Journal of
Linear and Topological Algebra
Vol. 08, *No.* 01, 2019, 63- 70

On $\beta-$topological vector spaces

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Received 1 September 2018; Revised 7 October 2018; Accepted 10 November 2018.

Communicated by Hamidreza Rahimi

Abstract. We introduce and study a new class of spaces, namely $\beta-$topological vector spaces via $\beta-$open sets. The relationships among these spaces with some existing spaces are investigated. In addition, some important and useful characterizations of $\beta-$topological vector spaces are provided.

Keywords: $\beta-$open sets, $\beta-$closed sets, $\beta-$topological vector spaces.

2010 AMS Subject Classification: 57N17, 54C08, 57N99.

1. Introduction

It is well-known that the advent of topological vector spaces brought a revolution in the study of various branches of functional analysis. Because of nice properties and usefulness, these spaces remain a fundamental notion in fixed point theory, operator theory and various other advanced branches of mathematics. In 2015, Khan et al. [4] introduced and studied the s-topological vector spaces which are a generalization of topological vector spaces. In 2016, Khan and Iqbal [5] introduced the irresolute topological vector spaces which are a particular brand of s-topological vector spaces but they are independent of topological vector spaces. Ibrahim [3] initiated the study of $\alpha-$topological vector spaces which are contained in the class of s-topological vector spaces. In this paper, we introduce a new class of spaces, namely, $\beta-$topological vector spaces. Some general properties of $\beta-$topological vector spaces along with their relationships with certain other types

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

of spaces are investigated. Furthermore, a broad characterizations of these spaces are presented.

2. Preliminaries

Let X be a topological space. For a subset A of X, the closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$ respectively. We represent the set of real numbers by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$. The notations ϵ and δ denote negligibly small positive real numbers.

Definition 2.1 [1, 6, 7] A subset A of a topological space X is called
(a) semi-open if $A \subseteq Cl(Int(A))$,
(b) $\alpha-$open if $A \subseteq Int(Cl(Int(A)))$,
(c) $\beta-$open if $A \subseteq Cl(Int(Cl(A)))$.

Clearly, every open set is $\alpha-$open and every $\alpha-$open set is semi-open and every semi-open set is $\beta-$open but, in general, the converse need not be true.

Example **2.2** Let $X = \mathbb{R}$ with the usual topology. Consider $A = [1, 2],\ B = (1, 2) \cap \mathbb{Q}$, where $\mathbb{Q}$ denotes the set of rational numbers. Then A is semi-open but it is neither open nor $\alpha-$open. Also, notice that B is $\beta-$open which is not semi-open.

The complement of a $\beta-$open set is said to be $\beta-$closed set. The intersection of all $\beta-$closed sets containing a subset $A \subseteq X$ is called the $\beta-$closure of A and is denoted by $\beta Cl(A)$. It is known that a subset A of X is $\beta-$closed if and only if $A = \beta Cl(A)$. A point $x \in \beta Cl(A)$ if and only if $A \cap U \neq \emptyset$ for each $\beta-$open set U in X containing x. The $\beta-$interior of a subset $A \subseteq X$ is the union of all $\beta-$open sets in X contained in A and is denoted by $\beta Int(A)$. A point x of X is called $\beta-$interior point of a subset A if there exists a $\beta-$open set U in X containing x such that $x \in U \subseteq A$. The set of all $\beta-$interior points of A is equal to $\beta Int(A)$. The family of all $\beta-$open (resp. $\beta-$closed) sets in X will be denoted by $\beta O(X)$ (resp. $\beta C(X)$).

Also we recall some definitions that will be used in the sequel.

Definition 2.3 Let L be a vector space over the field F ($\mathbb{R}$ or $\mathbb{C}$). Let T be a topology on L such that

(1) For each $x, y \in L$ and each open neighborhood W of $x + y$ in L, there exist open neighborhoods U and V of x and y respectively, in L such that $U + V \subseteq W$,

(2) For each $\lambda \in F,\ x \in L$ and each open neighborhood W of λx in L, there exist open neighborhoods U of λ in F and V of x in L such that $U.V \subseteq W$.

Then the pair $(L_{(F)}, T)$ is called topological vector space.

