Journal of
Linear and Topological Algebra
Vol. 09*, No.* 04*,* 2020, 291- 299

Operators reversing b-Birkhoff orthogonality in 2-normed linear spaces

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Received 20 June 2020; Revised 25 December 2020; Accepted 30 December 2020.

Communicated by Tatjana Dosenović

Abstract. In this paper, we discuss the relationships between 2-functionals and existence of b-Birkhoff orthogonal elements in 2-normed linear spaces. Moreover, we obtain some characterizations of 2-inner product spaces by b-Birkhoff orthogonality. Then we study the operators reversing b-Birkhoff orthogonality in 2-normed linear spaces.

Keywords: b-Birkhoff orthogonal, 2-functionals, 2-hyperplane, 2-inner product, 2-normed linear spaces.

2010 AMS Subject Classification: 46C05.

1. Introduction and preliminaries

The concept of 2-normed linear spaces has been investigated by Gähler in 1960's [7] and has been developed extensively in different subjects by many authors (for example, see [11–13]).

Let X be a linear space of dimension greater than 1. Suppose $\|.,.\|$ is a real-valued function on $X \times X$ satisfying the following conditions:

(1) $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors,
(2) $\|x, y\| = \|y, x\|$ for all $x, y \in X$,
(3) $\|\lambda x, y\| = |\lambda|\|x, y\|$ for all $\lambda \in \mathbb{R}$ and all $x, y \in X$,
(4) $\|x + y, z\| \leqslant \|x, z\| + \|y, z\|$ for all $x, y, z \in X$.

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

Then $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a 2-normed linear space. A 2-norm is non-negative and the basic property of a 2-norm is $\|x, y+\alpha x\| = \|x, y\|$ for all $x, y \in X$ and all $\alpha \in \mathbb{R}$. Note that $(X, \|.,.\|)$ with the formula $\|x, y\| = \|x\|\|y\|$ for each $x, y \in X$ is not a 2-normed space. So the relationship $\|x, y+\alpha x\| = \|x, y\|$ is not valid. For example, let $x \neq 0$ and $\alpha \neq 0$. Then

$$0 = \|x, 0\| = \|x, 0+\alpha x\| = \|x, \alpha x\| = \|x\|\|\alpha x\| = |\alpha|\|x\|^2 > 0.$$

***Example* 1.1** [19] Let $X = \mathbb{R}^3$ with 2-norm defined as follow:

$$\|(x_1, x_2, x_3), (y_1, y_2, y_3)\| = |x_1y_2 - x_2y_1| + |x_1y_3 - x_3y_1| + |x_2y_3 - x_3y_2|$$

for all $(x_1, x_2, x_3), (y_1, y_2, y_2) \in X$. Let vector addition and scalar multiplication be defined componentwise. Then the 2-norm properties are satisfied.

***Example* 1.2** Let $X = E^3$ be an Euclidean 3-dimensional linear space. The formula $\|x, y\| = |x \times y|$ defines a 2-norm on X, where x, y are two vector in E^3 and $x \times y$ means the vector product of x and y.

The following elementary proposition is proved in [10].

Proposition 1.3 Let $(X, \|.,.\|)$ be a 2-normed space. Then

(1) $\|x+y, x\| = \|x, y\|$ for all x, y in X,
(2) if for two linearly independent x and y in E, $\|z, x\| = \|z, y\| = 0$ for $z \in X$, then $z = 0$.

Every 2-normed space is a locally convex topological vector space. In fact, for a fixed $b \in X$, $p_b(x) = \|x, b\|$ for all $x \in X$ is a semi-norm and the family $P = \{p_b : b \in X\}$ of semi-norms generates a locally convex topology on X. As an example of a 2-normed space, take $X = \Re^2$ equipped with $\|x, y\| =$ which is defined as the area of the parallelogram spanned by the vectors x and y (i.e. the parallelogram whose adjacent sides are the vectors a and b) which may be given explicitly by the formula $\|x, y\| =| x_1y_2 - x_2y_1 |$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$ ([16]).

