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A $(2-\varepsilon)$-approximation ratio for vertex cover problem on special graphs

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Abstract. The vertex cover problem is a famous combinatorial problem, and its complexity has been heavily studied. It is known that it is hard to approximate to within any constant factor better than 2, while a 2-approximation for it can be trivially obtained. In this paper, new properties and new techniques are introduced which lead to approximation ratios smaller than 2 on special graphs; e.g. graphs for which their maximum cut values are less than $0.999|E|$. In fact, we produce a (1.9997)-approximation ratio on graph G, where the (0.878)-approximation algorithm of the Goemans-Williamson for the maximum cut problem on G produces a value smaller than $0.877122|E|$.

Keywords: Discrete Optimization, vertex cover problem, complexity theory, NP-complete problems.

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1. Introduction

The vertex cover problem (VCP) is a famous NP-complete problem, where the set of vertices in graph G should be partitioned into two sets, one that includes none of the edges of the graph and the other that includes at least one endpoint of every edge of the graph. The VCP problem cannot be approximated within a factor of 1.36 [6], unless P=NP, while a 2-approximation factor for it can be trivially obtained by taking all the vertices of a maximal matching in the graph. However, improving this simple 2-approximation algorithm has been a quite hard task [11, 12].

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The minimum VCP problem is one of the fundamental problems in the combinatorial optimization and it has received a lot of attentions, and, many approximation algorithms have been proposed to construct vertex cover in different ways. The best constant approximation ratio known is 2, and this is the best-known worst-case one [5]. Several approximation algorithms were proposed by various authors to deal with the problem of minimum VCP. Some of them can be classified as direct algorithms.

The first algorithm in this class is the maximum degree greedy algorithm, where, it introduces few changes in the previously existing greedy heuristic algorithm for set-cover problem by Chavatal [3]. The other algorithms in this class which have some improvements for classical algorithms and calculation techniques are the depth first search algorithm [13], the edge deletion algorithm [8], the ListLeft algorithm [1], the ListRight algorithm [4], the iterated local search algorithm [17] and so on.

Another class of VCP approximation algorithms is intelligent algorithms. In this class, Singh et al. [15] proposed a hybrid approach for the problem combining with steady-state genetic algorithm and Greedy Heuristic. Xu et al. [18] presented an efficient simulated annealing algorithm and simulated on several benchmark graphs. Stefan et al. [16] applied a modified reactive tabu search approach with simulated annealing for the minimum weight vertex cover problem. Bhasin [2] applied the theory of natural selection via genetic algorithms for solving the problem.

Ant colony optimization algorithm is also used to discuss the problem [10, 14]. For a detailed literature review and comparative analysis of some well-known approximation algorithms for Minimum VCP problem, see [7]. Contribution of that paper is the provision of small benchmark graphs on which the given approximation algorithms fail to provide optimal results.

In this paper, we introduce a $(2-\varepsilon)$-approximation ratio on special graphs. The rest of the paper is structured as follows: Section 2 is about the vertex cover problem and introduces new properties and techniques which lead to a $(2-\varepsilon)$-approximation ratio on special graphs. Finally, Section 3 is allocated to conclusions.

2. Introducing a $(2-\varepsilon)$-approximation ratio on special graphs

In the mathematical discipline of graph theory, a vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. The problem of finding a minimum vertex cover is a typical example of an NP-complete optimization problem. In this section, new properties and new techniques are introduced which lead to approximation ratios smaller than 2 on special problems.

Let $G=(V,E)$ be an undirected graph on vertex set V and edge set E, where $|V|=n$. Throughout this paper, suppose that the vertex cover problem on G is hard with an optimal value $z^*_{VCP} \geqslant \frac{n}{2}$, and we have produced an arbitrary feasible solution for the problem with vertex partitioning $V = V_{1G} \cup V_{-1G}$ and objective value $|V_{1G}|$ (V_{1G} is a vertex cover of graph G).

By defining the decision variables x_{oj} and x_{ij} as follows:

$$x_{oj} = \begin{cases} +1 & (if\ j \in V^*_{1G}) \\ -1 & (if\ j \in V^*_{-1G}) \end{cases}$$

$$x_{ij} = \begin{cases} +1 & (if\ i,j \in V^*_{-1G}\ or\ i,j \in V^*_{1G}) \\ -1 & (Otherwise) \end{cases}$$

And by addition of triangle inequalities to the constraints of vertex cover problem, we can introduce the following ILP model for the minimum vertex cover problem:

$$min_{s.t} \quad z = \sum_{j \in V} \frac{1 + x_{oj}}{2}$$

$$x_{oi} + x_{oj} - x_{ij} = +1 \quad ij \in E$$

$$x_{ij} + x_{ik} + x_{jk} \geqslant -1 \quad i, j, k \in V$$

$$x_{ij} - x_{ik} - x_{jk} \geqslant -1 \quad i, j, k \in V$$

$$x_{oj}, x_{ij} \in \{-1, +1\} \quad i, j \in V$$

Here, triangle inequalities are as cutting plane inequalities. By relaxing the last constraints and addition of the constraint $X \succeq 0$, we have a well-known semidefinite programming formulation.