Definition 2.4 [4] Let L be a vector space over the field F ($\mathbb{R}$ *or* $\mathbb{C}$) and let T be a topology on L such that

(1) For each $x, y \in L$ and each open set W in L containing $x+y$, there exist semi-open sets U and V in L containing x and y respectively such that $U + V \subseteq W$,

(2) For each $\lambda \in F,\ x \in L$ and each open set W in L containing λx, there exist semi-open sets U in F containing λ and V in L containing x such that $U.V \subseteq W$.

Then the pair $(L_{(F)}, T)$ is called s-topological vector space.

Definition 2.5 [5] Let L be a vector space over the field F ($\mathbb{R}$ *or* $\mathbb{C}$) and T be a topology on L such that

(1) For each $x, y \in L$ and each semi-open set W in L containing $x + y$, there exist semi-open sets U and V in L containing x and y respectively such that $U + V \subseteq W$,

(2) For each $\lambda \in F,\ x \in L$ and each semi-open set W in L containing λx, there exist semi-open sets U in F containing λ and V in L containing x such that $U.V \subseteq W$.
Then the pair $(L_{(F)}, T)$ is called irresolute topological vector space.

Definition 2.6 [3] Let L be a vector space over the field F ($\mathbb{R}$ *or* $\mathbb{C}$) and T be a topology on L such that

(1) For each $x, y \in L$ and each $\alpha-$open set W in L containing $x+y$, there exist $\alpha-$open sets U and V in L containing x and y respectively such that $U + V \subseteq W$,

(2) For each $\lambda \in F,\ x \in L$ and each $\alpha-$open set W in L containing λx, there exist $\alpha-$open sets U in F containing λ and V in L containing x such that $U.V \subseteq W$.
Then the pair $(L_{(F)}, T)$ is called $\alpha-$topological vector space.

3. $\beta-$topological vector spaces

The purpose of this section is to define and investigate some basic properties of $\beta-$topological vector spaces.

Definition 3.1 Let E be a vector space over the field K, where $K = \mathbb{R}$ *or* $\mathbb{C}$ with standard topology. Let τ be a topology on E such that the following conditions are satisfied:

(1) For each $x, y \in E$ and each open set $W \subseteq E$ containing $x + y$, there exist $\beta-$open sets U and V in E containing x and y respectively, such that $U + V \subseteq W$,

(2) For each $\lambda \in K,\ x \in E$ and each open set $W \subseteq E$ containing λx, there exist $\beta-$open sets U in K containing λ and V in E containing x such that $U.V \subseteq W$.
Then the pair $(E_{(K)}, \tau)$ is called $\beta-$topological vector space (written in short, βTVS).

First of all we present some examples of $\beta-$topological vector spaces and then these examples will be used in the sequel for investigating the relationships of $\beta-$topological vector spaces with certain other types of spaces.

***Example* 3.2** Consider the field $K = \mathbb{R}$ with the standard topology. Let $E = \mathbb{R}$ be the real vector space, is also endowed with the standard topology. Then $(E_{(K)}, \tau)$ is $\beta-$topological vector space.

After tasting this example, an immediate question that comes into mind is that is there any other topology on $\mathbb{R}$ which turn it out a $\beta-$topological vector space. The answer is in affirmative. In fact, there are topologies on $\mathbb{R}$ other than the standard topology which turn it out a $\beta-$topological vector space. Let us present some examples of them.

***Example* 3.3** Consider $F = \mathbb{R}$ with the standard topology. Let $E = \mathbb{R}$ be the vector space of real numbers over the field F, is endowed with the topology $\tau = \{\emptyset, D, \mathbb{R}\}$, where D denotes the set of irrational numbers. Then

(1) For each $x, y \in E$, we have two cases:

Case (I) If $x + y$ is rational, then the only open neighborhood of $x + y$ in E is $\mathbb{R}$. So, there is nothing to prove.

Case (II) If $x + y$ is irrational, then for open neighborhood $W = D$ of $x + y$ in E. We have following sub-cases:

Sub-case (i) If both x and y are irrational, we can choose $\beta-$open sets $U = \{x\}$ and $V = \{y\}$ in E such that $U + V \subseteq W$.

Sub-case (ii) If one of x or y is rational, say y. Then, for the selection of $\beta-$open sets $U = \{x\}$ and $V = \{p, y\}$ in E, where $p \in D$ such that $p + x \in D$, we have $U + V \subseteq W$.

