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Topics on a class of pseudo-Michael algebras

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Abstract. In this paper, we first generalize the Gelfand-Mazur theorem for pseudo-Michael Q-algebras. Then some applications of the spectral mapping theorem are also investigated in k-Banach algebras.

Keywords: Pseudo-Michael algebra, k-Banach algebra, idempotent, spectrum, uniformly continuous.

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1. Introduction and preliminaries

Non-normed topological algebras were initially introduced around the year 1950 for the investigation of certain classes of these algebras that appeared naturally in mathematics and physics. Some results concerning such topological algebras had been obtained earlier in 1950. It was in 1952 that Arens and Michael [4, 6] independently published the first systematic study on locally m-convex algebras, which constitutes an important class of non-normed topological algebras. Here, we would like to mention about the predictions made by the famous Soviet mathematician Naimark, an expert in the area of Banach algebras, in 1950 regarding the importance of non-normed algebras and the development of their related theory. During his study concerning cosmology, Lassner [4] realized that the theory of normed topological algebras was insufficient for his study purposes.

Pseudo-Michael algebras, in particular, k-Banach algebras are an important class of non-normed topological algebras and play a crucial role in functional analysis.

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In this paper, we first generalize the Gelfand-Mazur theorem for pseudo-Michael Q-algebras. Then some applications of the spectral mapping theorem are also investigated in k-Banach algebras. Throughout this paper, all algebras will be assumed unital and the units will be denoted by e.

2. Definitions and known results

In this section, we present a collection of definitions and known results, which are included in the list of our references.

Definition 2.1 Let A be an algebra. The set of all invertible elements of A is denoted by $Inv(A)$, and the complement of $Inv(A)$ by $(Inv(A))^c$.

The following Lemma follows immediately from Exercise D.1.3 of [8].

Lemma 2.2 Let A be an algebra and $x, y \in A$. If at least two points of the set $\{x, y, xy, yx\}$ belong to $Inv(A)$, then $\{x, y, xy, yx\} \subseteq Inv(A)$.

Definition 2.3 For an algebra A, the spectrum $sp_A(x)$ of an element $x \in A$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda e - x$ is not invertible in A. The spectral radius $r_A(x)$ of an element $x \in A$ is defined by $r_A(x) = \sup\{|\lambda| : \lambda \in sp_A(x)\}$.

Definition 2.4 [1] Let A be an algebra. An element $x \in A$ is said to be nilpotent if $x^m = 0$ for some natural number $m \geqslant 1$. It is said to be quasinilpotent if $sp_A(x) = \{0\}$.

Lemma 2.5 [3] If A is an algebra, then

$$Rad(A) = \{x \in A : r_A(xy) = 0; \text{for any } y \in A\},$$

where $Rad(A)$ is the Jacobson radical of A.

Definition 2.6 By a topological algebra we mean an algebra over $\mathbb{C}$ endowed with a topology that makes the multiplication separately continuous.

Definition 2.7 A topological algebra A is said to be a Q-algebra if and only if $Inv(A)$ is open.

Corollary 2.8 [5] If A is a Q-algebra, then $sp_A(x)$ is compact for each $x \in A$.

Definition 2.9 [2] A k-seminorm on A with $k \in (0, 1]$ is a function $p : A \to \mathbb{R}^+ \cup \{0\}$ such that, for each $x, y \in A$ and $\lambda \in \mathbb{C}$,

$$p(x + y) \leqslant p(x) + p(y), \tag{1}$$

$$p(\lambda x) \leqslant |\lambda|^k p(x). \tag{2}$$

If, in addition, the function satisfies

$$p(xy) \leqslant p(x)p(y), \tag{3}$$

then the k-seminorm is called submultiplicative.

A k-seminorm p is also called a pseudo-seminorm and k is called the homogenity index of p. A pseudo-seminorm p is a pseudo-norm if $p(x) = 0$ implies $x = 0$.

