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Proximity spaces via hereditary classes

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Abstract. A hereditary class on a set X is a nonempty collection of subsets of X which is closed under subsets. In this paper, we present a new structure of proximity spaces by using a hereditary class, called $\mathcal{H}$-proximity spaces, as a generalization of Efremovič proximity spaces, I-proximity spaces and coarse proximity spaces. Some properties of this proximity structure and generalized topology induced by it are studied.

Keywords: $\mathcal{H}$-proximity space, generalized topology, hereditary class.

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1. Introduction and Preliminaries

The concept of proximity or nearness in topology was described by Riesz in 1908 but ignored at the time [15]. It was rediscovered by Efremovič in 1934, but not published until 1951 [4]. He axiomatically characterized the proximity relation "A is near B" for subsets A, B of any set X. The set X together with this relation was called an infinitesimal (proximity) space. Proximity space is a natural generalization of a metric space and of a topological group. Every proximity δ on a set X induces a topology τ_δ on X by defining the closure of a subset A to be the set $\{x \mid \{x\}\delta A\}$. Conversely, Efremovič showed that if (X, τ) is any completely regular space, then there exists a proximity δ on X such that $\tau = \tau_\delta$. In fact, the proximity δ is defined by $A \not\delta B$ if and only if A and B are functionally distinguishable, i.e., there exists a continuous map $f : X \to [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Some authors have worked with weaker axioms than those of Efremovič and some types of proximity structures were introduced, such as quasi-proximity [14], paraproximity [7],

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pseudo-proximity [6], local proximity [11], I-proximity [10], coarse proximity [5] and μ-proximity [12]. In this paper, we present a new structure of proximity spaces by using a hereditary class, called $\mathcal{H}$-proximity spaces, as a generalization of Efremovič proximity spaces, I-proximity spaces and coarse proximity spaces. The relationships between $\mathcal{H}$-proximity spaces and generalized topological spaces induced by them in the sense of Császár are investigated.

We first recall some basic results and definitions of proximity structures.

Definition 1.1 [13] Let X be a set and $P(X)$ be the power set of X. A (Efremovič) proximity on a set X is a relation δ on $P(X)$ satisfying the following axioms for all $A, B, C \in P(X)$:

(1) $A\delta B$ implies $B\delta A$,
(2) $A\delta B$ implies $A \neq \emptyset$ and $B \neq \emptyset$,
(3) $A \cap B \neq \emptyset$ implies $A\delta B$,
(4) $(A \cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$,
(5) $A \not\delta B$ implies that there exists a subset E such that $A \not\delta E$ and $(X - E) \not\delta B$;

where $A \not\delta B$ means $A\delta B$ is not true. If $A\delta B$, then we say that A is close to (or near) B. Axiom 4 is called the union axiom and axiom 5 is called the strong axiom. A pair (X, δ), where X is a set and δ is a proximity on X, is called a proximity space.

Definition 1.2 [1, 5] A bornology $\mathcal{B}$ on a nonempty set X is a family of subsets of X satisfying:

(1) $\{x\} \in \mathcal{B}$ for all $x \in X$,
(2) $A \subseteq B$ and $B \in \mathcal{B}$ implies $A \in \mathcal{B}$ (i.e., it is closed under taking subsets),
(3) if $A, B \in \mathcal{B}$, then $A \cup B \in \mathcal{B}$ (i.e., it is closed under taking finite unions).

Definition 1.3 [3, 9] A nonempty collection $\mathcal{H}$ of subsets of a set X is called an ideal if it is closed under taking subsets and finite unions; and it is called a hereditary class if it is closed under taking subsets only.

Remark 1 *It is clear that every bornology is an ideal and every ideal is a hereditary class. Ideal is a fundamental concept in topological spaces and plays an important role in the study of topological spaces [9]. Similarly, hereditary classes are important in the study of generalized topological spaces [3]. Bornologies play an important role in the theory of locally convex spaces [8], boundedness in metric spaces [1] and coarse geometry [5].*

***Example* 1.4** The following families are bornologies on a nonempty set X:

(1) the finite subsets of X,
(2) the countable subsets of X,
(3) the power set $P(X)$,
(4) the bounded subsets of a metric space X,
(5) the totally bounded subsets of a metric space X,
(6) the subsets of a metric space X with compact closure.

***Example* 1.5** Let X be a nonempty set and $A \subsetneq X$. Then the collections $\mathcal{H}_1 = \{\emptyset\}$ and $\mathcal{H}_2 = \{B \subseteq X \mid B \subseteq A\}$ are ideals but not bornologies. Indeed, $\emptyset \in \mathcal{H}$ for any hereditary class $\mathcal{H}$, so $\mathcal{H}_1$ is the smallest hereditary class on X. Also, the collection $\mathcal{H}_p = \{\emptyset\} \bigcup \{\{x\} \mid x \in X\}$ is a hereditary class (called hereditary class of points) but not an ideal if X has at least two elements.

In the following, we recall some notions and notations defined in [2, 3]. Let X be a set. A subset μ of $P(X)$ is called a generalized topology (briefly GT) on X and the pair

(X, μ) is called a generalized topological space (briefly GTS) if $\emptyset \in \mu$ and any union of elements of μ belongs to μ. A GTS (X, μ) is called strong if $X \in \mu$. A set $A \subseteq X$ is said to be μ-open if $A \in \mu$ and μ-closed if $X - A \in \mu$. A mapping $f : (X, \mu_X) \to (Y, \mu_Y)$ between GTS's is said to be μ-continuous if $f^{-1}(B) \in \mu_X$ whenever $B \in \mu_Y$.

A mapping $\lambda : P(X) \to P(X)$ is said to be monotone provided that $A \subseteq B \subseteq X$ implies $\lambda A \subseteq \lambda B$, where we write λA for $\lambda(A)$.