Along with the 2-norm, we have the standard 2-inner product space. Let X be a real vector space of dimension$\geqslant 2$. The real-valued function $\langle ., .|.\rangle : X \times X \times X \to \mathbb{R}$, which satisfies the following properties on X^3 is called 2-inner product on X:

(1) $\langle x, x|z\rangle \geqslant 0$ for every $x, z \in X$ and $\langle x, x|z\rangle = 0$ if and only if x and z are linearly dependent,
(2) $\langle x, y|z\rangle = \langle y, x|z\rangle$ for every $x, y, z \in X$,
(3) $\langle x, x|z\rangle = \langle z, z|x\rangle$ for every $x, z \in X$,
(4) $\langle \alpha x, y|z\rangle = \alpha\langle x, y|z\rangle$ for every $x, y, z \in X$ and $\alpha \in \mathbb{R}$,
(5) $\langle x_1 + x_2, y|z\rangle = \langle x_1, y|z\rangle + \langle x_2, y|z\rangle$ for every $x_1, x_2, y, z \in X$.

Under these conditions, the pair $(X, \langle ., .|.\rangle)$ is called an inner product space [3, 4, 6]. Also, by the formula

$$\langle x, y|z\rangle := \begin{vmatrix} \langle x, y\rangle & \langle x, z\rangle \\ \langle z, y\rangle & \langle z, z\rangle \end{vmatrix},$$

we observe that $\|x, y\| = \langle x, x|y\rangle^{1/2}$ and the Cauchy-Schwarz inequality $\langle x, y|z\rangle^2 \leqslant \|x, z\|^2\|y, z\|^2$ for every $x, y, z \in X$ is valid.

Now, let $(X, \|.,.\|)$ be a 2-normed space and W_1 and W_2 be two subspaces of X. A map $f : W_1 \times W_2 \to \mathbb{R}$ is called a bilinear 2-functional ([15]) on $W_1 \times W_2$ whenever for all $x_1, x_2 \in W_1, y_1, y_2 \in W_2$ and all $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

(1) $f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_2, y_2) + f(x_2, y_1) + f(x_2, y_2)$,
(2) $f(\lambda_1 x_1, \lambda_2 y_1) = \lambda_1 \lambda_2 f(x_1, y_1)$.

A bilinear 2-functional $f : W_1 \times W_2 \to \mathbb{R}$ is called bounded if there exists a non-negative real number M (M is called a Lipschitz constant for f) such that $|f(x, y)| \leqslant M\|x, y\|$ for all $x \in W_1$ and all $y \in W_2$. Also, the norm of a bilinear 2-functional is defined by

$$\|f\| = \inf\{M \geqslant 0 : M\, is\, a\, Lipschitz\, constant\, for\, f\}.$$

It is known that [12]

$$\begin{aligned} \|f\| &= \sup\{|f(x, y)| : (x, y) \in W_1 \times W_2, \|x, y\| \leqslant 1\} \\ &= \sup\{|f(x, y)| : (x, y) \in W_1 \times W_2, \|x, y\| = 1\} \\ &= \sup\{|f(x, y)|/\|x, y\| : (x, y) \in W_1 \times W_2, \|x, y\| \neq 0\}. \end{aligned}$$

For a 2-normed space $(X, \|.,.\|)$ and $0 \neq b \in X$, we denote by X_b^* the Banach space of all bounded bilinear 2-functionals on $X \times \langle b\rangle$, where $\langle b\rangle$ is the subspace of X generated by b ([12]).

***Example* 1.4** [19] Let $(E^3, \|, \|)$ be the 2-normed space with $\|x, y\| = |x \times y|$. Define $f(x, y) = x \cdot y$, where $x \cdot y$ is the dot product of vector analysis. Then f is an unbounded linear 2-functional. Now, define

$$f(x, y) = (|x|^2|y|^2 - |(x.y)|^2)^{\frac{1}{2}},$$

where $|a|$ denotes the length of a. Since $|x|^2|y|^2 - |(x.y)|^2 = |x \times y|^2$, then f is a bounded 2-functional

2. Types of orthogonality

When we say that a normed linear space is Euclidean, we mean that it is an inner product space. In particular, a two-dimensional (real) inner product space is referred to as the Euclidean plan. There are many different ways to characterize inner product spaces among normed linear spaces ([1]).

