Journal of
Linear and Topological Algebra
Vol. 08, *No.* 04, 2019, 229- 235

Measures of maximal entropy

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Received 28 February 2019, Accepted 17 March 2019.
Communicated by Hamidreza Rahimi

Abstract. We extend the results of Walters on the uniqueness of invariant measures with maximal entropy on compact groups to an arbitrary locally compact group. We show that the maximal entropy is attained at the left Haar measure and the measure of maximal entropy is unique.

Keywords: Infinite invariant measure, Haar measure, maximal entropy, locally compact group.

2010 AMS Subject Classification: 37B40,37L40.

1. Introduction and preliminaries

The ergodic theory and dynamics is usually studied in the context of probability spaces, or compact metric spaces in the topological setting. The main reason is the examples which suggest lack of finite invariant measure in the non compact case, yet it is known that there might be infinite invariant measure [1, 11]. The other concern is the uniqueness of infinite invariant measures with certain characteristic property. For instance, it is known that the normalized Haar measure of a compact group is the unique probability measure of maximal entropy on the group. The main purpose of the current paper is to show a similar result for arbitrary locally compact group (Theorem 3.2).

The paper is organized as follows. In section 2, we briefly review part of the existing literature on infinite invariant measures for non compact dynamics [2]. We recall a result on the existence of invariant measures for transformation groups from [2]. Section 3 is

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

devoted to the extension of the results on uniqueness of invariant measures with maximal entropy on locally compact groups [3].

2. infinite invariant measures

This section briefly reviews certain results on infinite invariant measures. Before doing this we briefly review the compact case.

2.1 *compact dynamics*

Let us take a look at the existence and uniqueness of invariant measures in compact dynamics.

If X is a set and $p \in X$, $f : X \to \mathbb{R}$ is a function and $T : X \to X$ is a bijection,

$$M(f,p,k) = f_k(p) := \frac{1}{k}\sum_{i=1}^{k} f(T^i p) \quad (k = 1, 2, \cdots)$$

and $M(f,p) = f^*(p) := \lim_{k\to\infty} M(f,p,k)$, when the limit exists. Also the *upper density* of a set $E \subseteq \mathbb{N}$ is by definition $\delta^*(E) := \limsup_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \chi_E(i)$.

Now if X is a compact metric space and T is a bijective homeomorphism of X, by the Krylov-Bogolioubov theorem, the system (X,T) admits an invariant probability Radon measure μ, and for any closed subset K of X, either $\mu(K) > 0$ for some invariant measure μ, or $M(\chi_K, p) = 0$, for every $p \in X$. Moreover, the set $M(X,T)$ of invariant regular Borel probability measures on X is convex and weak*-compact.

A system (X,T) is *uniquely ergodic* if it has a unique invariant probability Borel measure μ as above (equivalently, if it has a unique ergodic set) and *strictly ergodic* (in the sense of Nemyckiĭ and Stepanov [7], which differs slightly with the same notion in [6]) if X consists of a single ergodic set.

By the unique ergodic theorem, if the system (X,T,μ) is uniquely ergodic then $f_k(p) \to \int_X f d\mu$ as $k \to \infty$, uniformly on X, for any $f \in C(X)$.

An alternative uniqueness type result is the uniqueness of invariant measure with maximal entropy. If α is any finite open cover of X, we let $N(\alpha)$ be the number of members in a subcover of α of minimal cardinality. For two covers α and β, we write $\alpha \vee \beta := \{U \cap V : U \in \alpha, V \in \beta\}$ and $\alpha \geqslant \beta$ if α is a refinement of β. Then $N(\alpha) \geqslant N(\beta)$ when $\alpha \geqslant \beta$ and $N(\alpha \vee \beta) \leqslant N(\alpha)N(\beta)$ and so the following limit exists

$$h(\alpha,T) = \lim_{n\to\infty} \frac{1}{n} \log N(\bigvee_{i=0}^{n-1} T^{-i}\alpha),$$