A monotone map $\lambda : P(X) \to P(X)$ is said to be:

(1) idempotent if $\lambda^2 A = \lambda\lambda A = \lambda A$ for all $A \subseteq X$,
(2) restricting if $\lambda A \subseteq A$ for all $A \subseteq X$,
(3) enlarging if $A \subseteq \lambda A$ for all $A \subseteq X$,
(4) $\vee$-additive if $\lambda(A \cup B) = \lambda A \cup \lambda B$ for all $A, B \subseteq X$.

Remark 2 *[2] If μ is a GT on X, then the interior operator $i_\mu : P(X) \to P(X)$ defined by $i_\mu A := \bigcup\{M \in \mu \mid M \subseteq A\}$ is monotone, idempotent and restricting; and the closure operator $c_\mu : P(X) \to P(X)$ defined by $c_\mu A := \bigcap\{N \mid A \subseteq N, X - N \in \mu\}$ is monotone, idempotent and enlarging. Moreover, i_μ and c_μ are conjugate, i.e., $c_\mu A = X - (i_\mu(X - A))$ for all $A \subseteq X$. Conversely, if $\lambda : P(X) \to P(X)$ is enlarging, monotone and idempotent, then the collection $\mu := \{A \mid \lambda(X - A) = X - A\}$ is a GT on X such that $c_\mu A = \lambda A$ for all $A \subseteq X$.*

Definition 1.6 [3] Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$ and $A \subseteq X$. Then

$$A^* := \{x \in X \mid O_x \cap A \notin \mathcal{H} \quad \text{for every } \mu\text{-open set } O_x \text{ containing } x\}$$

is called the local function of A with respect to $\mathcal{H}$ and μ.

Theorem 1.7 [3] Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$. Then the operator $c^* : P(X) \to P(X)$ defined by $c^*(A) = A \cup A^*$ is monotone, idempotent and enlarging. Hence the collection $\mu^* := \{A \subseteq X \mid c^*(X - A) = X - A\}$ is a GT on X, called the GT induced by $(\mu, \mathcal{H})$.

Theorem 1.8 [3] Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$. Then the following statements hold:

(1) $\mu \subseteq \mu^*$.
(2) If $\mathcal{H} = \{\emptyset\}$, then $c^*(A) = c_\mu(A) = A^*$ and $\mu = \mu^*$.
(3) The collection $\{M - H \mid M \in \mu, H \in \mathcal{H}\}$ is a base for μ^*.
(4) If $M_\mu := \bigcup\{M \mid M \in \mu\}$, then $H^* = X - M_\mu$ for any $H \in \mathcal{H}$.
(5) If $X \in \mu$ and $H \in \mathcal{H}$, then $H^* = \emptyset$. Hence H is μ^*-closed.

2. $\mathcal{H}$-Proximity Spaces

In this section, we introduce a new approach of proximity structure based on a hereditary class. Some results on these spaces and generalized topological spaces induced by them are investigated.

Definition 2.1 Let $\mathcal{H}$ be a hereditary class on a nonempty set X. A binary relation θ on $P(X)$ is called an $\mathcal{H}$-proximity on X if it satisfies the following conditions for all $A, B, C \in P(X)$:

$(\mathcal{A}_1)$ $A\theta B$ implies $B\theta A$,
$(\mathcal{A}_2)$ $A\theta B$ implies $A \notin \mathcal{H}$ and $B \notin \mathcal{H}$,

($\mathcal{A}_3$) $A \cap B \notin \mathcal{H}$ implies $A\theta B$,
($\mathcal{A}_4$) $A\theta B$ or $A\theta C$ implies $A\theta(B \cup C)$,
($\mathcal{A}_5$) $A \not\theta B$ implies that there exist subsets C and D such that $A \not\theta (X - C)$, $(X - D) \not\theta B$ and $C \cap D \in \mathcal{H}$.

A triple $(X, \mathcal{H}, \theta)$, where X is a set, $\mathcal{H}$ is a hereditary class on X and θ is an $\mathcal{H}$-proximity on X, is called an $\mathcal{H}$-proximity space. Axiom ($\mathcal{A}_4$) is called the weak union axiom and ($\mathcal{A}_5$) is called the $\mathcal{H}$-strong axiom.

Lemma 2.2 Let $\mathcal{H}$ be a hereditary class on a nonempty set X and θ a binary relation on $P(X)$ satisfying the axioms ($\mathcal{A}_1$)–($\mathcal{A}_3$). Then the following statements hold.

(1) The strong axiom implies the $\mathcal{H}$-strong axiom.
(2) If θ also satisfies the union axiom, then the axioms strong and $\mathcal{H}$-strong are equivalent.

Proof. To prove (1), let $A \not\theta B$. Then there exists a subset D such that $A \not\theta D$ and $(X - D) \not\theta B$. Since $D \not\theta A$, there exists a subset C such that $D \not\theta C$ and $(X - C) \not\theta A$. Thus by ($\mathcal{A}_3$), we have $C \cap D \in \mathcal{H}$. To prove (2), let the $\mathcal{H}$-strong axiom holds and $A \not\theta B$. Then there exist subsets C and D such that $A \not\theta (X - C)$, $(X - D) \not\theta B$ and $C \cap D \in \mathcal{H}$. Set $E = X - C$ and $H = C \cap D$, we have $A \not\theta E$ and $H \in \mathcal{H}$. Now we show that $C \not\theta B$. For contradiction assume that $C\theta B$. Since $C \subseteq (X - D) \cup H$, by the union axiom we have $((X - D) \cup H)\theta B$. Again by the union axiom, $(X - D)\theta B$ or $H\theta B$, which is a contradiction to $(X - D) \not\theta B$ or $H \in \mathcal{H}$, respectively. Thus the result holds. ■

