t=0}^{\frac{1}{n}} |\Psi(t)| \left| \frac{1}{2\pi} \sum_{m=0}^n p_m \epsilon_m \frac{\cos(n-m+\frac{1}{2})t}{\sin \frac{t}{2}} \right| dt \\
&= \int_{t=0}^{\frac{1}{n}} |\Psi(t)| \left| \frac{1}{2\pi} \sum_{m=0}^n p_m \epsilon_m \left\{ \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} - \frac{\cos \frac{t}{2} - \cos(n-m+\frac{1}{2})t}{\sin \frac{t}{2}} \right\} \right| dt \\
&\leq \frac{1}{2\pi} \sum_{m=0}^n p_m \epsilon_m |\Psi(t) \cos \frac{t}{2}| dt + \int_{t=0}^{\frac{1}{n}} |\Psi(t)| \left| \frac{1}{2\pi} \left[ \sum_{m=0}^n p_m \epsilon_m \left\{ \frac{\cos \frac{t}{2} - \cos(n-m+\frac{1}{2})t}{\sin \frac{t}{2}} \right\} \right] \right| dt \\
&= o(p_n) \cdot o(1) + (np_n) \int_{t=0}^{\frac{1}{n}} |\Psi(t)| dt, \quad (\text{by (9), (16) and (28)}) \\
&= o(p_n) + o(np_n) \cdot o\left[\frac{\mu(n)p_n}{\nu(p_n)}\right] \\
&= o(p_n) + o\left[p_n \frac{\mu(n)p_n}{\nu(p_n)}\right] \\
&= o(p_n) + o(p_n), \quad \text{using (10)} \\
&= o(p_n) \text{ as } n \rightarrow \infty. \tag{29}
\end{aligned}$$

Also, for  $\frac{1}{n} \leq t \leq \delta$ ,

$$|J_2| \leq \int_{t=\frac{1}{n}}^{\delta} |\Psi(t)| |\bar{K}_n(t)| dt = o\left[\int_{t=\frac{1}{n}}^{\delta} |\Psi(t)| \frac{p_\tau}{P_n(t)} dt\right] = o(P_n) \text{ as } n \rightarrow \infty. \tag{30}$$

Moreover,

$$|J_3| = o(P_n) \tag{31}$$

by virtue of Riemann-Lebesgue theorem and the regularity of the method of summation. Hence, combining (27), (29), (30) and (31), we get the required result in our (14). This completes the proof of Theorem 3.3.

#### 4. Conclusion

In this paper, we have characterized the Nörlund summability of Fourier series and their conjugate series. We deduced the special case to obtain the necessary conditions for the Nörlund summability of series and their conjugate.

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