If p is a k-seminorm (k-norm) on a linear space A, then the resulting topological linear space $A = (A, p)$ is called a k-seminormed (k-normed) linear space. A topological algebra whose topology is induced by a k-seminorm (k-norm) p is called a k-seminormed (k-normed) algebra. We generally denote a k-norm p by the symbole $\|\cdot\|_k$. A complete k-normed algebra is called a k-Bunach algebra.

Definition 2.10 [2] A locally pseudo-convex space A is a topological linear space equipped with a family $\mathcal{P} = (p_\alpha)_{\alpha \in I}$ of pseudo-seminorms on A which define its topology. If each $p_\alpha \in \mathcal{P}$ is a k-seminorm, then A is called a locally k-convex space.

A locally pseudo-convex algebra A is a topological algebra such that its underlying topological linear space is locally pseudo-convex. If its underlying topological linear space is locally k-convex, then A is called a locally k-convex algebra. A is called a locally m-pseudo-convex algebra (or locally m-(k-convex) algebra) if p_α is submultiplicative for each $\alpha \in I$.

Definition 2.11 [2] We call a complete Hausdorff locally m-pseudo-convex algebra A as a pseudo-Michael algebra.

Corollary 2.12 [2] Every pseudo-Michael algebra A is spectral, i.e. $sp_A(x) \neq \emptyset$ for each $x \in A$.

Definition 2.13 [1] The Hausdorff distance is defined by

$$\Delta(K_1, K_2) = \max\left(\sup_{z \in K_2} \operatorname{dist}(z, K_1), \sup_{z \in K_1} \operatorname{dist}(z, K_2)\right)$$

for K_1, K_2 compact subsets of $\mathbb{C}$. Let $r > 0$ and K be a compact subset of $\mathbb{C}$. If $K + r$ denotes $\{z : \operatorname{dist}(z, K) \leqslant r\}$, then obviously $K_1 \subseteq K_2 + \Delta(K_1, K_2)$ and $K_2 \subseteq K_1 + \Delta(K_1, K_2)$.

As in [1, p. 48], we have the following definition.

Definition 2.14 Let A be a k-Banach algebra and $E \subseteq A$. The function $x \longmapsto sp_A(x)$ is said to be continuous at $a \in A$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|x - a\|_k < \delta$ implies $\Delta(sp_A(x), sp_A(a)) < \epsilon$. As usual we say that $x \longmapsto sp_A(x)$ is continuous on E if it is continuous at every point of E. If for a given $\epsilon > 0$, the number $\delta > 0$ is independent of a on E, we say that $x \to sp_A(x)$ is uniformly continuous on E.

3. Generalization of the Gelfand-Mazur theorem

In this section, we generalize some results, in particular, the Gelfand-Mazur theorem for pseudo-Michael Q-algebras.

Lemma 3.1 Let A be a pseudo-Michael Q-algebra. Then $\mathrm{Rad}(A) = (\mathrm{Inv}(A))^c$ if and only if $sp_A(x) = \{0\}$ for all $x \in \partial(\mathrm{Inv}(A))$ the boundary of $\mathrm{Inv}(A)$ in A.

Proof. Assume that $\mathrm{Rad}(A) = (\mathrm{Inv}(A))^c$. Let $x \in \partial(\mathrm{Inv}(A))$. Since (A) is open in A, $x \in ((A))^c$. Hence $x \in (A)$ and so $r_A(x) = 0$ by Lemma 2.5. Therefore $sp_A(x) = 0$ because $sp_A(x)$ is a non-empty set in $\mathbb{C}$ by Corollary 2.12.