Remark 3 *If $\mathcal{H}$ is a bornology (an ideal) on a nonempty set X and θ a binary relation on $P(X)$ such that satisfies the axioms ($\mathcal{A}_1$)–($\mathcal{A}_3$) and also the axioms union and strong, then the triple $(X, \mathcal{H}, \theta)$ is called a coarse proximity space [5] (an I-proximity space [10]). Thus by Lemma 2.2, every coarse proximity space is an I-proximity space and every I-proximity space is an $\mathcal{H}$-proximity space. Also, every proximity space is an $\mathcal{H}$-proximity space, where $\mathcal{H} = \{\emptyset\}$.*

***Example* 2.3** Let $\mathcal{H}$ be a hereditary class on a nonempty set X. For any subsets A and B of X, define

$$A\theta B \Longleftrightarrow A \cap B \notin \mathcal{H}.$$

Then θ is an $\mathcal{H}$-proximity on X. Indeed, one easily sees that θ satisfies the axioms ($\mathcal{A}_1$)–($\mathcal{A}_4$). To show the $\mathcal{H}$-strong axiom, set $E = B$. Then the strong axiom holds and hence by Lemma 2.2, the result follows.

***Example* 2.4** Let $\mathcal{H}$ be a hereditary class on a nonempty set X. For any subsets A and B of X, define

$$A\theta B \Longleftrightarrow A, B \notin \mathcal{H}.$$

Then θ is an $\mathcal{H}$-proximity on X. Indeed, one easily sees that θ satisfies the axioms ($\mathcal{A}_1$)–($\mathcal{A}_4$). To show axiom the $\mathcal{H}$-strong axiom, assume $A \not\theta B$. It follows that $A \in \mathcal{H}$ or $B \in \mathcal{H}$. If $A \in \mathcal{H}$, let $E = X - A$. If $B \in \mathcal{H}$, let $E = B$. Then the strong axiom holds and hence by Lemma 2.2, the result follows.

Similar to the proofs of proximity spaces, we have the following lemma and so the proof is omitted.

Lemma 2.5 Let $(X, \mathcal{H}, \theta)$ be an $\mathcal{H}$-proximity space. Then the following statements hold.

(1) If $A\theta B$, $A \subseteq C$ and $B \subseteq D$, then $C\theta B$.
(2) If there exists an x such that $A\theta x$ and $x\theta B$, then $A\theta B$.
(3) If $A \notin \mathcal{H}$, then $A\theta A$.

Theorem 2.6 Let $(X, \mathcal{H}, \theta)$ be an $\mathcal{H}$-proximity space. Then the operator $(-)^\theta : P(X) \to P(X)$ defined by $A^\theta = \{x \in X \mid x\theta A\}$ satisfies the following conditions:

(1) $A \subseteq B \Rightarrow A^\theta \subseteq B^\theta$,
(2) $A^\theta \cup B^\theta \subseteq (A \cup B)^\theta$ and $(A \cap B)^\theta \subseteq A^\theta \cap B^\theta$,
(3) $A \in \mathcal{H} \Rightarrow A^\theta = \emptyset$,
(4) if $\mathcal{H}_p \subseteq \mathcal{H}$, then $A^\theta = \emptyset$ for every subset A of X.

Proof. Property (1) follows from Lemma 2.5, and property (2) follows from property (1). To see (3) and (4), if $A \in \mathcal{H}$ or $\mathcal{H}_p \subseteq \mathcal{H}$, then $x \not\theta A$ for any $x \in X$. Hence $A^\theta = \emptyset$. ■

Unlike proximity spaces, the following example shows that in $\mathcal{H}$-proximity spaces the operator $(-)^\theta$ need not be $\vee$-additive and $A \nsubseteq A^\theta$, in general. Also, the converse of the weak union axiom need not be true, i.e., the union axiom need not be true.

***Example* 2.7** Let $X = \{a, b, c\}$, $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$ and θ be the $\mathcal{H}$-proximity relation defined in Example 2.4. If $A = \{a\}$, $B = \{b\}$ and $C = \{c\}$, then $(A \cup B)^\theta = C$ but $A^\theta \cup B^\theta = \emptyset$. Also, $C\theta(A \cup B)$ but $C \not\theta A$, $C \not\theta B$; and $A \nsubseteq A^\theta = \emptyset$.

Theorem 2.8 Let $(X, \mathcal{H}, \theta)$ be an $\mathcal{H}$-proximity space and $A, B \subseteq X$. Then the following statements hold.

(1) $B \not\theta A$ implies $A^\theta \subseteq X - B$.
(2) $B \not\theta A$ implies $B \not\theta A^\theta$.
(3) $B \not\theta A$ implies $B^\theta \not\theta A^\theta$.
(4) $A^\theta = A^{\theta\theta}$, i.e., $(-)^\theta$ is idempotent.

Proof. (1): Let $B \not\theta A$ and $A^\theta \cap B \neq \emptyset$. Then there exists $x \in B$ such that $x\theta A$. By Lemma 2.5, $B\theta A$, which is a contradiction. Thus $A^\theta \cap B = \emptyset$ and hence $A^\theta \subseteq X - B$.

(2): Let $B \not\theta A$. Then by the $\mathcal{H}$-strong axiom, there exist subsets C and D such that $B \not\theta (X - C)$, $(X - D) \not\theta A$ and $C \cap D \in \mathcal{H}$. By part (1) we have $A^\theta \subseteq D$. Now, we show that $A^\theta \subseteq (X - C)$. Suppose $x \in A^\theta$, then $x\theta A$. If $x \in C$, then $x \in C \cap D$ and hence $\{x\} \in \mathcal{H}$, which is a contradiction to $x\theta A$. Thus $A^\theta \subseteq (X - C)$. Since $B \not\theta (X - C)$, by Lemma 2.5, we have $B \not\theta A^\theta$.