This verifies the first condition of $\beta-$topological vector spaces.

(2) Let $\lambda \in \mathbb{R}$ and $x \in E$. If λx is rational, then it is straightforward to prove. Suppose λx is irrational. Let $W = D$ be an open neighborhood of λx. The following cases arise:

Case (I) If both λ and x are irrational, then, choose $\beta-$open sets $U = [(\lambda - \epsilon, \lambda + \epsilon) \cap \mathbb{Q}] \cup \{\lambda\}$ in $\mathbb{R}$ containing λ and $V = \{x\}$ in E containing x, we see that $U.V \subseteq W$.

Case (II) If λ is rational and x is irrational, then for the selection of $\beta-$open sets $U = (\lambda - \epsilon, \lambda + \epsilon) \cap \mathbb{Q}$ in $\mathbb{R}$ containing λ and $V = \{x\}$ in E containing x, we have $U.V \subseteq W$.

Case (III) Finally, suppose λ is irrational and x is rational. Choose $\beta-$open sets $U = (\lambda - \epsilon, \lambda + \epsilon) \cap D$ of $\mathbb{R}$ and $V = \{x, p\}$ of E such that $p \in D$ with pq is irrational for each $q \in U$, we find that $U.V \subseteq W$.
This proves that $(E_{(\mathbb{R})}, \tau)$ is $\beta-$topological vector space.

***Example* 3.4** Let $E = \mathbb{R}$ be the vector space of real numbers over the field K, where $K = \mathbb{R}$ with standard topology and the topology τ on E be generated by the base $\mathcal{B} = \{(a, b), [c, d) : a, b, c$ and d are real numbers with $0 < c < d\}$. We show that $(E_{(K)}, \tau)$ is β-topological vector space. For which we have to verify the following two conditions:

(1) Let $x, y \in L$. Then, for open neighborhood $W = [x + y, x + y + \epsilon)$ (resp. $(x + y - \epsilon, x + y + \epsilon)$) of $x + y$ in E, we can opt for $\beta-$open sets $U = [x, x + \delta)$ (resp. $(x - \delta, x + \delta)$) and $V = [y, y + \delta)$ (resp. $(y - \delta, y + \delta)$) of E containing x and y respectively, such that $U + V \subseteq W$ for each $\delta < \frac{\epsilon}{2}$.

(2) Let $x \in E$ and $\lambda \in K$. Consider open neighborhood $W = [\lambda x, \lambda x + \epsilon)$ (resp. $(\lambda x - \epsilon, \lambda x + \epsilon)$) of λx in E. We have following cases:

Case (1). If $\lambda > 0$ and $x > 0$, then clearly $\lambda x > 0$. We can choose $\beta-$open sets $U = [\lambda, \lambda + \delta)$ (resp. $(\lambda - \delta, \lambda + \delta)$) in K containing λ and $V = [x, x + \delta)$ (resp. $(x - \delta, x + \delta)$) in E containing x such that $U.V \subseteq W$ for each $\delta < \frac{\epsilon}{\lambda + x + 1}$.

Case (II). If $\lambda < 0$ and $x < 0$, then $\lambda x > 0$. We can choose $\beta-$open sets $U = (\lambda - \delta, \lambda]$ (resp. $(\lambda - \delta, \lambda + \delta)$) in K and $V = (x - \delta, x]$ (resp. $(x - \delta, x + \delta)$) in E such that $U.V \subseteq W$ for sufficiently appropriate $\delta \leqslant \frac{-\epsilon}{\lambda + x - 1}$.

Case (III). If $\lambda = 0$ and $x > 0$ (resp. $\lambda > 0$ and $x = 0$). Then $\lambda x = 0$. Consider any open neighborhood $W = (-\epsilon, \epsilon)$ of 0 in E. We can opt for $\beta-$open sets $U = (-\delta, \delta)$ (resp. $U = (\lambda - \delta, \lambda + \delta)$) of $\mathbb{R}$ containing λ and $V = (x - \delta, x + \delta)$ (resp. $V = (-\delta, \delta)$) of E containing x such that $U.V \subseteq W$ for each $\delta < \frac{\epsilon}{x+1}$ (resp. $\delta < \frac{\epsilon}{\lambda + 1}$).