The *topological entropy* of T is defined by $h_{top}(T) := \sup h(\alpha,T)$, where the supremum is taken over all finite open covers of X. On the other hand, for $\mu \in M(X,T)$ and a finite measurable partition α of X, we write,

$$H_\mu(\alpha) = -\sum_{A\in\alpha} \mu(A)\log\mu(A) \;\text{ and }\; h_\mu(\alpha,T) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha).$$

The *measure-theoretic entropy* of T is defined as $h_\mu(T) = \sup h_\mu(\alpha, T)$, where the supremum is taken over all finite measurable partitions of X. It is known that

$$h(T) = \sup_{\mu \in M(X,T)} h_\mu(T).$$

This indeed holds for any (non metrizable) compact Hausdorff space X [4].

There are examples in which the above supremum is not attained [5] and cases where there is a unique $\mu \in M(X, T)$ at which the supremum is attained [13].

2.2 *dynamics on complete metric spaces*

The existence of a finite invariant measures for transformations on (complete) metric spaces imposes certain restriction on the transformation. Even in the compact case, the invariant measures may be quite trivial (confined to a finite set of points) [8]. For the existence of an invariant measures, we have the following result of Oxtoby-Ulam: Let T be an automorphism of a complete separable metric space X. There exist a finite invariant measure in X iff for some point p and compact subset K, $M(\chi_K, p) < \infty$, where χ_K is the characteristic function of K. Also, it is sufficient that the limit superior is positive [8, Theorem 1]. If we insist that the finite invariant measure is zero at points, then it exists iff T has uncountably many periodic points and there is a compact set K consisting of non-periodic points, one of which returns to K with positive frequency under iterations of T [8, Theorem 2].

Next for infinite invariant measures, let T be an automorphism of a complete, separable metric space X, of second category in itself. Then there exists a (possibly infinite) measure, invariant under T, defined for Borel sets, zero for points, positive (possibly infinite) for non-void open sets, such that it is finite and strictly positive on some Borel set. If we only require that the measure is positive on uncountable open sets (and X itself is uncountable), we may drop the category condition on X [8, Theorem 4].

2.3 *transformation groups*

In this subsection we review the existence of invariant measures for transformation groups based on [2].

Let G be a locally compact group and (X, d) be a metric space. Let G act continuously on X. The action is uniformly expansive if there are constants $\delta > 0$ and $C \geqslant 1$ such that $d(t \cdot x, t \cdot y) \leqslant Cd(x, y)$, for each $t \in G$, whenever $d(x, y) < \delta$. The action is topologically transitive if there is a dense orbit at each point. For flows on compact metrizable spaces, topological transitivity is equivalent to the average shadowing property (possibly with respect to some other equivalent metric) [9, Theorem 2].

The next theorem is the main result of [2].

Theorem 2.1 Let G be a locally compact group acting continuously on a metric space X. If the action is uniformly expansive and topologically transitive then there is an G-invariant Radon measure on X.

There are non transitive actions which are locally transitive. A concrete example is the irrational rotation on the unit circle $\mathbb{S}^1$, which is not transitive, but has dense orbits.

3. invariant measures of maximal entropy

In this section we prove uniqueness results for invariant measures. In the first subsection we show uniqueness of invariant measures for homeomorphisms of locally compact groups. Unlike the compact case [3], here the invariant measure is not a probability measure and uniqueness is only up to a positive scalar (unless we make some normalization). Then we move to intrinsically ergodic systems, in the sense of Weiss [14]. We show that in the non compact case, some of the results of [14] are still valid. The final subsection involves the main results of this paper. Following Walters [13], we study systems with unique invariant measures of maximal entropy. We adapt the results of [13] to the setup of [12].

3.1 *automorphisms on groups*

Let T be a continuous proper transformation of a locally compact metrizable group (sometimes we need further to assume that T is a homeomorphism, in which case we specify this). We assume that T has a (not necessarily finite) positive invariant measure. By the main result of [2], this holds when T is positively expansive continuous transformation with dense orbits.