(3): By part (2) and axiom $(\mathcal{A}_1)$, the result holds.

(4): If $x \notin A^\theta$, then $x \not\theta A$. By part (2), $x \not\theta A^\theta$ and hence $x \notin A^{\theta\theta}$. Thus $A^{\theta\theta} \subseteq A^\theta$. Conversely, if $x \in A^\theta$, then $\{x\} \notin \mathcal{H}$. So $x\theta x$ and hence by Lemma 2.5, $x\theta A^\theta$, i.e., $x \in A^{\theta\theta}$. ■

In the following, we consider the GT on X which is induced by an $\mathcal{H}$-proximity on X, and study its elementary properties. For this purpose, we first give the concept of an admissible $\mathcal{H}$-proximity space.

Definition 2.9 An $\mathcal{H}$-proximity space $(X, \mathcal{H}, \theta)$ is said to be admissible if θ satisfies the following condition for all $A, B \subseteq X$:

$$A\theta(B \cup B^\theta) \Longrightarrow A\theta B.$$

Lemma 2.10 Let $(X, \mathcal{H}, \theta)$ be an admissible $\mathcal{H}$-proximity space and $A, B \subseteq X$. Then the following statements hold.

(1) $A\theta(B \cup B^\theta) \Longleftrightarrow A\theta B$.

(2) $A^\theta = (A \cup A^\theta)^\theta$.

Proof. Let $A\theta B$. Then by the weak union axiom $A\theta(B \cup B^\theta)$, so part (1) holds. To see (2), we have $x \in A^\theta \Leftrightarrow x\theta A \Leftrightarrow x\theta(A \cup A^\theta) \Leftrightarrow x \in (A \cup A^\theta)^\theta$. ■

Remark 4 *Notice that by the union axiom and Theorem 2.8, proximity spaces, coarse proximity spaces and I-proximity spaces are admissible $\mathcal{H}$-proximity spaces.*

***Example* 2.11** The $\mathcal{H}$-proximity spaces defined in Examples 2.3 and 2.4 are admissible. For θ given in 2.3, we have $B^\theta = \{x \mid x \in B \quad \text{and} \quad \{x\} \notin \mathcal{H}\} \subseteq B$ for any $B \subseteq X$. Thus the condition of admissibility holds. For θ given in 2.4, let $A\theta(B \cup B^\theta)$. Then $A \notin \mathcal{H}$ and $B \cup B^\theta \notin \mathcal{H}$. If $B \in \mathcal{H}$, then $B^\theta = \emptyset$. So $B = B \cup B^\theta \notin \mathcal{H}$, a contradiction. Hence $B \notin \mathcal{H}$, this implies $A\theta B$. Thus the condition of admissibility holds.

***Example* 2.12** Let $X = \{a, b, c, d\}$, $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$ and $D = \{c\}$, $D' = \{a, b\}$, $F = \{a, b, c\}$. Suppose $\theta_1 = \{(D, D'), (D', D), (F, X - F), (X - F, F)\}$ and $\theta_2 = \{(A, B) \mid A \cap B \notin \mathcal{H}\}$, i.e., θ_2 is the relation given in 2.3. Now, we define $\theta = \theta_1 \cup \theta_2$ and show that θ is an $\mathcal{H}$-proximity relation on X but not admissible. It is clear that θ satisfies the axioms $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$. To see axiom $\mathcal{A}_4$, if $A\theta_2 B$ and $A\theta_2 C$, then $A\theta_2(B \cup C)$. For case $D\theta_1 D'$, if $D\theta_2 B$ for some $B \subseteq X$, then $D \cap B \notin \mathcal{H}$. Thus $(D' \cup B) \cap D = D \cap B \notin \mathcal{H}$ and hence $D\theta_2(D' \cup B)$. Similarly, for cases $D'\theta_1 D$, $F\theta_1(X - F)$ and $(X - F)\theta_1 F$ the result holds. To see axiom $\mathcal{A}_5$, let $A \not\theta B$. Then $A \cap B \in \mathcal{H}$ and $A \not\theta_1 B$. If $B = F$ or $X - F$, then $F \neq A \neq X - F$, put $E = X - A$. Otherwise $F \neq B \neq X - F$, put $E = B$. Thus there exists a subset E of X such that $A \not\theta E$ and $(X - E) \not\theta B$. Finally, let $A = D'$. Then $A^\theta = D$ and $(A \cup A^\theta)^\theta = F^\theta = \{c, d\}$. Thus $A^\theta \subsetneq (A \cup A^\theta)^\theta$, which shows that $(X, \mathcal{H}, \theta)$ is not admissible by Lemma 2.10.

Theorem 2.13 Let $(X, \mathcal{H}, \theta)$ be an admissible $\mathcal{H}$-proximity space. Then the operator $c_\theta : P(X) \to P(X)$ defined by $c_\theta(A) = A \cup A^\theta$ is enlarging, monotone and idempotent. Hence the collection $\mu_\theta := \{A \mid c_\theta(X - A) = X - A\}$ is a strong GT on X, called the GT induced by $(\theta, \mathcal{H})$.

Proof. It is clear that c_θ is enlarging, and monotone by Theorem 2.6. By the admissibility, we have $c_\theta c_\theta(A) = A \cup A^\theta \cup (A \cup A^\theta)^\theta = A \cup A^\theta = c_\theta(A)$ for any $A \subseteq X$. Thus c_θ is idempotent. Also, $c_\theta(\emptyset) = \emptyset$ and $c_\theta(X) = X$, so $\emptyset, X \in \mu_\theta$. ■

***Example* 2.14** Let $(X, \mathcal{H}, \theta)$ be the admissible $\mathcal{H}$-proximity space given in 2.3. Then $A^\theta \subseteq A$ for any subset A of X. Thus $c_\theta(A) = A$ for any subset A of X. Hence $\mu_\theta = P(X)$ is the discrete topology.