Case (IV). If $\lambda = 0$ and $x < 0$ (resp. $\lambda < 0$ and $x = 0$). Consider any open neighborhood $W = (-\epsilon, \epsilon)$ of 0 in E. Then, for the selection of $\beta-$open sets $U = (-\delta, \delta)$ (resp. $U = (\lambda - \delta, \lambda + \delta)$) in $\mathbb{R}$ and $V = (x - \delta, x + \delta)$ (resp. $V = (-\delta, \delta)$) in E, we have $U.V \subseteq W = (-\epsilon, \epsilon)$ for each $\delta < \frac{\epsilon}{1 - x}$ (resp. $(\delta < \frac{\epsilon}{1 - \lambda})$).

Case (V). If $\lambda = 0$ and $x = 0$. Consider any open neighborhood $W = (-\epsilon, \epsilon)$ of 0 in E, we can find $\beta-$open sets $U = (-\delta, \delta)$ of $\mathbb{R}$ and $V = (-\delta, \delta)$ of E, such that $U.V \subseteq W$ for each $\delta < \sqrt{\epsilon}$.

Case (VI). If $\lambda < 0,\ x > 0$ (resp. $\lambda > 0,\ x < 0$). In this case, there is only one type of open neighborhood $W = (\lambda x - \epsilon, \lambda x + \epsilon)$ of λx in E. Choose $\beta-$open sets $U = (\lambda - \delta, \lambda + \delta)$ in $\mathbb{R}$ and $V = (x - \delta, x + \delta)$ in E, we have $U.V \subseteq W$ for each $\delta < \frac{\epsilon}{x - \lambda + 1}$ (resp. $\delta < \frac{\epsilon}{\lambda - x + 1}$).
Hence, $(E_{(\mathbb{R})}, \tau)$ is $\beta-$topological vector space.

The definitions clarify that every s-topological vector space is $\beta-$topological vector space but the converse is not true because, in general, Example 3.3 is not s-topological vector space.

From here on, E denotes a $\beta-$topological vector space $(E_{(K)}, \tau)$ unless stated explicitly and by a scalar we mean an element of the associated field K of a $\beta-$topological vector space $(E_{(K)}, \tau)$. Now, we discuss some basic properties of $\beta-$topological vector spaces.

Theorem 3.5 Let A be any open subset of a $\beta-$topological vector space E. Then the following are true:

(i) $x + A \in \beta O(E)$ for each $x \in E$,
(ii) $\lambda A \in \beta O(E)$ for each non-zero scalar λ.

Proof. (i) Let $y \in x + A$. Then there exist $\beta-$open sets $U, V \in \beta O(E)$ containing $-x$ and y, respectively, such that

$$U + V \subseteq A \Rightarrow -x + V \subseteq U + V \subseteq A \Rightarrow V \subseteq x + A \Rightarrow y \in Int_\beta(x + A).$$

Hence, $x + A = Int_\beta(x + A)$. This proves that $x + A$ is $\beta-$open set in E.

(ii) Let $x \in \lambda A$. By the definition of $\beta-$topological vector spaces, there exist $\beta-$open sets U in K containing $\frac{1}{\lambda}$ and V in E containing x such that

$$U.V \subseteq A \ \Rightarrow x \in V \subseteq \lambda A \ \Rightarrow x \in Int_\beta(\lambda A) \ \Rightarrow \lambda A = Int_\beta(\lambda A).$$

Thus, $\lambda A \in \beta O(E)$. ■

Corollary 3.6 For any open subset A of a $\beta-$topological vector space E, the following are true:
(i) $x + A \subseteq Cl(Int(Cl(x + A)))$ for each $x \in E$,
(ii) $\lambda A \subseteq Cl(Int(Cl(\lambda A)))$ for each non-zero scalar λ.

Theorem 3.7 Let F be any closed subset of a $\beta-$topological vector space E. Then the following are true:
(i) $x + F \in \beta C(E)$ for each $x \in E$,
(ii) $\lambda F \in \beta C(E)$ for each non-zero scalar λ.