We let m denote the left Haar measure of G. We assume that there is a Borel subset E_0 of G with $0 < m(E_0) = m(T^{-1}E_0) < \infty$. Let $\mathfrak{B}$ be the Borel σ-field of G. Let $M_T(G)$ denote the collection of Radon measures on G which are invariant under T. Let π, π_1 and π_2 be the multiplication map and projections onto the first and second components from $G \times G$ onto G. The Borel σ-field of $G \times G$ is $\mathfrak{B} \times \mathfrak{B}$. We have three sub-$\sigma$-fields $\mathfrak{B}_1 := \pi_1^{-1}(\mathfrak{B})$, $\mathfrak{B}_2 := \pi_2^{-1}(\mathfrak{B})$, and $\mathfrak{B}_c := \pi^{-1}\mathfrak{B}$ (c stands for convolution) of $\mathfrak{B} \times \mathfrak{B}$. Let $\mu * \nu$ be the convolution of Radon measures μ and ν on G, given by $\mu * \nu(E) = \mu \times \nu(\pi^{-1}(E))$, for $E \in \mathfrak{B}$.

First let us observe that $\mathfrak{B}_1 \vee \mathfrak{B}_c = \mathfrak{B} \times \mathfrak{B}$. Consider the measurable map $\sigma : (G \times G, \mathfrak{B} \times \mathfrak{B}) \to (G, \mathfrak{B})$ defined by $\sigma(x, y) = y^{-1}x$. Then $\pi_1 : (G \times G, \mathfrak{B}_1 \vee \mathfrak{B}_c) \to (G, \mathfrak{B})$ is measurable, and the same holds for π_1 replaced by π. Therefore, this also holds for π_1 replaced by $\pi_2 = \sigma \circ (\pi \times \pi_1)$, and the equality between σ-fields follows. A similar argument shows that $\mathfrak{B}_2 \vee \mathfrak{B}_c = \mathfrak{B} \times \mathfrak{B}$.

Lemma 3.1 With the above notations,
(*i*) $m \in M_T(G)$,
(*ii*) When T is also a group homomorphism, $\mu * \nu \in M_T(G)$ for $\mu, \nu \in M_T(G)$.

Proof. (*i*) Since T is proper $f \circ T$ has compact support for each $f \in C_c(G)$ and

$$\begin{aligned}\int_G f(gx)d(m \circ T^{-1})(x) &= \int_G f(T(gx))dm(x)\\ &= \int_G f \circ T(gx)dm(x)\\ &= \int_G f \circ T(x)dm(x)\\ &= \int_G f(x)d(m \circ T^{-1})(x),\end{aligned}$$

hence by the uniqueness of the left Haar measure, there is a constant $c > 0$ such that $m \circ T^{-1} = cm$. On the other hand, for the Borel set E_0 as above, $0 < m \circ T^{-1}(E_0) =$

$m(E_0) < \infty$, thus $c = 1$.

(ii) The fact that T is a group homomorphism could be written as $T\pi = \pi(T \times T)$. If $\mu, \nu \in M_T(G)$, for any $E \in \mathfrak{B}$,

$$\begin{aligned} \mu * \nu(T^{-1}E) &= \mu \times \nu(\pi^{-1}T^{-1}E) = \mu \times \nu((T^{-1} \times T^{-1})\pi^{-1}E) \\ &= \mu \times \nu(\pi^{-1}E) = \mu * \nu(E). \end{aligned}$$

On the other hand, if $\mu, \nu \in M_T(G)$, take Borel sets E_1, E_2 with $0 < \mu(E_1) < \infty$ and $0 < \nu(E_2) < \infty$, and put $E_0 = \pi(E_1 \times E_2)$, then

$$\mu * \nu(E_0) = \mu \times \nu(\pi^{-1}E_0) = \mu \times \nu(\pi^{-1}\pi(E_1 \times E_2)) \geqslant \mu \times \nu(E_1 \times E_2) > 0.$$

