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Weak separation axioms via almost-ID-sets

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Abstract. The purpose of this paper is to introduce some new classes of almost ideal topological spaces by using the notion of almost-I-open sets and study some of their fundamental properties. We study some low separation axioms in almost ideal topological spaces.

Keywords: Ideal topological spaces, almost-I-open set, almost-ID-set.

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1. Introduction and preliminaries

Ideals in topological spaces have been considered since 1930 by Kuratowski [6, 7]. The topic has won its importance by the paper of Vaidyanathaswamy [14] in 1945. The ideal concept has been studied in different fileds [3, 4, 8–11, 13]. A non-empty collection of subsets of X with hereditary and finite additivity conditions is called an ideal or a dual filter on X. Namely a non-empty family $I \subseteq P(X)$, where $P(X)$ is the set of all subsets of X, is called an ideal which satisfies (i) $A \in I$ gives $P(A) \subseteq I$ and (ii) $A, B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X, a set operator $(.)^* : P(X) \to P(X)$, is called a local function [14] of A with respect to τ and I is defined as follows: for $A \subseteq X, A^*(I, \tau) = \{x \in X : U \cap A \notin I, for \quad every \quad U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. Kuratowski closure operator $Cl^*(.)$ for a topology

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$\tau^*(I,\tau)$ is called the $*$-topology finer than τ is defined by $Cl^*(A) = A \cup A^*(I,\tau)$ ([14]). Where there is no chance for confusion, we will simply write A^* for $A^*(I,\tau)$ and τ^* for $\tau^*(I,\tau)$. If I is an ideal on X, then the space (X,τ,I) is called an ideal topological space. By a space, we always mean a topological space (X,τ) with no separation properties assumed. If $A \subseteq X$, $Cl(A)$ and $Int(A)$ will denote the closure and interior of A in (X,τ), respectively. The almost-I-open (briefly AI-open) and almost-I-closed (briefly AI-closed) sets are presented by the first author and others in [2]. Utilizing these new concepts the class of almost-I-continuous functions have been obtained. Both of almost-I-openness and almost-I-continuity are considered as generalizations of those I-openness and I-continuity of Janković and Hamlett [5] and studied in [1, 12]. A subset S of an ideal topological space (X,τ,I) is said to be AI-open [2] if $S \subseteq Cl(Int(S^*))$, $X\backslash S$ is called almost-I-Closed. The collection of all AI-open sets of (X,τ) will be denoted by $AIO(X,\tau)$. Also, $AIO(X,x)$ denotes the class of all AI-open sets containing $x \in X$.

2. *AID*-sets and its separation axioms

Definition 2.1 A subset S of an ideal topological (X,τ,I) is called almost-ID-set (AID) if there exists $U,V \in AIO(X)$ such that $U \neq X$ and $A = U\backslash V$.

Observe that every AI-open set U different from X is an AID set with $S = U$ and $V = \emptyset$.

Definition 2.2 An ideal topological space (X,τ,I) is called AID_0 (resp. AIT_0) if for any distinct pair of points x and y of X, there exists an AID set of (X,τ,I) containing x but not y or AID set (resp. AI-open set) of (X,τ,I) containing y but not x.

Definition 2.3 An ideal topological space (X,τ,I) is called AID_1 (resp. AIT_1) if for any distinct pair of points x and y of X, there exists an AID set (resp. an AI-open set) of X containing x but not y and an AID set (resp. AI-open set) of X containing y but not x.

Definition 2.4 An ideal topological space (X,τ,I) is called AID_2 (resp. AIT_2) if for nay distinct pair of points x and y of X, there exists disjoint AID sets (resp. an AI-open set) of (X,τ,I) containing x and y, respectively.

Remark 1

(i) If (X,τ,I) is AIT_i, then it is AID_i, where $i = 0,1,2$.
(ii) If (X,τ,I) is AID_i, then it is AID_{i-1}, where $i = 1,2$.

***Example* 2.5** Let $X = \{a,b,c\}$, $\tau = \{\emptyset,\{a,c\},X\}$ and $I = \{\emptyset,\{a\}\}$. Then the ideal topological space (X,τ,I) is both AID_2 and AID_1 but none of AIT_2 and AIT_1.

Theorem 2.6 For an ideal topological space (X,τ,I), the following statements are true:

(1) (X,τ,I) is AID_0 if and only if it is AIT_0.
(2) (X,τ,I) is AID_1 if and only if it is AIT_2.

Proof.

(1) We prove only the necessary condition since the sufficiency is stated in Remark 1 (i).
Necessity. Let (X,τ,I) be AID_0. Then for each distinct pair of points $x,y \in X$, at least one of x,y say x, belongs to an AID set G where $y \notin G$. Let $G = U_1\backslash U_2$

such that $U_1 \neq X$ and $U_1, U_2 \in AIO(X)$. Then $x \in U_1$, and for $y \notin G$, we have two cases:
(a) $y \notin U_1$; (b) $y \in U_1$ and $y \in U_2$. In case (a), $x \in U_1$ but $y \notin U_1$; In case(b), $y \in U_2$ but $x \notin U_2$. Hence X is AIT_0.

