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Suzuki-Berinde type fixed-point and fixed-circle results on S-metric spaces

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Abstract. In this paper, the notions of a Suzuki-Berinde type F_S-contraction and a Suzuki-Berinde type F_C^S-contraction are introduced on a S-metric space. Using these new notions, a fixed-point theorem is proved on a complete S-metric space and a fixed-circle theorem is established on a S-metric space. Some examples are given to support the obtained results.

Keywords: S-metric, Suzuki-Berinde type contraction, fixed point, fixed circle.

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1. Introduction

The fixed-point theory was started with the classical Banach contraction principle [2]. This principle has been generalized using different approaches. One of these approaches is to generalize the used contractive conditions (for example, see [3, 4, 8, 18, 20, 25, 26]). Another approach is to generalize the used metric spaces. For example, the concept of S-metric space was introduced for this purpose as a generalization of metric spaces [23]. Using this space, new fixed-point theorems were obtained with various approaches such as generalized Banach's contractive conditions, Rhoades' condition, Wardowski's condition and etc (for more details, see [5–7, 9, 10, 14–16, 19, 21–24]).

Recently, the fixed-circle problem has been considered and studied a new direction of the extensions of the fixed-point results on metric and S-metric spaces. For example, in [13], some fixed-circle theorems were proved using Caristi's inequality with existence and uniqueness conditions on metric spaces. In [12], using a family of some functions, a

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fixed-circle result was given with discontinuity application. Therefore, some fixed-circle results were studied using different approaches on S-metric spaces (see [11, 17]).

Motivated by the above studies, in this paper we prove a fixed-point theorem and a fixed-circle theorem using the Suzuki-Berinde type contractive conditions on S-metric spaces. In Section 2, we recall some definitions, results and examples related to S-metric spaces. In Section 3, we define two new notions of a Suzuki-Berinde type F_S-contraction and a Suzuki-Berinde type F_C^S-contraction. Using these contractive conditions, we present a fixed-point theorem and a fixed-circle theorem with some illustrative examples on S-metric spaces.

2. Preliminaries

In this section, we recall some necessary notions and results about S-metric spaces.

Definition 2.1 [23] Let X be a nonempty set and $\mathcal{S} : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions for all $u, v, w, a \in X$:
$(S1)$ $\mathcal{S}(u, v, w) = 0$ if and only if $u = v = w$,
$(S2)$ $\mathcal{S}(u, v, w) \leqslant \mathcal{S}(u, u, a) + \mathcal{S}(v, v, a) + \mathcal{S}(w, w, a)$.
Then $\mathcal{S}$ is called a S-metric on X and the pair $(X, \mathcal{S})$ is called a S-metric space.

Definition 2.2 [23] Let $(X, \mathcal{S})$ be a S-metric space and $\{u_n\}$ be a sequence in this space.

(1) A sequence $\{u_n\} \subset X$ converges to $u \in X$ if $\mathcal{S}(u_n, u_n, u) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have $\mathcal{S}(u_n, u_n, u) < \varepsilon$.
(2) A sequence $\{u_n\} \subset X$ is a Cauchy sequence if $\mathcal{S}(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geqslant n_0$ we have $\mathcal{S}(u_n, u_n, u_m) < \varepsilon$.
(3) The S-metric space $(X, \mathcal{S})$ is complete if every Cauchy sequence is a convergent sequence.

Lemma 2.3 [23] Let $(X, \mathcal{S})$ be a S-metric space and $u, v \in X$. Then we have

$$\mathcal{S}(u, u, v) = \mathcal{S}(v, v, u).$$

The relationships between a metric and a S-metric were studied in different papers such as [6, 7, 16]. In [7], a formula of a S-metric space which is generated by a metric d was given as follows:

Let (X, d) be a metric space. Then the function $\mathcal{S}_d : X \times X \times X \to [0, \infty)$ defined by $\mathcal{S}_d(u, v, w) = d(u, w) + d(v, w)$ for all $u, v, w \in X$ is a S-metric on X. The S-metric $\mathcal{S}_d$ is called the S-metric generated by d [16]. We note that there exists a S-metric which is not generated by any metric d as seen in the following example.