■

Now as in part (ii) above, assume that T is a group homomorphism. Then for T-invariant measures μ, ν, the convolution $\mu * \nu$ is also T-invariant. A second observation is that the multiplication map π induces a conjugacy between the systems $(G \times G, T \times T, \mathfrak{B}_c, \mu \times \nu)$ and $(G, T, \mathfrak{B}, \mu * \nu)$. Since $h_{\mu\times\nu}(T \times T) = h_\mu(T) + h_\nu(T)$, we get

$$h_{\mu*\nu}(T) \leqslant h_\mu(T) + h_\nu(T).$$

Next assume that G is separable, then $\mathfrak{B}$ is separable, hence there are increasing sequences of finite algebras $\mathfrak{B}_n$ and $\mathfrak{C}_n$ with $\bigvee_{n=1}^\infty \mathfrak{B}_n = \mathfrak{B}_c$ and $\bigvee_{n=1}^\infty \mathfrak{C}_n = \mathfrak{B}_1$. For a field $\mathfrak{D}$, let us put $\mathfrak{D}^{-} = \bigvee_{i=1}^\infty T^{=i}\mathfrak{D}$, then as for the case of probability measures, we still have

$$H_{\mu\times\nu}(\mathfrak{B}_n \vee \mathfrak{C}_n \big| (\mathfrak{B}_n \vee \mathfrak{C}_n)^{-}) \leqslant H_{\mu\times\nu}(\mathfrak{B}_n | (\mathfrak{B}_n)^{-}) + H_{\mu\times\nu}(\mathfrak{C}_n | (\mathfrak{C}_n)^{-}),$$

taking limit on n we get

$$h_{\mu\times\nu}(T \times T, \mathfrak{B} \times \mathfrak{B}) \leqslant h_{\mu\times\nu}(T \times T, \mathfrak{B}_c) + h_{\mu\times\nu}(T \times T, \mathfrak{B}_1).$$

By the above equivalence, $h_\mu(T) + h_\nu(T) \leqslant h_{\mu*\nu}(T) + h_\mu(T)$. In particular, if $h_\mu(T) < \infty$, we get $h_\nu(T) \leqslant h_{\mu*\nu}(T)$. Similarly, if $h_\nu(T) < \infty$, we get $h_\mu(T) \leqslant h_{\mu*\nu}(T)$. For $\nu = m$, we have $\mu * m = m$ (since m is left translation invariant, this holds for the case that μ is a finite linear combination of point masses, and the general case follows by taking an strict limit), therefore, if we know that $h_m(T) < \infty$, we get $h_\mu(T) \leqslant h_m(T)$. This shows that in this case, m is a measure with maximal entropy. To have a case, where m is the unique measure with maximal entropy, we further need an ergodicity condition.

We prove the following result, which extends [13, Theorem 2.1].

Theorem 3.2 Let G be a σ-compact, locally compact Polish group with left Haar measure m and $T : G \to G$ be an automorphism of G. If $h_m(T) < \infty$ then m is the unique invariant Radon measure of maximal entropy on G iff m is ergodic.

To prepare for an argument based on ergodicity, let us first observe that $\mathfrak{B}_1$ and $\mathfrak{B}_c$

are $(m \times m)$-independent subfields of $\mathfrak{B} \times \mathfrak{B}$. Given Borel subsets E, F of G,

$$\begin{aligned} m \times m(\pi_1^{-1}(E) \cap \pi^{-1}(F)) &= \int_E \int_{x^{-1}F} dm(y)dm(x) \\ &= \int_E m(x^{-1}F)dm(x) \\ &= m(F)m(E), \end{aligned}$$

including the case that some of the terms above might be infinite. Conversely, if $\mathfrak{B}_1$ and $\mathfrak{B}_c$ are $(m \times \nu)$-independent, for some $\nu \in M(G)$, then

$$\begin{aligned} m \times \nu(\pi_1^{-1}(E) \cap \pi^{-1}(F)) &= m \times \nu(\pi_1^{-1}(E)) m \times \nu(\pi^{-1}(F)) \\ &= m(E) m * \nu(F) \\ &= m(E)m(F) \\ &= m \times m(\pi_1^{-1}(E) \cap \pi^{-1}(F)), \end{aligned}$$

thus $\nu = m$.