Proof. (i) Suppose that $y \in \beta Cl(x+F)$. Consider $z = -x+y$ and let W be any open set in E containing z. Then there exist $\beta-$open sets U and V in E such that $-x \in U,\ y \in V$ and $U + V \subseteq W$. Since $y \in \beta Cl(x + F)$, $(x + F) \cap V \neq \emptyset$. So, there is $a \in (x + F) \cap V$. Now,

$$-x + a \in F \cap (U + V) \subseteq F \cap W \ \Rightarrow F \cap W \neq \emptyset \ \Rightarrow z \in Cl(F) = F \ \Rightarrow y \in x + F.$$

Hence, $x + F = \beta Cl(x + F)$. This proves that $x + F$ is $\beta-$closed set in E.

(ii) Assume that $x \in \beta Cl(\lambda F)$ and let W be any open neighborhood of $y = \frac{1}{\lambda}x$ in E. Since E is βTVS, there exist $\beta-$open sets U in K containing $\frac{1}{\lambda}$ and V in E containing x such that $U.V \subseteq W$. By hypothesis, $(\lambda F) \cap V \neq \emptyset$. Therefore, there is $a \in (\lambda F) \cap V$. Now,

$$\frac{1}{\lambda}a \in F \cap (U.V) \subseteq F \cap W \ \Rightarrow F \cap W \neq \emptyset \ \Rightarrow y \in Cl(F) = F \ \Rightarrow x \in \lambda F$$

and thereby, $\lambda F = \beta Cl(\lambda F)$. Hence, $\lambda F \in \beta C(E)$. ■

Corollary 3.8 For any closed subset F of a $\beta-$topological vector space E, the following are true:
(i) $Int(Cl(Int(x + F))) \subseteq x + F$ for each $x \in E$,
(ii) $Int(Cl(Int(\lambda F))) \subseteq \lambda F$ for each non-zero scalar λ.

Theorem 3.9 Let A and B be any subsets of a $\beta-$topological vector space E. Then $\beta Cl(A) + \beta Cl(B) \subseteq Cl(A + B)$.

Proof. Let $x \in \beta Cl(A),\ y \in \beta Cl(B)$ and let W be any open neighborhood of $x+y$ in E. Then, by the definition of $\beta-$topological vector spaces, there exist $\beta-$open sets $U, V \in \beta O(E)$ such that $x \in U,\ y \in V$ and $U+V \subseteq W$. By assumption, there are $a \in A\cap U$ and $b \in B\cap V$. Consequently, $a+b \in (A+B)\cap(U+V) \subseteq (A+B)\cap W \ \Rightarrow (A+B)\cap W \neq \emptyset$

and as a result, $x+y \in Cl(A+B)$. Therefore, $\beta Cl(A)+\beta Cl(B) \subseteq Cl(A+B)$. ■

4. Characterizations

In this section, we obtain some useful characterizations of $\beta-$topological vector spaces.

Theorem 4.1 For a subset A of a $\beta-$topological vector space E, the following are valid:
(a) $\beta Cl(x+A) \subseteq x+Cl(A)$ for each $x \in E$,
(b) $x+\beta Cl(A) \subseteq Cl(x+A)$ for each $x \in E$,
(c) $x+Int(A) \subseteq \beta Int(x+A)$ for each $x \in E$,
(d) $Int(x+A) \subseteq x+\beta Int(A)$ for each $x \in E$.

Proof. (a) Let $y \in \beta Cl(x+A)$ and consider $z=-x+y$ in E. Let W be any open neighborhood of z. Then we get $\beta-$open sets U containing $-x$ and V containing y in E such that $U+V \subseteq W$. Whence we find that $(x+A)\cap V \neq \emptyset \Rightarrow$ there is $a \in E$ such that $a \in (x+A)\cap V$. Now, $-x+a \in A\cap(U+V) \subseteq A\cap W \Rightarrow A\cap W \neq \emptyset$ and hence, $z \in Cl(A)$; that is, $y \in x+Cl(A)$. Therefore, $\beta Cl(x+A) \subseteq x+Cl(A)$.

(b) Let $z \in x+\beta Cl(A)$. Then $z=x+y$ for some $y \in \beta Cl(A)$. Notice that for any open neighborhood W of z, there exist $\beta-$open sets $U,V \in \beta O(E)$ such that $x \in U$, $y \in V$ and $U+V \subseteq W$. Since $y \in \beta Cl(A)$, $A\cap V \neq \emptyset \Rightarrow$ there is $a \in A\cap V$. Now,

$$x+a \in (x+A)\cap(U+V) \subseteq (x+A)\cap W \Rightarrow (x+A)\cap W \neq \emptyset \Rightarrow z \in Cl(x+A).$$

Hence, the assertion follows.