Now, we give an example of admissible $\mathcal{H}$-proximity spaces such that the induced GT μ_θ need not be a topology, in general.

***Example* 2.15** Let $(X, \mathcal{H}, \theta)$ be the admissible $\mathcal{H}$-proximity space given in 2.7. Then $A^\theta = \{c\}$ for any $A \notin \mathcal{H}$. It is easily verified that $c_\theta(A) = A$ for any $A \subseteq X$ such that $A \neq \{a, b\}$ and $c_\theta(\{a, b\}) = X$. Hence $\mu_\theta = P(X) - \{\{c\}\}$, which is not a topology on X.

Theorem 2.16 Let $(X, \mathcal{H}, \theta)$ be an admissible $\mathcal{H}$-proximity space. Then the following statements hold.

(1) $A \subseteq X$ is μ_θ-closed if and only if $A^\theta \subseteq A$.
(2) A^θ is μ_θ-closed for any $A \subseteq X$.
(3) If $\mathcal{H}_p \subseteq \mathcal{H}$, then $\mu_\theta = P(X)$.
(4) $\mathcal{H} = \{\emptyset\}$ if and only if $c_\theta(A) = A^\theta$ for any $A \subseteq X$.

Proof. Part (1) is clear. Since $A^{\theta\theta} = A^\theta$, it follows that A^θ is a μ_θ-closed set. If $\mathcal{H}_p \subseteq \mathcal{H}$, then by Theorem 2.6, $A^\theta = \emptyset$. Thus $c_\theta(A) = A$ for any $A \subseteq X$, so that $\mu_\theta = P(X)$. To show part (4), let $\mathcal{H} = \{\emptyset\}$ and $A \subseteq X$. If $x \in A$, then $x\theta A$ and hence $x \in A^\theta$. Thus $A \subset A^\theta$, so that $c_\theta(A) = A^\theta$. Conversely, since $X \subseteq X^\theta$, it follows that $x\theta X$ for any $x \in X$. Thus $\{x\} \notin \mathcal{H}$ for any $x \in X$. Hence $\mathcal{H} = \{\emptyset\}$. ■

Corollary 2.17 A subset G of an admissible $\mathcal{H}$-proximity space $(X, \mathcal{H}, \theta)$ is μ_θ-open if and only if $x \not\theta (X - G)$ for every $x \in G$.

Proof. By part (1) of the above theorem, the result holds. ■

Corollary 2.18 Let $(X, \mathcal{H}, \theta)$ be an admissible $\mathcal{H}$-proximity space and $A, B \subseteq X$. Then

$$A\theta B \Longleftrightarrow c_\theta(A)\theta c_\theta(B).$$

Proof. By Lemma 2.10, we have

$$A\theta B \Leftrightarrow A\theta c_\theta(B) \Leftrightarrow c_\theta(B)\theta A \Leftrightarrow c_\theta(B)\theta c_\theta(A) \Leftrightarrow c_\theta(A)\theta c_\theta(B).$$

■

3. Alternative description of $\mathcal{H}$-proximity spaces

In this section, we first introduce the concept of an $\mathcal{H}$-proximity neighborhood and explore several of its basic properties. Then we give a definition of an $\mathcal{H}$-proximity in terms of $\mathcal{H}$-proximity neighborhoods.

Definition 3.1 Let $(X, \mathcal{H}, \theta)$ be an $\mathcal{H}$-proximity space. Given subsets $A, B \subseteq X$, we say that B is an $\mathcal{H}$-proximity neighborhood of A, denoted $A \ll B$, if $A \not\theta (X - B)$.

Theorem 3.2 Let $(X, \mathcal{H}, \theta)$ be an $\mathcal{H}$-proximity space. Let A, B, C and D be subsets of X. Then the relation $\ll$ satisfies the following properties:

- (P_1) $X \ll (X - H)$ for all $H \in \mathcal{H}$,
- (P_2) $A \ll B$ implies that there exists $H \in \mathcal{H}$ such that $(A - H) \subseteq B$,
- (P_3) $A \subseteq B \ll C \subseteq D$ implies $A \ll D$,
- (P_4) $A \ll B$ if and only if $(X - B) \ll (X - A)$,
- (P_5) $A \ll B$ implies that there exists $F \subseteq X$ such that $A \ll F$ and $F - H \ll B$ for some $H \in \mathcal{H}$.

Proof. By Axiom $\mathcal{A}_2$, $X \not\theta H$ for any $H \in \mathcal{H}$. This means that $X \not\theta (X - (X - H))$, or equivalently $X \ll (X - H)$ for any $H \in \mathcal{H}$, which is the property (P_1). To show (P_2), notice that if $A \cap (X - B) \notin \mathcal{H}$, then $A\theta(X - B)$, a contradiction to $A \ll B$. So if $H = A \cap (X - B)$, then $H \in \mathcal{H}$ and $(A - H) \subseteq B$. To show (P_3), for contradiction assume that $A \not\ll D$, i.e., $A\theta(X - D)$. The weak union axiom implies that $B\theta(X - D)$. Since $(X - D) \subseteq (X - C)$, again by the weak union axiom we get $B\theta(X - C)$, a contradiction to $B \ll C$. To show (P_4), we have

$$A \ll B \Leftrightarrow A \not\theta (X - B) \Leftrightarrow (X - B) \not\theta (X - (X - A)) \Leftrightarrow (X - B) \ll (X - A).$$

To show (P_5), let $A \ll B$, i.e., $A \not\theta (X - B)$. By the $\mathcal{H}$-strong axiom there exist $C, D \subseteq X$ such that $A \not\theta (X - C)$, $(X - D) \not\theta (X - B)$ and $C \cap D \in \mathcal{H}$. Set $F = C$ and $C \cap D = H$, we have $A \ll F$ and $F - H \subseteq (X - D)$. Thus $(F - H) \not\theta (X - B)$ and hence $F - H \ll B$. ■

Theorem 3.3 Let X be a set with a hereditary class $\mathcal{H}$. If $\ll$ is a binary relation on $P(X)$ satisfying (P_1) through (P_5) of Theorem 3.2 and θ is a relation on $P(X)$ defined by

$$A \not\theta B \quad \text{if and only if} \quad A \ll (X - B).$$

Then θ is an $\mathcal{H}$-proximity on X. Also, B is an $\mathcal{H}$-proximity neighborhood of A if and only if $A \ll B$.