Proof of Theorem 3.2. Suppose that $h(m) < \infty$ and m is ergodic. Following [13, Theorem 2.1] (which proves the result for compact groups) we want to show that m is the unique invariant measure with maximal entropy. Take any invariant measure μ. First assume that μ is also ergodic. Since (G, T, m) is mixing, $(G \times G, T \times T, m \times \mu)$ is $(\mathfrak{B} \times \mathfrak{B})$-ergodic, and so it is also $\mathfrak{B}_1$-ergodic and $\mathfrak{B}_c$-ergodic. Both these systems have finite entropy, hence there are measurable partitions which give these entropies. On the other hand,

$$\begin{aligned} h_m(T, \mathfrak{B}) + h_\mu(T, \mathfrak{B}) &= h_{m \times \mu}(T \times T, \mathfrak{B} \times \mathfrak{B}) \\ &= h_{m \times \mu}(T \times T, \mathfrak{B}_1 \vee \mathfrak{B}_c) \\ &\leqslant h_{m \times \mu}(T \times T, \mathfrak{B}_1) + h_{m \times \mu}(T \times T, \mathfrak{B}_c) \\ &= h_m(T, \mathfrak{B}) + h_{m * \mu}(T, \mathfrak{B}) \\ &= 2h_m(T, \mathfrak{B}), \end{aligned}$$

and if $h_m(T, \mathfrak{B}) = h_\mu(T, \mathfrak{B})$, then all the terms above are equal. In particular,

$$h_{m \times \mu}(T \times T, \mathfrak{B}_1 \vee \mathfrak{B}_c) = h_{m \times \mu}(T \times T, \mathfrak{B}_1) + h_{m \times \mu}(T \times T, \mathfrak{B}_c),$$

which implies that $\mathfrak{B}_1$ and $\mathfrak{B}_c$ are $(m \times \nu)$-independent, therefore $\mu = m$.

Next, for a not necessarily ergodic, but probability measure μ, we use the Jacob's barrycentric decomposition, which is valid for our dynamic when G is a complete separable metrizable group [10, Theorem 9.6.2]. The ergodic decomposition of μ consists of ergodic invariant probability measures μ_P, $P \in \mathcal{P}$, and there is a probability measure $\hat{\mu}$ on $\mathcal{P}$ such that

$$h_\mu(T) = \int_{\mathcal{P}} h_{\mu_P}(T) d\hat{\mu}(P),$$

where the left side is finite (by the above observations) and so is the integrand (every-where) in the right side. We may assume that $\mu_P \neq m$ (otherwise we are back to the compact case), and so by the above observation, $h_{\mu_P}(T) < h_m(T)$, for each P, thus $h_\mu(T) < h_m(T)$.

Finally we deal we the general case, without any restriction on the invariant Radon measure μ. The above cited theorem is proved in [10] for probability measures, however the proof is based on the fact that $\mathcal{P}$ (being the same as the convex set of probability Borel measures on a complete separable metrizable space) is complete, separable and metrizable (in the weak* topology). When G is a σ-compact Polish group, this could be recovered for non probability measure μ. First, by [1, 2.2.9], there is a standard probability space (Y, λ), a collection of Borel invariant ergodic measures $\{\mu_y\}_{y\in Y}$ such that $y \mapsto \mu_y(E)$ is λ-measurable and $\mu(E) = \int_Y \mu_y(E)d\lambda(y)$, for each $E \in \mathfrak{B}$. Since Y is a standard Borel space, we may apply the same proof as in [10, Theorem 9.6.2] (plus the above argument) to get

$$h_\mu(T) = \int_Y h_{\mu_y}(T)d\lambda(y) < h_m(T).$$

This shows that if m is ergodic then m is the unique invariant measure of maximal entropy. Conversely, if m is not ergodic, then applying the above analog of the Jacob's barrycentric decomposition to m (instead of μ) we conclude that $h_m(T) > h_\nu(T)$ could not hold for any ergodic invariant measure ν, that is m is not unique among invariant measures of maximal entropy.

Acknowledgements

The original idea of this work was stemmed from the lecture of Krerley Oliveira of Universidade Federal de Alagoas in Tarbiat Modares University. This work has been supported by the Center for International Scientific Studies & Collaboration (CISSC), Ministry of Science Research and technology (MSRT).

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