In a real normed space $(X, \|.\|)$ one can define orthogonality of two vectors x and y in many different ways. For example, the following definitions of Pythagorean, Isosceles, and the Birkhoff-James orthogonality are known [5, 17].

P-orthogonality: x is P-orthogonal to y (denoted by $x \perp_P y$) if and only if

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

I-orthogonality: x is I-orthogonal to y (denoted by $x \perp_I y$) if and only if

$$\|x + y\| = \|x - y\|.$$

BJ-orthogonality: x is BJ-orthogonal to y ($x \perp_{BJ} y$) if and only if $\|x + \alpha y\| \geqslant \|x\|$ for every $\alpha \in \mathbb{R}$.

Note that in an inner product space $(X, \langle ., . \rangle)$; $x \perp_P y$, $x \perp_I y$, and $x \perp_{BJ} y$ are all equivalent to the condition $\langle x, y \rangle = 0$ for which we have the usual orthogonality in a normed space which is not an inner product space, however, one does $x \perp y$. not imply another. For further properties of these orthogonalities and related results (for example, see [5, 17]).

Cho and Kim [2] defined the condition of G-orthogonality of two vectors in a 2-inner product space of dimension 3 or higher as follows:

In an arbitrary 2-inner product space $(X, \langle ., .|. \rangle)$; $x \perp_P y$, $x \perp_I y$ and $x \perp_{BJ} y$ are equivalent to the condition

$$\langle x, y|z \rangle = 0, \quad for \quad every \quad x \notin span\{x, y\}. \tag{1}$$

In [9], Khan and Siddiqui defined the notion of P, I and BJ-orthogonality in 2-normed spaces $(X, \|., .\|)$ as follows:

P-orthogonality: $x \perp_P y$ if only if $\|x + y, z\|^2 = \|x, z\|^2 + \|y, z\|^2$ for every z.

I-orthogonality: $x \perp_I y$ if only if $\|x + y, z\| = \|x - y, z\|$ for every $z \neq 0$.

BJ-orthogonality: $x \perp_{BJ} y$ if only if $\|x + \alpha y, z\| \geqslant \|x, z\|$ for every $z \neq 0$ and $\alpha \in \mathbb{R}$.

Also we have the following definition [15].

Definition 2.1 Let $(X, \|., .\|)$ be a 2-normed space and $x, y \in X$. If there exists $b \in X$ such that $\|x, b\| = 0$ and $\|x, b\| \geqslant \|x + \alpha y, b\|$ for each scalar $\alpha \in \Re$, then x is b-orthogonal to y (denoted by $x \perp_b y$).

In this paper, we discuss the relationships between 2-functionals and existence of b-Birkhoff orthogonal elements in 2-normed linear spaces. Moreover, we obtain some characterizations of 2-inner product spaces by b-Birkhoff orthogonality. Then we study the operators reversing b-Birkhoff orthogonality in 2-normed linear spaces.

3. 2-functionals in 2-normed linear spaces and existence of b-Birkhoff orthogonal elements

Let X be a 2-normed linear space. Also, let $0 \neq b \in X$ and $0 \neq f$ be a nonzero bilinear 2-functional on $X \times \langle b \rangle$. Then we define the 2-hyperplane H through the origin by $H = \{x \in X; f(x, b) = 0\}$.

We start this section with the following useful theorem.

Theorem 3.1 Under the above conditions, $|f(x, b)| = \|f\| \|x, b\|$ if and only if $x \perp_b H$, where H is a 2-hyperplane of all h for which $f(h, b) = 0$.