Theorem 2.1 Suppose that for a feasible solution $V_{1G} \cup V_{-1G}$ we have $|V_{1G}| \leqslant k|V_{-1G}|$. Then, based on such a solution, we have an approximation ratio $\frac{|V_{1G}|}{z^*_{VCP}} \leq \frac{2k}{k+1}$.

Proof. We know that $|V_{1G}| + |V_{-1G}| = n$. Therefore, there exist $t \leqslant k$, for which $|V_{1G}| = t|V_{-1G}| = t\frac{n}{t+1}$. Then, $z^*_{VCP} \geqslant \frac{n}{2} = \frac{t+1}{2t}|V_{1G}|$ which concludes that $\frac{|V_{1G}|}{z^*_{VCP}} \leqslant \frac{2t}{t+1} \leqslant \frac{2k}{k+1}$. ■

Therefore, for bounded values of k, we have some approximation ratios smaller than 2. For example,

If $|V_{1G}| \leqslant 2|V_{-1G}|$ then $\frac{|V_{1G}|}{z^*_{VCP}} \leqslant \frac{2k}{k+1} \leqslant \frac{4}{3} < 1.36$,

If $|V_{1G}| \leqslant 3|V_{-1G}|$ then $\frac{|V_{1G}|}{z^*_{VCP}} \leqslant \frac{2k}{k+1} \leqslant \frac{3}{2}$,

$\vdots$

If $|V_{1G}| \leqslant 10^6|V_{-1G}|$ then $\frac{|V_{1G}|}{z^*_{VCP}} \leqslant \frac{2k}{k+1} \leqslant \frac{2000000}{1000001} < 2 - 10^{-6}$,

But, if $k \to \infty$ then $\frac{|V_{1G}|}{z^*_{VCP}} \to 2$. Hence, we don't still have an approximation ratio better than 2.

Corollary 2.2 Suppose that we know $v \in V^*_{-1G}$, where $deg_v \geqslant \frac{n}{500}$. Then, by construction of graph H from graph G, after addition of $\frac{n}{500}$ copies of vertex v, we have $V^*_{1G} = V^*_{1H}$ and $z^*_{VCP}(G) = z^*_{VCP}(H)$. Therefore, we can introduce V_{-1H} by v and $\frac{n}{500}$ copies of it, to produce a feasible solution $V_{1H} \cup V_{-1H}$, correspondingly, where $|V_{1H}| \leqslant 500|V_{-1H}|$. Hence, we have a performance ratio $\frac{|V_{1G}|}{z^*_{VCP}(G)} = \frac{|V_{1H}|}{z^*_{VCP}(H)} \leqslant \frac{2 \times 500}{500+1} < 1.997$.

Corollary 2.3 We can remove each vertex v with $deg_v \geqslant \frac{n}{500}$ (i.e. we can consider $v \in V^*_{1G}$) to produce an approximation ratio on $G_{V-\{v\}}$. Otherwise, we can produce a feasible solution $V^*_{-1G} = \{v\}$, $V^*_{1G} = V - \{v\}$ with approximation ratio $\frac{|V_{1G}|}{z^*_{VCP}(G)} = \frac{|V_{1H}|}{z^*_{VCP}(H)} < 1.997$.

Theorem 2.4 For a fixed value of a, suppose that we can select an arbitrary matching with cardinality $\frac{n}{a}$ (not a maximal matching). Then, we can produce a feasible solution with an approximation ratio of $\frac{|V_{1G}|}{z^*_{VCP}} \leqslant \max\{\frac{4a}{3a-2}, \frac{2a}{a+0.002}\} < 2$, or more than 0.999 of the selected edges are cuts and have only one vertex in the optimal vertex cover.

Proof. After removing $2\frac{n}{a}$ selected vertices, let H be the remaining subgraph for which we have $|V_H| = n - 2\frac{n}{a} = \frac{a-2}{a}n$.

If the vertex cover problem on graph H is easy (i.e. $|V^*_{-1H}| \geqslant \frac{|V_H|}{2} = \frac{a-2}{2a}n$), then we can produce a suitable feasible solution $V_{1G} \cup V_{-1G}$, correspondingly, where $|V_{-1G}| = |V^*_{-1H}| \geqslant \frac{|V_H|}{2} = \frac{a-2}{2a}n$. Hence, $|V_{1G}| \leqslant n \leqslant \frac{2a}{a-2}|V_{-1G}|$ and $\frac{|V_{1G}|}{z^*_{VCP}} \leqslant \frac{2\times\frac{2a}{a-2}}{\frac{2a}{a-2}+1} = \frac{4a}{3a-2} < 2$.