(c) Let $y \in x+Int(A)$. Then $U+V \subseteq Int(A)$ where $U,V \in \beta O(E)$ such that $-x \in U$ and $y \in V$. Whence we have $-x+V \subseteq U+V \subseteq A \Rightarrow V \subseteq x+A$. Since V is $\beta-$open, $y \in \beta Int(x+A)$ and consequently, $x+Int(A) \subseteq \beta Int(x+A)$.

(d) Let $y \in Int(x+A)$. Then $y=x+a$ for some $a \in A$. Since E is βTVS, there exist $U,V \in \beta O(E)$ such that $x \in U$, $a \in V$ and $U+V \subseteq Int(x+A)$. Now $x+V \subseteq U+V \subseteq Int(x+A) \subseteq x+A$ implies that $y \in x+\beta Int(A)$. Therefore, the assertion follows. ■

The following is the analog of Theorem 4.1.

Theorem 4.2 For a subset A of a $\beta-$topological vector space E, the following are valid:
(a) $\beta Cl(\lambda.A) \subseteq \lambda.Cl(A)$ for each non-zero scalar λ,
(b) $\lambda.\beta Cl(A) \subseteq Cl(\lambda.A)$ for each non-zero scalar λ,
(c) $\lambda.Int(A) \subseteq \beta Int(\lambda.A)$ for each non-zero scalar λ,
(d) $Int(\lambda.A) \subseteq \lambda.\beta Int(A)$ for each non-zero scalar λ.

Theorem 4.3 Let A be any subset of a $\beta-$topological vector space E. Then
(a) $Int(Cl(Int(x+A))) \subseteq x+Cl(A)$ for each $x \in E$,
(b) $x+Int(Cl(Int(A))) \subseteq Cl(x+A)$ for each $x \in E$,
(c) $x+Int(A) \subseteq Cl(Int(Cl(x+A)))$ for each $x \in E$,
(d) $Int(x+A) \subseteq x+Cl(Int(Cl(A)))$ for each $x \in E$.

Proof. (a) Since $Cl(A)$ is closed, by Theorem 3.7, $x+Cl(A)$ is $\beta-$closed. Consequently, $Int(Cl(Int(x+A))) \subseteq x+Cl(A)$.

(b) In view of Theorem 3.7, $-x+Cl(x+A)$ is $\beta-$closed and hence $Int(Cl(Int(A))) \subseteq -x+Cl(x+A)$. Thereby the assertion follows.

(c) In consequence of Theorem 3.5, $x+Int(A)$ is $\beta-$open. Therefore, $x+Int(A) \subseteq Cl(Int(Cl(x+Int(A)))) \subseteq Cl(Int(Cl(x+A)))$. Hence the assertion follows.

(d) Obvious. ■

The analog of Theorem 4.3 is the following:

Theorem 4.4 Let A be any subset of a $\beta-$topological vector space E. Then
(a) $Int(Cl(Int(\lambda A))) \subseteq \lambda Cl(A)$ for each non-zero scalar λ,
(b) $\lambda Int(Cl(Int(A))) \subseteq Cl(\lambda A)$ for each non-zero scalar λ,
(c) $\lambda Int(A) \subseteq Cl(Int(Cl(\lambda A)))$ for each non-zero scalar λ,
(d) $Int(\lambda A) \subseteq \lambda Cl(Int(Cl(A)))$ for each non-zero scalar λ.

Theorem 4.5 For any open set U in a $\beta-$topological vector space E, $x + Int(Cl(U)) \subseteq Cl(x + U)$ for each $x \in E$.

Proof. On account of Theorem 4.3(b), $x + Int(Cl(Int(U))) \subseteq Cl(x + U)$. Since U is open, we have $x + Int(Cl(U)) \subseteq Cl(x + U)$. This completes the proof. ■

Theorem 4.6 For any closed set F in a $\beta-$topological vector space E, $Int(x + F) \subseteq x + Cl(Int(F))$ for each $x \in E$.

Proof. In view of Theorem 4.3(d), $Int(x + F) \subseteq x + Cl(Int(Cl(F))) = x + Cl(Int(F))$ because F is closed. Hence the proof is finished. ■

Definition 4.7 [1] A mapping $f : X \to Y$ from a topological space X to a topological space Y is called $\beta-$continuous if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists a $\beta-$open set U in X containing x such that $f(U) \subseteq V$.