***Example* 2.4** [16] Let $X = \mathbb{R}$. If we consider the function $\mathcal{S} : X \times X \times X \to [0, \infty)$ defined by $\mathcal{S}(u, v, w) = |u - w| + |u + w - 2v|$ for all $u, v, w \in X$, then $\mathcal{S}$ is a S-metric on X which is not generated by any metric d.

Also in [6], it was shown that every S-metric defines a metric $d_S(u, v) = \mathcal{S}(u, u, v) + \mathcal{S}(v, v, u)$ for all $u, v \in X$. But the function d_S does not always define a metric since the triangle inequality does not satisfied for all elements of X.

***Example* 2.5** [16] Let $X = \{1, 2, 3\}$. If we consider the function $\mathcal{S} : X \times X \times X \to [0, \infty)$

defined by

$$\mathcal{S}(1,1,2) = \mathcal{S}(2,2,1) = 5,$$
$$\mathcal{S}(2,2,3) = \mathcal{S}(3,3,2) = \mathcal{S}(1,1,3) = \mathcal{S}(3,3,1) = 2,$$
$$\mathcal{S}(u,v,w) = 0 \text{ if } u = v = w,$$
$$\mathcal{S}(u,v,w) = 1 \text{ otherwise.}$$

for all $u, v, w \in X$, then $\mathcal{S}$ is a S-metric on X which is not generated by any metric d and does not generate a metric d_S.

Also the relationship between a b-metric defined in [1] and a S-metric was proved in the following theorem.

Theorem 2.6 [21] Let $(X, \mathcal{S})$ be a S-metric space and $d^S(u,v) = \mathcal{S}(u,u,v)$ for all $u, v \in X$. Then we have

(1) d^S is a b-metric on X,
(2) $u_n \to u$ in $(X, \mathcal{S})$ if and only if $u_n \to u$ in (X, d^S),
(3) $\{u_n\}$ is a Cauchy sequence in $(X, \mathcal{S})$ if and only if $\{u_n\}$ is a Cauchy sequence in (X, d^S).

The metric d^S is called the b-metric generated by $\mathcal{S}$. From the above relationships, it was important to study new fixed-point results on S-metric spaces.

3. New fixed-point and fixed-circle results on S-metric spaces

In this section, using the Suzuki-Berinde and Wardowski's techniques, we give a fixed-point theorem and a fixed-circle theorem on S-metric spaces. Some illustrative examples are also presented for the validity of our results. For this purpose, we use the following known family of functions and a lemma.

Let Δ_F be the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions [26]:
(F_1) F is strictly increasing,
(F_2) For all sequence $\{u_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} u_n = 0$ if and only if $\lim_{n\to\infty} F(u_n) = -\infty$,
(F_3) There exists $0 < k < 1$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Lemma 3.1 [20] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be an increasing mapping and $\{u_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then the followings hold:
(a) If $\lim_{n\to\infty} F(u_n) = -\infty$, then $\lim_{n\to\infty} u_n = 0$.
(b) If $\inf F = -\infty$ and $\lim_{n\to\infty} u_n = 0$, then $\lim_{n\to\infty} F(u_n) = -\infty$.

After that, Secelean replaced the condition (F_2) by (F_2') as follows:
(F_2') $\inf F = -\infty$ or (F_2'') There exists a sequence $\{u_n\}_{n=1}^{\infty}$ of positive real numbers such that $\lim_{n\to\infty} F(u_n) = -\infty$.

Further, Piri et al. [18] used the following condition (F_3') instead of the condition (F_3) to obtain some new fixed-point results.
(F_3') F is continuous on $(0, \infty)$.
In the sequel, we consider $\mathcal{F}$ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying in conditions (F_1), (F_2') and (F_3').

At first, we define the notion of Suzuki-Berinde type F_S-contraction on S-metric spaces.