Otherwise, the vertex cover problem on H is hard and $|V^*_{1H}| \geqslant \frac{|V_H|}{2} \geqslant \frac{a-2}{2a}n$. Produce G' by addition of $\frac{n}{a}$ new vertices and connection of endpoints of each edge of the selected matching to one of these vertices to construct $\frac{n}{a}$ triangles in G'. We have $z'^*_{VCP} = z^*_{VCP} + \alpha\frac{n}{a}$ $(0 \leqslant \alpha \leqslant 1)$. Moreover, due to these triangles, we have $z'^*_{VCP} \geqslant \frac{2n}{a}_{(for\ triangles)} + \frac{a-2}{2a}n_{(for\ H)} = \frac{(a+2)n}{2a} = \frac{n}{2} + \frac{n}{a}$, and $\frac{n}{z'^*_{VCP}} \leqslant \frac{2a}{a+2}$. Therefore, We have $\frac{n}{z'^*_{VCP}} = \frac{n}{z^*_{VCP}+\alpha\frac{n}{a}} \leqslant \frac{2a}{a+2}$, which concludes that $\frac{n}{z^*_{VCP}} \leqslant \frac{2a}{a+2-2\alpha}$.

Therefore, if $\alpha \leqslant 0.999$ then for all feasible solutions $V_{1G} \cup V_{-1G}$, we have $\frac{|V_{1G}|}{z^*_{VCP}} < \frac{n}{z^*_{VCP}} < \frac{2a}{a+2-1.998} < 2$. Otherwise (i.e. $\alpha > 0.999$), more than 0.999 of the selected edges have only one vertex in the optimal vertex cover. ∎

Corollary 2.5 For different values of $3 \leqslant a \leqslant 10$ and by consideration of disjoint sets of matchings with cardinality $\frac{n}{a}$, where their unions are equal to the set E, we can conclude that

all feasible solutions $V_{1G} \cup V_{-1G}$ have a ratio $\frac{|V_{1G}|}{z^*_{VCP}} \leqslant \max\{\frac{4a}{3a-2}, \frac{2a}{a+0.002} |\ 3 \leqslant a \leqslant 10\} = \frac{2\times 10}{10+0.002} < 1.9997$,

or more than 0.999 of the edges of E have only one vertex in optimal vertex cover; i.e. The optimal value for the maximum cut problem is more than $0.999|E|$.

Corollary 2.6 By executing the (0.878)-approximation algorithm of the Goemans-Williamson [9] for the maximum cut problem on G,

if $Z_{GW} < 0.878 \times 0.999|E| = 0.877122|E|$ then less than 0.999 of the edges of E have only one vertex in the optimal vertex cover, and (based on Corollary 2.5), we can produce a ratio $\frac{|V_{1G}|}{z^*_{VCP}} \leqslant 1.9997$.

But, if $Z_{GW} \geqslant 0.877122|E|$ then we cant ensure to produce a $(2-\varepsilon)$-approximation ratio for vertex cover problem on G.

Therefore, we could introduce $(2-\varepsilon)$-approximation ratio on special graphs with suitable characteristics; e.g. Graphs for which their cut values introduced by Goemans-Williamson algorithm are less than $0.877122|E|$.

Algorithm (To produce a vertex cover solution with a factor $\rho \leqslant 1.999$ on graphs with maximum cut value smaller than $0.999|E|$)

Step 1. Produce an arbitrary feasible solution $V_{1G} \cup V_{-1G}$.

Step 2. If $|V_{1G}| < 0.999n$ then (based on Theorem 2.1) stop and return the current solution $V_{1G} \cup V_{-1G}$ with a ratio factor of 1.998. Otherwise, go to Step 3.

Step 3. In different iterations and for different values of $3 \leqslant a \leqslant 10$, introduce disjoint sets of matchings with cardinality $\frac{n}{a}$, where their unions are equal to the set E. In each iteration, if the vertex cover problem on the remaining subgraph H is easy, then (based on H and Theorem 2.4) produce a suitable feasible solution $V_{1G} \cup V_{-1G}$, which has an approximation ratio $\frac{|V_1|}{z^*_{VCP}} \leqslant \frac{4a}{3a-2} < \frac{4\times3}{3\times3-2} < 1.715$. Otherwise, go to Step 4.