(2) **Sufficiency**. Follows directly from Remark 1 (ii).
Necessity. Suppose (X, τ, I) is AID_1 space. Then for each distinct pair $x, y \in X$, we have AID sets G_1, G_2 such that $x \in G_1$, $y \notin G_1$; $y \in G_2$,$x \notin G_2$. Let $G_1 = U_1 \backslash U_2$, $G_2 = U_3 \backslash U_4$. From $x \notin G_2 = U_3 \backslash U_4$, we have either $x \notin U_3$ or ($x \in U_3$ and $x \in U_4$). Now we consider two cases:

(i) $x \notin U_3$. By $y \notin G_1$ we have two subcases:
(a) $y \notin U_1$. By $x \in U_1 \backslash U_2$, it follows that $x \in U_1 \backslash (U_2 \cup U_3)$ and by $y \in U_3 \backslash U_4$, we have $y \in U_3 \backslash (U_2 \cup U_4)$. Hence $(U_1 \backslash (U_2 \cup U_3)) \cap (U_3 \backslash (U_2 \cup U_4)) = \emptyset$.
(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 \backslash U_2, y \in U_2$ such that $(U_1 \backslash U_2) \cap U_2 = \emptyset$.

(ii) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \backslash U_4, x \in U_4$ such that $(U_3 \backslash U_4) \cap U_4 = \emptyset$. Therefore, X is AID_2.

■

Definition 2.7 A point $x \in X$ which has only X as the AI-neighborhood is called an AI-neat point.

Theorem 2.8 For an AIT_0 ideal topological space (X, τ, I) the following are equivalent:

(i) (X, τ, I) is AID_1.
(ii) (X, τ, I) has no AI-neat point.

Proof. $(i) \Rightarrow (ii)$: Since (X, τ, I) is AID_1, then each point of X is contained in an AID set $G = U \backslash V$ and thus in U. By definition $U \neq X$. This implies that x is not an AI-neat point.
$(ii) \Rightarrow (i)$: If X is AIT_0, then for each distinct pair of points $x, y \in X$, at least one of them, x (say) has an AI-neighborhood U containing x and not y. Thus U which is different from X is an AID set. If X has no an AI-neat point then y is not an AI-neat point. This means that there exists an AI-neighborhood V of y such that $V \neq X$. Thus $y \in V \backslash U$ but not x and $V \backslash U$ is AID. Hence (X, τ, I) is AID_1. ■

Definition 2.9 [2] A function $f : (X, \tau, I) \longrightarrow (Y, \sigma, J)$ is said to be AI-irresolute if $f^{-1}(V) \in AIO(X)$, for every $V \in AIO(Y)$.

Theorem 2.10 If $f : (X, \tau, I) \longrightarrow (Y, \sigma, J)$ is an AI-irresolute surjective function and E is an AID set in (Y, σ, J), then the inverse image of E is an AID set in (X, τ, I).

Proof. Let E be an AID set in (Y, σ, J). Then, there are AI-open sets U_1 and U_2 in (Y, σ, J) such that $E = U_1 \backslash U_2$ and $U \neq Y$. By the AI-irresoluteness of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are AI-open in (X, τ, I). Since $U_1 \neq Y$, we have $f^{-1}U_1 \neq X$. Hence $f^{-1}(E) = f^{-1}U_1 \backslash f^{-1}U_2$ is an AID set. ■

Theorem 2.11 If (Y, σ, J) is AID_1 and $f : (X, \tau, I) \longrightarrow (Y, \sigma, J)$ is AI-irresolute and bijective, then (X, τ, I) is AID_1.

Proof. Suppose that Y is an AID_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is AID_1, there exist AID sets G_x and G_y of Y containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin G_x$ and $f(x) \notin G_y$. By Theorem 2.10, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are AID sets in (X, τ, I) containing x and y, respectively. This implies that (X, τ, I) is an AID_1 space. ■

Theorem 2.12 An ideal topological space (X, τ, I) is AID_1 if and only if for each pair of distinct points $x, y \in X$, there exists an AI-irresolute surjective function $f : (X, \tau, I) \longrightarrow (Y, \sigma, J)$, where (Y, σ, J) is an AID_1 space such that $f(x)$ and $f(y)$ are distinct.

Proof. Necessity. For every pair of distinct points of X, it suffices to take the identity function on X.
Sufficiency. Let x and y be any pair of distinct points of X. By hypothesis, there exists an AI-irresolute, surjective function f from an ideal topological space (X, τ, I) onto an AID_1 space (Y, σ, J) such that $f(x) \neq f(y)$. Therefore, there exist disjoint AID sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is AI-irresolute and surjective, by Theorem 2.10, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint AID sets in X containing x and y, respectively. Hence the space (X, τ, I) is an AID_1 space. ■

Problem 1. Find an AID_0 space which is not AIT_0.
Problem 2. Find an ideal topological space AID_{i-1} which is not AID_i, where $i = 1, 2$.

3. Conclusion and further works

We hope this paper is just a beginning of a new structure. It will inspire many to contribute to the cultivation of ideal topology in the filed of mathematical structure of approximations. As further works, using the results in this paper, we will further study the following research areas: extended almost-I-open sets structures, ideals topological data analysis, geographical model, big data analysis and statistics analysis.

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