Proof. Let H be the 2-hyperplane consisting of all elements h for which $f(h, b) = 0$. Also, let $|f(x, b)| = \|f\| \|x, b\|$. Since $f(h, b) = 0$, we have $f(\alpha h, b) = 0$ for each $\alpha \in \mathbb{R}$. So, we have

$$|f(x + \alpha h, b)| = |f(x, b) + f(\alpha h, b)| = |f(x, b)| = \|f\| \|x, b\|.$$

On the other hand,

$$f(x + \alpha h, b)| \leqslant \|f\| \|(x + \alpha h, b)\|, \quad \forall \alpha \in \mathbb{R}.$$

So, we have

$$\|x + b\| \leqslant \|x + \alpha h, b\|, \quad \forall h \in H, \forall \alpha \in \mathbb{R}.$$

That is $x \perp_b H$. Conversely, suppose $x \perp_b H$ and $|f(x,b)| = a\|x,b\|$. So

$$\|x,b\| \leqslant \|x+\alpha h, b\|, \quad \forall h \in H, \forall \alpha \in \mathbb{R}.$$

Hence, for each $h \in H$ and $\alpha \in \mathbb{R}$, we have

$$|f(x+\alpha h, b)| = |f(x,b)| = a\|x,b\| \leqslant a\|x+\alpha h, b\|.$$

Since H is a hyperplane through the origin, it follows that

$$|f(y,b)| \leqslant a\|y,b\|, \quad \forall y \in X.$$

That is $a = \|f\|$ and $|f(x,b)| = \|f\|\|x,b\|$. ∎

***Example* 3.2** Let $X = (E^3, \|,\|)$ be the 2-normed space with $\|x,y\| = |x \times y|$. Suppose $b = (1,0,0)$ and define $f : X \times < b > \to \mathbb{R}$ with $f(x,y) = |x \times y|$, where $x \in X$ and $y \in < b >$. So $\|f\| = 1$ so for each $x \in X$, we have $|f(x,b)| = \|f\|\|x,b\|$. On the other hand, the 2-hyperplane H through the origin is as follows:

$$H = \{x \in X; f(x,b) = 0\} = \{x \in X; |x \times b| = 0\} = \{x \in X; x = (a,0,0), \forall a \in \mathbb{R}\}.$$

Now, for each $\alpha \in \mathbb{R}$, $(x,y,z) \in X$ and $h = (a,0,0) \in H$, we have

$$\|x+\alpha h, b\| = \|(x+\alpha a, y, z), (1,0,0)\| = \sqrt{z^2+y^2} = \|x,b\|.$$

That means $x \perp_b H$.

Now, let X be a 2-normed linear space. For $X_0 \subseteq X$, put

$$M^b_{X_0} = \{f \in X^*_b; \|f\| = 1, f(x,b) = \|x,b\|, \forall x \in X_0\}.$$

One can find the proof of the following theorem in [15].

Theorem 3.3 Let X be a 2-normed linear space, $b \in X$, $y \in X$ and $x \in X \setminus \langle b \rangle$. Then $x \perp_b y$ if and only if there exists $f \in M^b_x$ such that $f(y,b) = 0$.

***Example* 3.4** Let $X = \mathbb{R}^3$, $W = \{(0,x,x), x \in \mathbb{R}\}$ and

$$\|(x_1,x_2,x_3),(y_1,y_2,y_3)\| = max\{|x_1y_2 - x_2y_1| + |x_1y_3 - x_3y_1|, |x_1y_2 - x_2y_1| + |x_2y_3 - x_3y_2|\}$$

for all $(x_1,x_2,x_3),(y_1,y_2,y_2) \in X$. Then $\|\cdot,\cdot\|$ is a 2-norm on X. If $x = (1,0,1)$ and $b = (2,2,0)$, it is clear that $x \perp_b W$.

In the following theorem we will show that there is an analogical relation between the existence of element orthogonal to given closed subsets and the existence of elements x with $|f(x,b)| = \|f\|\|x,b\|$ for given linear functionals f.

Theorem 3.5 Let X be a 2-normed linear space and $(0 \neq)b \in X$. Then there exist an element b-orthogonal to each closed 2-linear subset of X if and only if for each bilinear 2-functional f defined on $X \times \langle b \rangle$, there is an element x with $f(x,b) = \|f\|\|x,b\|$.