Assume that $sp_A(x) = \{0\}$ for each $x \in \partial((A))$. Let $x \in (A)$. Then $r_A(x) = 0$ by Lemma 2.5. We show that $x \in ((A))^c$. Suppose that $x \in (A)$. Then $xy = e = yx$ for some $y \in A$, and hence we have

$$1 = r_A(e) = r_A(xy) \leqslant r_A(x) r_A(y) = 0,$$

which is impossible. Therefore, $(A) \subseteq ((A))^c$. Now, we claim that $((A))^c \subseteq (A)$. Let $z \in ((A))^c$. Since $sp_A(z)$ is non-empty and compact [5, Proposition 4.2], we can choose $\lambda_0 \in sp_A(z)$ such that $\lambda_0 \in \partial(sp_A(z))$. Then $z - \lambda_0 e \in \partial((A))$ by continuity. First we prove that $sp_A(z) = \{0\}$. Let $\lambda \in sp_A(z)$, then

$$z - \lambda e = (z - \lambda_0 e) - (\lambda - \lambda_0)e.$$

Hence, $\lambda - \lambda_0 \in sp_A(z - \lambda_0 e) = \{0\}$. Thus, $\lambda = \lambda_0$. Since $z \in ((A))^c$, $\lambda_0 = 0$. So $sp_A(z) = \{0\}$ for all $z \in (A)$. Let $y \in A$. We claim that $zy \in ((A))^c$ or $yz \in ((A))^c$. Assume otherewise. Then, by Lemma 2.2, we have $\{zy, yz\}, \{z, y, zy, yz\} \subseteq (A)$, which is impossible, because $z \in ((A))^c$. Hence, $zy \in ((A))^c$ or $yz \in ((A))^c$. By the above argument, we get $r_A(zy) = 0$ or $r_A(yz) = 0$. Since $r_A(zy) = r_A(yz)$ for all $y, z \in A$ [2, Lemma 1.8.12], we have $r_A(zy) = r_A(yz) = 0$. Since y is arbitrary in A, by Lemma 2.5, $z \in (A)$. Therefore $((A))^c \subseteq (A)$. Hence $(A) = ((A))^c$. ■

Theorem 3.2 Let A be a semisimple commutative pseudo-Michael Q-algebra. If A has only the trivial idempotents and $sp_A(x)$ is countable for each $x \in \partial((A))$, then $A = \mathbb{C}e$.

Proof. Let $x \in \partial((A))$. Then $sp_A(x)$ is a connected subset of $\mathbb{C}$ by Corollary 8.6.16 of [2]. Since $sp_A(x)$ is a connected separable metric space, it has only one point or an uncountable number of points of $\mathbb{C}$. Our hypothesis implies that $sp_A(x)$ has only one point. Since $x \in ((A))^c$, $sp_A(x) = \{0\}$. It follows that $(A) = ((A))^c$ by Lemma 3.1. Since A is semisimple, $((A))^c = \{0\}$. By [2, Lemma 6.5.1], we have $A = \mathbb{C}e$. ■

Corollary 3.3 If A is a commutative pseudo-Michael Q-algebra with $((A))^c = \{0\}$, then $A = \mathbb{C}e$.

Proof. Since $\{0\} \subseteq (A) \subseteq ((A))^c$, so $(A) = \{0\}$. Therefore A is semisimple. Let x be an idempotent element in A. Then $x(x - e) = 0$. Hence $x \in ((A))^c$ or $x - e \in ((A))^c$. So $x = 0$ or $x = e$. Let $x \in \partial((A))$. Then $x \in ((A))^c$, hence $x = 0$ and so $sp_A(x) = \{0\}$; thus $sp_A(x)$ is countable for each $x \in \partial((A))$. Hence, by Theorem 3.2, the result follows. ■

Corollary 3.4 If A is a semisimple commutative pseudo-Michael Q-algebra such that $sp_A(x) = \{0\}$ for each $x \in \partial((A))$, then $A = \mathbb{C}e$.

Proof. By Lemma 3.1, it follows. ■

Remark 1 *Every k-Banach algebra is a pseudo-Michael algebra [1]. Thus all the above theorems and results which are true for pseudo-Michael algebras, also hold for k-Banach algebras.*

4. Some applications of the spectral mapping theorem

In this section, we investigate some applications of the spectral mapping theorem [2, Theorem 7.5.13] in k-Banach algebras.