Proof. To show axiom $(\mathcal{A}_1)$, assume $A \not\theta B$. Then $A \ll (X - B)$, property (P_4) implies that $B \ll (X - A)$, i.e., $B \not\theta A$. To show axiom $(\mathcal{A}_2)$, notice that properties (P_1) and (P_3) imply that $A \ll (X - H)$ for all $H \in \mathcal{H}$, i.e., $A \not\theta H$ for all $H \in \mathcal{H}$. By symmetry proven in axiom $(\mathcal{A}_1)$, this implies axiom $(\mathcal{A}_2)$. To show axiom $(\mathcal{A}_3)$, assume $A \not\theta B$, i.e., $A \ll (X - B)$. By property (P_2), there exists $H \in \mathcal{H}$ such that $(A - H) \subseteq (X - B)$. Thus $A \cap B \subseteq H$, which shows that $A \cap B \in \mathcal{H}$. To show the weak union axiom, assume $(A \cup B) \not\theta C$, i.e., $(A \cup B) \ll (X - C)$. Property (P_3) implies that $A \ll (X - C)$ and $B \ll (X - C)$, i.e., $A \not\theta C$ and $B \not\theta C$. To show the $\mathcal{H}$-strong axiom, assume $A \not\theta B$, i.e., $A \ll (X - B)$. By property (P_5), there exist $F \subseteq X$ and $H \in \mathcal{H}$ such that $A \ll F$ and $F - H \ll (X - B)$. Let $C = F$ and $D = (X - (F - H))$. Then $A \not\theta (X - C)$ and $(X - D) \not\theta B$ and $C \cap D = F \cap H \in \mathcal{H}$. Finally, B is an $\mathcal{H}$-proximity neighborhood of A if and only if $A \not\theta (X - B)$ if and only if $A \ll (X - (X - B))$ if and only if $A \ll B$. ■

Theorem 3.4 Let $(X, \mathcal{H}, \theta)$ be an $\mathcal{H}$-proximity space. Let A, B and C be subsets of X. Then the following statements hold.

(1) If $A \ll (B \cap C)$, then $A \ll B$ and $A \ll C$.
(2) If $(B \cup C) \ll A$, then $B \ll A$ and $C \ll A$.
(3) If $A \in \mathcal{H}$, then $A \ll E$ for any $E \subseteq X$.
(4) If $A \ll B$, then $A - B \in \mathcal{H}$.
(5) If $A - B \in \mathcal{H}$ and $B \ll C$, then $A - H \ll C$ for some $H \in \mathcal{H}$.
(6) If $A \ll B$ and $B \ll C$, then $A - H \ll C$ for some $H \in \mathcal{H}$.

Proof. Parts (1) and (2) follow from property (P_3). To show (3), let $A \in \mathcal{H}$. Then $A \not\theta (X - E)$ for any $E \subseteq X$, so the result holds. To show (4), assume $A \ll B$, by property (P_2) there exists $H \in \mathcal{H}$ such that $A - H \subseteq B$. Thus $A - B \subseteq H$, and hence $A - B \in \mathcal{H}$. To show (5), let $A - B \in \mathcal{H}$ and $B \ll C$. Set $H = A - B$, then $A - H \subseteq B$. Since $B \not\theta (X - C)$, it follows that $A - H \not\theta (X - C)$, i.e., $A - H \ll C$. Part (6) follows from parts (4) and (5). ■

Corollary 3.5 Let $(X, \mathcal{H}, \theta)$ be an admissible $\mathcal{H}$-proximity space and $A, B \subseteq X$ such that $A \ll B$. Then

(1) $c_\theta(A) \ll B$,
(2) $A \ll i_{\mu_\theta}(B)$.

Proof. Since $A \not\theta (X - B)$, by Corollary 2.18, we have $c_\theta(A) \not\theta (X - B)$ and $A \not\theta c_\theta(X - B)$. Since $c_\theta(X - B) = X - i_{\mu_\theta}(B)$, the result follows. ■

Corollary 3.6 Let $(X, \mathcal{H}, \theta)$ be an admissible $\mathcal{H}$-proximity space and $A, B \subseteq X$ such that $A \not\theta B$. Then

(1) $c_\theta(A) - H \subseteq (X - B)$ for some $H \in \mathcal{H}$,
(2) $A - H \subseteq i_{\mu_\theta}(X - B)$ for some $H \in \mathcal{H}$.

Proof. Since $A \not\theta (X - (X - B))$, it follows that $A \ll (X - B)$. Thus by the previous corollary and Theorem 3.2, the result follows. ■

4. Compatible $\mathcal{H}$-proximities

In this section, we introduce the concepts of complete regularity and normality for a GTS with respect to a hereditary $\mathcal{H}$ and study its relationships to some $\mathcal{H}$-proximities defined on these spaces.

Definition 4.1 Let $\mathcal{H}$ be a hereditary class on a nonempty set X. If there exists a GT μ and an $\mathcal{H}$-proximity θ on X such that $\mu = \mu_\theta$, then μ and θ are said to be compatible.