Theorem 4.8 For a $\beta-$topological vector space E, the following are true:
(a) the translation mapping $f_x : E \to E$ defined by $f_x(y) = x + y$ for all $y \in E$ is $\beta-$continuous,
(b) the mapping $f_\lambda : E \to E$ defined by $f_\lambda(x) = \lambda x$ for all $x \in E$ is $\beta-$continuous, where λ is a fixed scalar.

Proof. (a) Let $y \in E$ and V be an open set in E containing $f_x(y) = x + y$. Then by the definition of $\beta-$topological vector spaces, we get $U,\ U' \in \beta O(E)$ such that $x \in U,\ y \in U'$ and $U + U' \subseteq V$ and consequently, $f_x(U') \subseteq V$. This proves that f_x is $\beta-$continuous.
(b) Let $x \in E$ be an arbitrary. Let W be any open set in E containing λx. Then there exist $\beta-$open sets U in K containing λ and V in E containing x such that $U.V \subseteq W$. Now $\lambda V \subseteq U.V \subseteq W \ \Rightarrow f_\lambda(x) \subseteq W$ and hence f_λ is $\beta-$continuous. ■

Theorem 4.9 Let E_1 be a $\beta-$topological vector space, E_2 be a topological vector space over the same field K. Let $f : E_1 \to E_2$ be a linear map such that f is continuous at 0. Then f is $\beta-$continuous everywhere.

Proof. Let x be any non-zero element of E_1 and V be an open set in E_2 containing $f(x)$. Since translation of an open set in topological vector spaces is open, $V - f(x)$ is open set in E_2 containing 0. Since f is continuous at 0, there exists an open set U in E_1 containing 0 such that $f(U) \subseteq V - f(x)$. Furthermore, linearity of f implies that $f(x+U) \subseteq V$. By Theorem 3.5, $x+U$ is $\beta-$open and hence f is $\beta-$continuous at x. By hypothesis, f is $\beta-$continuous at 0. This reflects that f is $\beta-$continuous. ■

Corollary 4.10 Let E be a $\beta-$topological vector space over the field K. Let $f : E \to K$ be a linear functional which is continuous at 0. Then the set $F = \{x \in E : f(x) = 0\}$ is $\beta-$closed.

Definition 4.11 [2] A topological space X is called $\beta-$compact if every cover of X by $\beta-$open sets of X has a finite subcover. A subset A of X is said to be $\beta-$compact relative to X if every cover of A by $\beta-$open sets of X has a finite subcover.

Theorem 4.12 Let A be any $\beta-$compact set in a $\beta-$topological vector space E. Then $x + A$ is compact for each $x \in E$.

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of $x + A$. Then $A \subseteq \cup_{\alpha\in\Lambda}(-x + U_\alpha)$. By hypothesis and Theorem 3.5, $A \subseteq \cup_{\alpha\in\Lambda_0}(-x + U_\alpha)$ for some finite $\Lambda_0 \subseteq \Lambda$. Whence we find that $x + A \subseteq \cup_{\alpha\in\Lambda_0}U_\alpha$. This shows that $x + A$ is compact. Hence, the proof is complete. ■

Theorem 4.13 Let A be any $\beta-$compact set in a $\beta-$topological vector space E. Then λA is compact for each scalar λ.

Proof. If $\lambda = 0$ we are nothing to prove. Assume that λ is non-zero. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of λA. Then $A \subseteq \cup_{\alpha\in\Lambda}(\frac{1}{\lambda}U_\alpha)$. In view of Theorem 3.1, $\frac{1}{\lambda}U_\alpha$ is $\beta-$open and consequently, by hypothesis, $A \subseteq \cup_{\alpha\in\Lambda_0}(\frac{1}{\lambda}U_\alpha)$ for some finite $\Lambda_0 \subseteq \Lambda$. Whence we find that $\lambda A \subseteq \cup_{\alpha\in\Lambda_0}U_\alpha$. This proves that λA is compact. ■

Acknowledgements

The authors would like to thank the referee and the anonymous reviewers for their valuable comments and suggestions. The second author is supported by UGC-India under the scheme of NET-JRF.

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