Theorem 4.1 Let A be a k-Banach algebra. If $x \in A$ is nilpotent, then x is quasinilpotent. The converse holds true if A is finite dimensional.

Proof. Suppose $x \in A$ is nilpotent with $x^n = 0$ for some $n \geqslant 1$. If $\lambda \in sp_A(x)$, then $\lambda^n \in sp_A(x^n) = sp_A(0) = \{0\}$ by the spectral mapping theorem, so that $\lambda = 0$. This shows that every nilpotent element is quasinilpotent.

Conversely, suppose that $x \in A$ is quasinilpotent. Since A is finite-dimensional, the powers of x must be linearly dependent; that is, x is algebraic over $\mathbb{C}$. Let $p(t)$ be the minimal polynomial of x, which is the unique monic polynomial of lowest degree such that $p(x) = 0$. By the spectral mapping theorem, the roots of $p(t)$ belong to $sp_A(x) = \{0\}$. But then $p(t) = t^n$ for some $n \geqslant 1$, which means that $x^n = 0$. ∎

Remark 2 *The converse of Therem 4.1, is false in general even in Banach algebras (see for instance, [9, Example 4.2]).*

Theorem 4.2 Let A be a k-Banach algebra. If a is a non-trivial idempotent and $\lambda \notin sp_A(a)$, then

$$(\lambda, sp_A(a)) = \frac{1}{r((\lambda e - a)^{-1})}$$

Proof. Suppose that a is a non-trivial idempotent. In this case $sp_A(a) = \{0, 1\}$ by the spectral mapping theorem. Since $\lambda \neq 0, 1$, so $(\lambda e - a)^{-1}$ exists. Now, we claim that

$$(\lambda e - a)^{-1} = \frac{1}{\lambda - 1} a + \frac{1}{\lambda}(e - a). \tag{4}$$

First we observe that

$$\begin{aligned}
\left(\frac{1}{\lambda - 1} a + \frac{1}{\lambda}(e - a)\right)(\lambda e - a) &= \frac{1}{\lambda - 1} a(\lambda e - a) + \frac{1}{\lambda}(e - a)(\lambda e - a) \\
&= \frac{1}{\lambda - 1}(\lambda a - a) + \frac{1}{\lambda}(\lambda e - a - \lambda a + a) \\
&= \frac{1}{\lambda - 1}(\lambda - 1)a + \frac{1}{\lambda}\lambda(e - a) \\
&= a + (e - a) \\
&= e.
\end{aligned}$$

similarly,

$$(\lambda e - a)\left(\frac{1}{\lambda - 1} a + \frac{1}{\lambda}(e - a)\right) = e.$$

Therefore, (4) follows.

Let $p(\mu) = \frac{1}{\lambda - 1}\mu + \frac{1}{\lambda}(1 - \mu)$ for every $\mu \in \mathbb{C}$. Then $p(x) = \frac{1}{\lambda - 1} x + \frac{1}{\lambda}(e - x)$ for every $x \in A$. If follows from the spectral mapping theorem that

$$sp_A((\lambda e - a)^{-1}) = sp_A\left(\frac{1}{\lambda - 1} a + \frac{1}{\lambda}(e - a)\right) = sp_A(p(a)) = p(sp_A(a)) = \left\{\frac{1}{\lambda}, \frac{1}{\lambda - 1}\right\}.$$

Hence,

$$r_A((\lambda e - a)^{-1}) = \sup\left\{\frac{1}{|\lambda|}, \frac{1}{|\lambda - 1|}\right\} = \frac{1}{(\lambda, sp_A(a))}.$$

∎

Now, we prove the following theorem without using the idempotent elements.