Step 4. Based on Step 3, we know that in all iterations, the vertex cover problem on the remaining subgraph H was hard. Therefore, if by executing the 0.878-approximation algorithm of the Goemans-Williamson, $Z_{GW} \geqslant 0.877122|E|$, then conclude that the maximum cut value on G is greater than $0.999|E|$ and we can't ensure to produce a $(2-\varepsilon)$-approximation ratio for vertex cover problem on G. Otherwise, (i.e. less than 0.999 of the edges of E have only one vertex in optimal vertex cover) return the current feasible solution $V_{1G} \cup V_{-1G}$ which has an approximation ratio of 1.9997, since based on Corollary 2.5, for all feasible solutions $V_{1G} \cup V_{-1G}$ we have a ratio $\frac{|V_{1G}|}{(z^*_{VCP}} \leqslant 1.9997$.

3. Conclusion

One of the open problems about the vertex cover problem is the possibility of introducing an approximation algorithm within any constant factor better than 2. Here, we proposed a new algorithm to introduce a (1.9997)-approximation algorithm for vertex cover problem on special graphs; i.e. Graphs for which their maximum cut values are less than $0.999|E|$.

Note that there is no need to explain the proven approach by small examples. For example, for all graphs G with $n < 1000$ vertices and $z^* > \frac{n}{2}$, we can introduce $V_{-1G} = \{v_1\}$ and $V_{1G} = V - \{v_1\}$ to have a feasible solution with a ratio of $\frac{|V_{1G}|}{z^*_{VCP}(G)} \leqslant \frac{n-1}{\frac{n}{2}} < 1.998 < 2$.

As an idea for extension of the approach for graphs with maximum cut values of more than $0.999|E|$, we can connect the endpoints of P_4s to produce a P_5-free graph with an objective value almost equal to the original graph. We know that the vertex cover problem is polynomially solvable on P_5-free graphs and this may lead to introducing a $(2-\varepsilon)$-approximation algorithm for vertex cover problem on all graphs.

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References

[1] D. Avis, T. Imamura, A list heuristic for vertex cover, Oper. Res. Lett. 35 (2007), 201-204.
[2] H. Bhasin, Harnessing genetic algorithm for vertex cover problem, Inter. J. Comput. Sci. Engin. 4 (2012), 218-223.
[3] V. Chvatal, A greedy heuristic for the set covering problem, Math. Oper. Res. 4 (3) (1979), 233-235.
[4] F. Delbot, C. Andlaforest, A better list heuristic for vertex cover, Inform. Proc. Lett. 107 (2008), 125-127.
[5] F. Delbot, C. Laforest, Analytical and experimental comparison of six algorithms for the vertex cover problem, ACM J. Experimental Algorithmics. 15 (2010), 1-26.
[6] I. Dinur, S. Safra, On the hardness of approximating minimum vertex cover, Annal. Math. 162 (2005), 439-485.
[7] M. Fayaz, S. Arshad, A. S. Shah, A. Shah, Approximate methods for minimum vertex cover fail to provide optimal results on small graph instances: A review, Inter. J. Control. Automation. 11 (2) (2018), 135-150.
[8] M. R. Garey, D. Andjohnson, Computers and Intractability, Freeman, New York, 1979.
[9] M. X. Goemans, D.P. Williamson, Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, J. ACM. 42 (6) (1995), 1115-1145.
[10] R. Jovanovic, M. Tuba, An ant colony optimization algorithm with improved pheromone correction strategy for the minimum weight vertex cover problem, Appl. Soft. Comput. 11 (2011), 5360-5366.
[11] S. Khot, On the Power of Unique 2-Prover 1-Round Games, Proceeding of 34th ACM Symposium on Theory of Computing, STOC, 2002.

[12] S. Khot, O. Regev, Vertex cover might be hard to approximate to within $2-\varepsilon$, J. Computer. System. Sciences. 74 (2008), 335-349.

[13] C. Savage, Depth-first search and the vertex cover problem, Inform. Proc. Lett. 14 (1982), 233-235.

[14] S. J. Shyu, P. Y. Yin, B. M. T. Lin, An ant colony optimization algorithm for the minimum weight vertex cover problem, Annal. Oper. Res. 131 (2004), 283-304.

[15] A. Singh, A. K. Gupta, A hybrid heuristic for the minimum weight vertex cover problem, Asia-Pacific J. Oper. Res. 23 (2006), 273-285.

[16] S. Voβ, A. Fink, A hybridized tabu search approach for the minimum weight vertex cover problem, J. Heuristics. 18 (2012), 869-876.

[17] C. Witt, Analysis of an iterated local search algorithm for vertex cover in sparse random graphs, Theore. Comput. Sci. 425 (2012), 117-125.

[18] X. Xu, J. Ma, An efficient simulated annealing algorithm for the minimum vertex cover problem, Neurocomputing. 69 (2006), 913-916.