Definition 3.2 Let $(X, \mathcal{S})$ be a S-metric space and $T : X \to X$ be a self-mapping. If there exist $F \in \mathcal{F}$, $\tau_1 > 0$ and $\tau_2 \geqslant 0$ such that for each $u, v \in X$ with $Tu \neq Tv$, we have

$$\frac{1}{3}\mathcal{S}(Tu, Tu, u) < \mathcal{S}(u, u, v)$$

implies

$$\tau_1 + F(\mathcal{S}(Tu, Tu, Tv)) \leqslant F(\mathcal{S}(u, u, v)) + \tau_2 \min\left\{\mathcal{S}(Tu, Tu, u), \mathcal{S}(Tv, Tv, u), \mathcal{S}(Tu, Tu, v)\right\},$$

then T is called a Suzuki-Berinde type F_S-contraction on X.

Using Definition 3.2, we prove the following fixed-point result.

Theorem 3.3 Let $(X, \mathcal{S})$ be a complete S-metric space and $T : X \to X$ be a self-mapping. If T is a Suzuki-Berinde type F_S-contraction on X, then T has a unique fixed point $u \in X$ and the sequence $\{T^n u_0\}$ converges to u for every $u_0 \in X$.

Proof. Let $u_0 \in X$ and the sequence $\{u_n\}$ be defined by $T^n u_0 = u_n$. If there exists $n_0 \in \mathbb{N}$ such that $u_{n_0+1} = u_{n_0}$, then u_{n_0} is a fixed point of T. Therefore, assume that $Tu_n = u_{n+1} \neq u_n$. Now, we have

$$\begin{aligned}\frac{1}{3}\mathcal{S}(Tu_n, Tu_n, u_n) &= \frac{1}{3}\mathcal{S}(u_n, u_n, u_{n+1}) \\ &< \mathcal{S}(u_n, u_n, Tu_n) \\ &= \mathcal{S}(Tu_n, Tu_n, u_n),\end{aligned}$$

for all $n \in \mathbb{N}$. Using the hypothesis, we get

$$\begin{aligned}\tau_1 + F(\mathcal{S}(Tu_n, Tu_n, u_n)) &= \tau_1 + F(\mathcal{S}(Tu_n, Tu_n, Tu_{n-1})) \\ &\leqslant F(\mathcal{S}(u_n, u_n, u_{n-1})) + \tau_2 \min\{\mathcal{S}(Tu_n, Tu_n, u_n), \\ &\quad \mathcal{S}(Tu_{n-1}, Tu_{n-1}, u_n), \mathcal{S}(Tu_n, Tu_n, u_{n-1})\}\end{aligned}$$

and so

$$\begin{aligned}F(\mathcal{S}(Tu_n, Tu_n, u_n)) &\leqslant F(\mathcal{S}(Tu_{n-1}, Tu_{n-1}, u_{n-1})) - \tau_1 \\ &\leqslant \cdots \leqslant F(\mathcal{S}(Tu_0, Tu_0, u_0)) - n\tau_1,\end{aligned}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$, we have $\lim\limits_{n\to\infty} F(\mathcal{S}(Tu_n, Tu_n, u_n)) = -\infty$ and so

$$\lim_{n\to\infty} \mathcal{S}(Tu_n, Tu_n, u_n) = 0, \tag{1}$$

since $F \in \mathcal{F}$. Now, we show that the sequence $\{u_n\}$ is Cauchy. On the contrary, $\{u_n\}$ is not a Cauchy sequence. Suppose that there exists $\varepsilon > 0$ and sequences $\{x(n)\}$ and $\{y(n)\}$ of natural numbers such that for $x(n) > y(n) > n$, we have

$$\mathcal{S}(u_{x(n)}, u_{x(n)}, u_{y(n)}) \geqslant \varepsilon. \tag{2}$$

Therefore, $\mathcal{S}(u_{x(n)-1}, u_{x(n)-1}, u_{y(n)}) < \varepsilon$ for all $n \in \mathbb{N}$. Using the inequality (2), Lemma

2.3 and the condition ($S2$), we obtain

$$\begin{aligned}\varepsilon &\leqslant \mathcal{S}(u_{x(n)}, u_{x(n)}, u_{y(n)}) \\ &\leqslant 2\mathcal{S}(u_{x(n)}, u_{x(n)}, u_{x(n)-1}) + \mathcal{S}(u_{y(n)}, u_{y(n)}, u_{x(n)-1}) \\ &< 2\mathcal{S}(u_{x(n)}, u_{x(n)}, u_{x(n)-1}) + \varepsilon.\end{aligned}$$