Definition 4.2 Let (X, μ) be a GTS and $A, B \subseteq X$. We say that A and B are μ-functionally distinguishable if there exists a μ-continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$, where $[0, 1]$ is endowed the GT generated by the base $\{[0, t) \mid t \in (0, 1)\} \bigcup \{(t, 1] \mid t \in (0, 1)\}$.

Notice that by definition, $\emptyset$ is μ-functionally distinguishable with any subset A of a GTS (X, μ).

Definition 4.3 Let μ and μ' be two GT's on a set X and $\mathcal{H}$ be a hereditary class on X. Then X is called (μ, μ')-completely regular if for any μ-closed sets F and any $x \in X$ such that $\{x\} \cap F \in \mathcal{H}$, $\{x\} - F$ and $F - \{x\}$ are μ'-functionally distinguishable.

Remark 5 *Notice that in the above definition if $x \in F$, then $\{x\} - F = \emptyset$ and hence $\{x\} - F$ and $F - \{x\}$ are μ'-functionally distinguishable. Also, if $\mathcal{H} = \{\emptyset\}$ and $\mu = \mu'$, then (μ, μ')-completely regular is exactly μ-completely regular in the general case, i.e., for any μ-closed sets F and any $x \in X$ such that $x \notin F$, $\{x\}$ and F are μ-functionally distinguishable.*

Theorem 4.4 Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$. Then the relation θ defined by $A \not\theta B$ if and only if there exists $H \in \mathcal{H}$ such that $A - H$ and $B - H$ are μ-functionally distinguishable, is an $\mathcal{H}$-proximity on X.

Proof. Axiom $(\mathcal{A}_1)$ is easily verified. To show $(\mathcal{A}_2)$, let $A \in \mathcal{H}$ and $B \subseteq X$. Then $A - A = \emptyset$ and $B - A$ are μ-functionally distinguishable, i.e., $A \not\theta B$. To show $(\mathcal{A}_3)$, let $A \not\theta B$. Then $A - H$ and $B - H$ are μ-functionally distinguishable for some $H \in \mathcal{H}$. Thus $(A - H) \cap (B - H) = \emptyset$ and hence $A \cap B \subseteq H$, which implies that $A \cap B \in \mathcal{H}$. The weak union axiom is easily verified. To prove the $\mathcal{H}$-strong axiom, let $A \not\theta B$. Then $A - H$ and $B - H$ are μ-functionally distinguishable for some $H \in \mathcal{H}$. Thus there exists a μ-continuous function $f : X \to [0, 1]$ such that $A - H \subseteq f^{-1}(\{0\})$ and $B - H \subseteq f^{-1}(\{1\})$. Let $E = \{x \in X \mid \frac{1}{2} \leqslant f(x) \leqslant 1\}$ and define $g : [0, 1] \to [0, 1]$ by $g(y) = 2y$ for $y \in [0, \frac{1}{2})$ and $g(y) = 1$ for $y \in [\frac{1}{2}, 1]$. Then $g \circ f : X \to [0, 1]$ is a μ-continuous function such that $(A - H) \subseteq (g \circ f)^{-1}(\{0\})$ and $(E - H) \subseteq (g \circ f)^{-1}(\{1\})$. Hence $A \not\theta E$. Similarly, if $X - E = \{x \in X \mid 0 \leqslant f(x) < \frac{1}{2}\}$, then we have $(X - E) \not\theta B$. Thus the strong axiom holds. ■

Theorem 4.5 Let (X, μ) be a (μ^*, μ)-completely regular strong GTS with a hereditary class $\mathcal{H}$ such that $\mathcal{H} \subseteq \mu^*$. Then the $\mathcal{H}$-proximity θ defined in Theorem 4.4 is compatible with μ^*, where μ^* is the GT induced by $(\mu, \mathcal{H})$.

Proof. Let $G \in \mu^*$ and $x \in G$. Since $X - G$ is μ^*-closed and $\{x\} \cap (X - G) = \emptyset \in \mathcal{H}$, by assumption $\{x\}$ and $(X - G)$ are μ-functionally distinguishable and hence $\{x\} - H$ and $(X - G) - H$ are μ-functionally distinguishable for $H = \emptyset \in \mathcal{H}$. Thus $x \not\theta (X - G)$

for any $x \in G$. Hence by Corollary 2.17, $G \in \mu_\theta$. Conversely, Let $G \in \mu_\theta$ and $x \in G$. Again by Corollary 2.17, $x \not\theta (X - G)$. So $\{x\} - H$ and $(X - G) - H$ are μ-functionally distinguishable for some $H \in \mathcal{H}$. If $x \in H$, then $\{x\} \in \mu^*$ and hence $x \in i_{\mu^*}(G)$. If $x \notin H$, then there exists a μ-continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $(X - G) - H) \subseteq f^{-1}(\{1\})$. Let $M = f^{-1}([0, \frac{1}{2}))$. Then $x \in M - H \subseteq G$ and $M \in \mu$. By Theorem 1.8, $x \in i_{\mu^*}(G)$. Thus $G = i_{\mu^*}(G)$ is μ^*-open, which shows that $\mu_\theta \subseteq \mu^*$. ■

Corollary 4.6 Let (X, μ) be a μ-completely regular strong GTS with the hereditary class $\mathcal{H} = \{\emptyset\}$ and θ be the $\mathcal{H}$-proximity defined in Theorem 4.4. Then θ is compatible with μ.

Proof. By Theorem 4.5 and Remark *5*, the result follows. ■

Definition 4.7 Let μ and μ' be two GT's on a set X and $\mathcal{H}$ be a hereditary class on X. Then X is called (μ, μ')-normal if for any μ-closed sets F_1 and F_2 such that $F_1 \cap F_2 \in \mathcal{H}$, there exist $G_1, G_2 \in \mu'$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 \in \mathcal{H}$.