Proof. Let $\|f\| \neq 0$ and set $H = \{x \in X; f(x,b) = 0\}$. Then H is a closed linear subset of X. By Theorem 3.1, each element x orthogonal to this set is such that $|f(x,b)| = \|f\|\|x,b\|$.

Conversely, suppose H is any closed linear subset of X. Define the 2-functional F as follow:

$$F(h,b)=0, \qquad \forall h\in H,$$
$$F(x_0,b)=1, \quad for\, some\, x_0\notin H.$$

So $\|F\| = 1$ and F is additive over the space obtained adjoining x to H. Since H is closed, F is continuous. Now, by Theorem 5.1 in [18], there is a bilinear 2-functional f over $X\times\langle b\rangle$ such that $f(x,b)=F(x,b)$ for all (x,b) for which F is defined. Also $\|f\|=\|F\|=1$. If there is an element x for which $f(x,b)=\|f\|\|x,b\|$, then we have $x\perp_b H$ (by Theorem 3.3). ■

Using Theorem 3.1, the above theorem says that if X is a 2-normed linear space with $\dim X=3$ and $x_1,x_2\in X$, then there is an element $y\in X$ b-orthogonal to the $\langle x_1,x_2\rangle$, where $\langle x_1,x_2\rangle$ is the linear span of x_1 and x_2.

Corollary 3.6 Any element of a 2-normed linear space X is b-orthogonal to some hyperplane through the origin for $0\neq b\in X$.

4. Characterization of 2-Inner Product Spaces by b-Birkhoff Orthogonality

First we define the notion of bilinear 2-operator as follow:

Definition 4.1 Let $(X,\|.,.\|)$, $(Y,\|.,.\|)$ be two 2-normed spaces, and W_1 and W_2 be two subspaces of X. A map $T: W_1\times W_2\longrightarrow Y$ is called a bilinear 2-operator on $W_1\times W_2$ whenever for all $x_1,x_2\in W_1$ and $y_1,y_2\in W_2$ and all $\lambda_1,\lambda_2\in\mathbb{R}$,

i) $T(x_1+x_2,y_1+y_2)=T(x_1,y_1)+T(x_1,y_2)+T(x_2,y_1)+T(x_2,y_2)$,
ii) $T(\lambda_1x_1,\lambda_2y_2)=\lambda_1\lambda_2T(x_1,y_1)$.

Note that if $Y=\mathbb{R}$, then T is called a bilinear 2-functional. Also, a bilinear 2-operator T is called a 2-projection if $T^2=T$.

The authors in [14] showed that a 2-normed space X is 2-inner product if and only if for all $x,y,z\in X$,

$$\|x+y,z\|^2+\|x-y,z\|^2=2(\|x,z\|^2+\|y,z\|^2). \tag{2}$$

On the other hand, a quite elementary proof similar to the proof given in [8] show that the relation (2) holds if and only if there is a 2-projection of norm 1 on any given closed linear subspace of X.

Theorem 4.2 Let X be a 2-normed linear space and $(0\neq b)\in X$. For any $x,y\in X$, there exists a number a such that $ax+y\perp_b x$. This number a is a value of k for which $\|kx+y,b\|$ takes on its absolute minimum.

Proof. By Definition 2.1, $ax+y\perp_b x$ if and only if

$$\|(ax+y)+kx,b\|\geqslant\|ax+y,b\| \quad \forall k,$$

or if and only if $\|ax+y,b\|$ is the smallest value of $\|kx+y,b\|$. Since $\|kx+y,b\|$ is continuous in k, it must take on its minimum. ■

Now we can prove the following theorem.

Theorem 4.3 Let X be a 2-normed space and $0 \neq b \in X$. If $\dim X \geqslant 3$, then b-orthogonality is symmetric if and only if a 2-inner product can be defined in X.