Theorem 4.3 Let A be a k-Banach algebra. Suppose that $x \in A$ and that $\lambda \notin sp_A(x)$. Then

$$(\lambda, sp_A(x)) = \frac{1}{r((\lambda e - x)^{-1})}.$$

Proof. Let Ω be an open set containing $sp_A(x)$, but not λ. Clearly the function $f(\alpha) = \dfrac{1}{\lambda - \alpha}$ is holomorphic on Ω. By the spectral mapping theorem we have

$$sp_A((\lambda e - x)^{-1}) = \left\{ \frac{1}{\lambda - \alpha} : \alpha \in sp_A(x) \right\}.$$

In particular,

$$r_A((\lambda e - x)^{-1}) = \sup \left\{ \frac{1}{|\lambda - \alpha|} : \alpha \in sp_A(x) \right\}.$$

Thus by the properties of the supremum and infimum

$$r_A((\lambda e - x)^{-1}) = \frac{1}{\inf\{|\lambda - \alpha| : \alpha \in sp_A(x)\}} = \frac{1}{(\lambda, sp_A(x))}.$$

■

Lemma 4.4 Let A be a k-Banach algebra. Suppose that $x, y \in A$ commute. Then

$$sp_A(y) \subseteq sp_A(x) + r_A(x - y).$$

Proof. Suppose that the inclusion is false. Then there exists $\lambda \in sp_A(y)$ such that $\lambda \notin sp_A(x) + r_A(x - y)$. Thus, by Definition 2.13, $r_A(x - y) < (\lambda, sp_A(x))$. This implies that $\lambda \notin sp_A(x)$. Hence, by Theorem 4.3,

$$(\lambda, sp_A(x)) = \frac{1}{r_A((\lambda e - x)^{-1})}.$$

So, $r_A((\lambda e - x)^{-1}) r_A(x - y) < 1$. Since $(\lambda e - x)^{-1}$ and $x - y$ commute, it follows from [2, Corollary 7.2.23] that $r_A((\lambda e - x)^{-1}(x - y)) < 1$. Hence, by [2, Corollary 3.3.20], $(e + (\lambda e - x)^{-1}(x - y))$ is invertible. But $\lambda e - y = \lambda e - x + x - y$. Since $\lambda e - x \in (A)$, it follows that

$$\lambda e - y = (\lambda e - x)(e + (\lambda e - x)^{-1}(x - y))$$

is also invertible. Clearly this is a contradiction and the result follows. ■

Theorem 4.5 Let A be a commutative k-Banach algebra. Then the spectrum function $x \longmapsto sp_A(x)$ is uniformly continuous on A.

Proof. Let $x, y \in A$. By Definition 2.13, we have

$$sp_A(x) + r_A(x - y) = \{z : (z, sp_A(x) \leqslant r_A(x - y)\}.$$

It follows from Lemma 4.4 that

$$\sup_{z \in sp_A(y)} (z, sp_A(x)) \leqslant r_A(x-y).$$

Similarly,

$$\sup_{z \in sp_A(x)} (z, sp_A(y)) \leqslant r_A(x-y).$$

Hence,

$$\max\left(\sup_{z \in sp_A(y)} (z, sp_A(x)), \sup_{z \in sp_A(x)} (z, sp_A(y))\right) \leqslant r_A(x-y).$$

Consequently, $\Delta(sp_A(x), sp_A(y)) \leqslant r_A(x-y)$. On the other hand, since $\|\cdot\|_k$ is submultiplicative, it follows from [7, Lemma 3.9] that $r_A(x-y) \leqslant \|x-y\|_k^{\frac{1}{k}}$. Thus,

$$\Delta(sp_A(x), sp_A(y)) \leqslant \|x-y\|_k^{\frac{1}{k}}.$$

This implies that the spectrum function is uniformly continuous. ■

Remark 3 *In Theorem 4.5, the commutativity of A is essential even in Banach algebra (see the example in [1, p. 48]).*

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