Remark 6 *If $\mathcal{H} = \{\emptyset\}$ and $\mu = \mu'$, then (μ, μ')-normal is exactly μ-normal in the general case, i.e., for any μ-closed sets F_1 and F_2 such that $F_1 \cap F_2 = \emptyset$, there exist $G_1, G_2 \in \mu$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$.*

Theorem 4.8 Let (X, μ) be a (μ^*, μ)-normal strong GTS with a hereditary class $\mathcal{H}$. Then the relation θ defined by

$$A\theta B \Longleftrightarrow c^*(A) \cap c^*(B) \notin \mathcal{H},$$

is an $\mathcal{H}$-proximity on X.

Proof. Axiom $(\mathcal{A}_1)$ is easily verified. To show $(\mathcal{A}_2)$, notice that if $A \in \mathcal{H}$, then by Theorem 1.8, $A^* = \emptyset$ and hence $c^*(A) = A$. Thus $c^*(A) \cap c^*(B) \subseteq A \in \mathcal{H}$, which shows that $A \not\theta B$. To show $(\mathcal{A}_3)$, let $A \cap B \notin \mathcal{H}$. Then $c^*(A) \cap c^*(B) \notin \mathcal{H}$ and hence $A\theta B$. To show the weak union axiom, let $A\theta B$ and $A\theta C$. Then $c^*(A) \cap c^*(B) \notin \mathcal{H}$ and $c^*(A) \cap c^*(C) \notin \mathcal{H}$. So $c^*(A) \cap c^*(B \cup C) \notin \mathcal{H}$, which shows that $A\theta(B \cup C)$. To prove the $\mathcal{H}$-strong axiom, let $A \not\theta B$. Then $c^*(A) \cap c^*(B) \in \mathcal{H}$, so by assumption there exist $G_1, G_2 \in \mu$ such that $c^*(A) \subseteq G_1$, $c^*(B) \subseteq G_2$ and $G_1 \cap G_2 \in \mathcal{H}$. Set $C = G_1$ and $D = G_2$, then we have $c^*(A) \cap c^*(X - C) \subseteq c^*(A) \in \mathcal{H}$ and $c^*(B) \cap c^*(X - D) \subseteq c^*(B) \in \mathcal{H}$. Thus $A \not\theta (X - C)$, $(X - D) \not\theta B$ and $C \cap D \in \mathcal{H}$. ■

Definition 4.9 A GTS (X, μ) with a hereditary class $\mathcal{H}$ is called (μ, μ')-T_4 if it is (μ, μ')-normal and μ-T_1.

Theorem 4.10 Let (X, μ) be a (μ^*, μ)-T_4 strong GTS with a hereditary class $\mathcal{H}$. Then the $\mathcal{H}$-proximity θ defined in Theorem 4.8 is compatible with μ^*.

Proof. It suffices to show that $c^*(A) = c_\theta(A)$ for any $A \subseteq X$. Let $x \in c_\theta(A)$. Then $x \in A$ or $x \in A^\theta$. If $x \in A$, then $x \in c^*(A)$. If $x \in A^\theta$, then $x\theta A$ and hence $c^*(\{x\}) \cap c^*(A) \notin \mathcal{H}$. Since (X, μ) is T_1 and $\mu \subseteq \mu^*$, it follows that $x \in c^*(A)$. Thus $c_\theta(A) \subseteq c^*(A)$. Conversely, let $x \notin c_\theta(A)$. Then $x \notin A$ and $x \not\theta A$. So $c^*(\{x\}) \cap c^*(A) \in \mathcal{H}$, by assumption there exist $G_1, G_2 \in \mu$ such that $x \in G_1$, $c^*(A) \subseteq G_2$ and $G_1 \cap G_2 \in \mathcal{H}$. Since $G_1 \cap A \subseteq (G_1 \cap G_2) \in \mathcal{H}$, it follows that $x \notin A^*$. Thus $x \notin c^*(A)$ and hence $c^*(A) \subseteq c_\theta(A)$. ■

Corollary 4.11 Let (X, μ) be a μ-T_4 strong GTS with the hereditary class $\mathcal{H} = \{\emptyset\}$ and θ be the $\mathcal{H}$-proximity defined in Theorem 4.8. Then θ is compatible with μ.

Proof. By Theorem 4.10 and Remark *6*, the result follows. ■

Theorem 4.12 Let (X, μ) be a strong GTS with an ideal $\mathcal{H}$ and θ be the relation defined in Theorem 4.8. If θ is an admissible $\mathcal{H}$-proximity on X such that is compatible with μ^* and $\mathcal{H} \subseteq \mu^*$, then X is (μ^*, μ^*)-normal.

Proof. If F_1 and F_2 are μ^*-closed sets such that $F_1 \cap F_2 \in \mathcal{H}$, then $F_1 \not\theta F_2$. By the $\mathcal{H}$-strong axiom, there exist subsets C and D such that $F_1 \not\theta (X - C)$, $(X - D) \not\theta F_2$ and $C \cap D \in \mathcal{H}$. By Corollary 3.6, there exist $H_1, H_2 \in \mathcal{H}$ such that $(F_1 - H_1) \subseteq i_{\mu_\theta}(C)$ and $(F_2 - H_2) \subseteq i_{\mu_\theta}(D)$. Set $G_1 = i_{\mu_\theta}(C) \cup H_1$ and $G_2 = i_{\mu_\theta}(D) \cup H_2$, then $F_1 \subseteq G_1$, $F_2 \subseteq G_2$, $G_1, G_2 \in \mu^*$ and $G_1 \cap G_2 \in \mathcal{H}$. Thus the result holds. ■

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