Proof. Suppose that $\dim X_0 = 3$, where X_0 is a subspace of X. Also, let x_1 and x_2 be any two elements of $X_0 \setminus (\langle b \rangle)$ and H_0 be the linear hull of x_1 and x_2. By Theorem 3.1 and Theorem 3.5, there is an element $y \in X_0$ that is b-orthogonal to H_0. Conversely, suppose that b-orthogonality is symmetric. Then $H_0 \perp_b y$ and by Theorem 4.2, there is a number a_z such that we can define $P : X_0 \times \langle b \rangle \longrightarrow H_0 \times \langle b \rangle$ by $P(z, b) = (z - a_z y, b)$ for each $z \in X_0$. So P is a bilinear 2-operator. Also, since H_0 is the linear hull of x_1 and x_2 and $H_0 \perp_b y$, we have

$$\|P(z, b)\| = \|z - a_z y, b\| \leqslant \|z, b\| \qquad \forall z \in X_0.$$

Thus, $\|P\| = 1$. In addition, since $P(a_z y, b) = 0$ for each $z \in X_0$, we have

$$P^2(z, b) = P(P(z, b)) = P(z - a_z y, b) = P(z, b) - P(a_z y, b) = P(z, b).$$

Therefore, P is a 2-projection of $X_0 \times \langle b \rangle$ on $H_0 \times \langle b \rangle$ with $\|P\| = 1$. Now, according to the points stated before this theorem, a 2-inner product can be defined in a 2-normed linear space of three or more dimensions if there is a 2-projection of norm 1 on any given closed linear subspace. Thus a 2-inner product can be defined in any three-dimensional subspace of X and hence in X itself. ■

Corollary 4.4 Let x and y be in a 2-normed space X with $\dim X \geqslant 3$, and $0 \neq b \in X$. If there exists a nonzero bilinear 2-functional f with $f(x, b) = \|f\|\|x, y\|$ and $f(y, b) = 0$, then there exists a nonzero bilinear 2-functional g such that $g(y, b) = \|g\|\|y, b\|$ and $g(x, b) = 0$.

Proof. Combine Theorem 4.3 and Theorem 3.5. ■

Corollary 4.5 Let X be a 2-normed space and $0 \neq b \in X$, and $x, y \in X$. If f is a bilinear 2-functional such that $f(x, b) = \|f\|\|x, b\|$, then $\|ax + y, b\|$ is minimum when $a = -\frac{f(y,b)}{f(x,b)}$.

Proof. Combine Theorem 4.3 and Theorem 2.7 in [15]. ■

5. Operators reversing b-Birkhoff orthogonality in 2-normed linear spaces

Definition 5.1 Let X and Y be two 2-normed linear spaces and $0 \neq b \in X$. Also, let $T : X \longrightarrow Y$ be a nonzero linear operator. If

$$x \perp_b y \Rightarrow T(y) \perp_{T(b)} T(x)$$

for each $x, y \in X$, then we say that T reverses b-Birkhoff orthogonality.

Definition 5.2 Let X be a 2-normed space and $0 \neq b \in X$. The subset $S_X^b = \{x \in X; \|x, b\| = 1\}$ is called the 2-unit sphere of X.

Lemma 5.3 Let X and Y be two 2-normed linear spaces and $0 \neq b \in X$. If $T : X \longrightarrow Y$ is a non-zero linear operator reversing b-Birkhoff orthogonality, then T is injective.

Proof. Since T is non-zero, there exists $0 \neq z \in X$ such that $T(z) \neq 0$. Set $x = \frac{z}{\|z,b\|}$ (note that z and b are non-zero, thus, $\|z,b\| \neq 0$). Therefore,

$$\|x,b\| = \|\frac{z}{\|z,b\|}, b\| = \frac{1}{\|z,b\|}\|z,b\| = 1.$$

So, $x \in S_X^b$ and $T(x) \neq 0$.

Now, suppose that T is not injective. Thus there exists $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $T(x_1) = T(x_2)$. So $T(x_1) - T(x_2) = T(x_1 - x_2) = 0$. Since $x_1 \neq x_2$, then $\|x_1 - x_2, b\| \neq 0$. Set $y = \frac{x_1 - x_2}{\|x_1 - x_2, b\|}$. Then $\|y,b\| = 1$ and therefore $y \in S_X^b$ and $T(y) = 0$.

Now, set $L = \text{span}\{x,y\}$. Let $u \in S_L^b$ be a point satisfying $\|u - y, b\| = \frac{1}{2}$. Then u and y are linearly independent. Because, if there exists $0 \neq \alpha \in \mathbb{R}$ such that $u = \alpha y$, then $u = rx + sy = \alpha y$ for some $r, s \in \mathbb{R}$. Since $T(y) = 0$, we have $rT(x) = 0$. But $T(x) \neq 0$. Therefore $r = 0$ and $u = sy$. On the other hands, $\|y,b\| = 1$ implies that

$$|s-1| = |s-1|\|y,b\| = \|(s-1)y,b\| = \|sy - y, b\| = \|u-y,b\| = \frac{1}{2}.$$

Thus, $s = \frac{1}{2}$ or $s = \frac{3}{2}$. If $s = \frac{1}{2}$ then we have $1 = \|u,b\| = \|sy,b\| = \frac{1}{2}\|y,b\| = \frac{1}{2}$. That is a contradiction. Similarly $s = \frac{3}{2}$ leads to a contradiction. So u, y are linearly independent. Also, $u \not\perp_b y$, because for $\lambda = -1$ we have $1 = \|u,b\| > \|u-y,b\| = \frac{1}{2}$. Now, by Corollary 3.6 (also Theorem 2.7 in [15]), there is $v \in S_L^b$ such that $u \perp_b v$, that means $\|u + \alpha v, b\| \geqslant \|u,b\|$ for each $\alpha \in \mathbb{R}$. We claim that v and y are linearly independent. Because if for some $r, s \in \mathbb{R}$, $v = cy$, choosing $\alpha = -\frac{1}{c}$ we have

$$\|u + \alpha v, b\| = \|u - \frac{1}{c}(cy), b\| = \|u - y, b\| = \frac{1}{2} < \|u,b\| = 1,$$

which is a contradiction with $u \perp_b v$. So v and y are linearly independent and there exist two numbers α, β (not both zero) such that $y = \alpha u + \beta v$. It follows that $T(u)$ and $T(v)$ are non-zero and $T(v) \perp_{T(b)} T(u)$. Now, $T(u)$ and $T(v)$ are linearly independent. On the other hands, $0 = T(y) = \alpha T(u) + \beta T(v)$. That means $T(u)$ and $T(v)$ are dependent. It is a contradiction and therefore T is injective. ■

Theorem 5.4 Let X and Y be two 2-normed linear spaces whose dimensions are at least 3 for $0 \neq b \in X$. Then there exists a non-zero linear operator $T : X \longrightarrow Y$ reverses b-orthogonality if and only if $T(X) \setminus \langle T(b) \rangle$ is a 2-inner product space.

Proof. Let $T : X \longrightarrow Y$ be a non-zero linear operator and T reverses b-orthogonality $0 \neq b \in X$. Without loss of generality, we may assume that T is surjective. So, by Lemma 5.3, T is bijective. By Theorem 4.3, it is suffices to show that b-orthogonality is symmetric in Y.

Let $0 \neq y_0 \in Y$. We can suppose $y_0 \in S_Y^{T(b)}$. So $\|y_0, T(b)\| = 1$ and since T is injective, $T^{-1}(y_0) \neq 0$. By Corollary 3.6, there exists a closed 2-hyperplane H' through the origin such that $T^{-1}(y_0) \perp_b H'$. Since T reverses b-orthogonality, we have $T(H') \perp_b y_0$. Set $H = T(H')$. Since T is linear and bijective, then H is a 2-hyperplane in Y such that $H \perp_b y_0$.

On the other hands, similar to the proof of the Theorem 3.5, we can define a bilinear 2-functional f on Y such that $\|f\| = 1$ and $f(y_0, T(b)) = \|f\|\|y_0, T(b)\|$. Therefore, by the Theorem 3.3, $y_0 \perp_b H$. That means $Y = T(X)$ is symmetric. Conversely, If $T(X) \setminus \langle T(b) \rangle$

is a 2-inner product space, then the b-orthogonality relation is symmetric and the identity mapping satisfies desired property. ■

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