Using the equality (1) and taking the limit, we get

$$\lim_{n\to\infty} \mathcal{S}(u_{x(n)}, u_{x(n)}, u_{y(n)}) = \varepsilon. \tag{3}$$

From (1) and (3), we can choose a natural number $n_0 \in \mathbb{N}$ such that

$$\frac{1}{3}\mathcal{S}(Tu_{x(n)}, Tu_{x(n)}, u_{x(n)}) < \frac{\varepsilon}{3} < \mathcal{S}(u_{x(n)}, u_{x(n)}, u_{y(n)}),$$

for all $n \geqslant n_0$. Using the hypothesis, we obtain

$$\begin{aligned}&\tau_1 + F(\mathcal{S}(Tu_{x(n)}, Tu_{x(n)}, Tu_{y(n)})) \\ &\leqslant F(\mathcal{S}(u_{x(n)}, u_{x(n)}, u_{y(n)})) + \tau_2 \min\left\{\begin{array}{c}\mathcal{S}(Tu_{x(n)}, Tu_{x(n)}, u_{x(n)}), \mathcal{S}(Tu_{y(n)}, Tu_{y(n)}, u_{x(n)}), \\ \mathcal{S}(Tu_{x(n)}, Tu_{x(n)}, u_{y(n)})\end{array}\right\} \\ &\leqslant F(\mathcal{S}(u_{x(n)}, u_{x(n)}, u_{y(n)})) + \tau_2 \min\left\{\begin{array}{c}\mathcal{S}(Tu_{x(n)}, Tu_{x(n)}, u_{x(n)}), \\ 2\mathcal{S}(Tu_{y(n)}, Tu_{y(n)}, Tu_{x(n)}) + \mathcal{S}(Tu_{x(n)}, Tu_{x(n)}, u_{x(n)}), \\ 2\mathcal{S}(Tu_{x(n)}, Tu_{x(n)}, u_{x(n)}) + \mathcal{S}(u_{y(n)}, u_{y(n)}, u_{x(n)})\end{array}\right\} \\ &= F(\mathcal{S}(u_{x(n)}, u_{x(n)}, u_{y(n)})) + \tau_2 \mathcal{S}(Tu_{x(n)}, Tu_{x(n)}, u_{x(n)}).\end{aligned}$$

Using the condition (F_3'), and (1) and (3), we get $\tau_1 + F(\varepsilon) \leqslant F(\varepsilon)$, which is a contradiction since $\tau_1 > 0$. Therefore, $\{u_n\}$ is a Cauchy sequence. From the completeness of X, there exists $u \in X$ such that $u_n \to u$ as $n \to \infty$. Thus we have $\lim\limits_{n\to\infty} \mathcal{S}(u_n, u_n, u) = 0$. Now, we claim that

$$\frac{1}{3}\mathcal{S}(Tu_n, Tu_n, u_n) < \mathcal{S}(u_n, u_n, u) \text{ or } \frac{1}{3}\mathcal{S}(T^2u_n, T^2u_n, Tu_n) < \mathcal{S}(Tu_n, Tu_n, u), \tag{4}$$

for all $n \in \mathbb{N}$. On the contrary, we assume that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{3}\mathcal{S}(Tu_m, Tu_m, u_m) \geqslant \mathcal{S}(u_m, u_m, u) \text{ and } \frac{1}{3}\mathcal{S}(T^2u_m, T^2u_m, Tu_m) \geqslant \mathcal{S}(Tu_m, Tu_m, u). \tag{5}$$

Thus, using Lemma 2.3, we get

$$\begin{aligned}3\mathcal{S}(u_m, u_m, u) &\leqslant \mathcal{S}(Tu_m, Tu_m, u_m) \\ &= \mathcal{S}(u_m, u_m, Tu_m) \\ &\leqslant 2\mathcal{S}(u_m, u_m, u) + \mathcal{S}(Tu_m, Tu_m, u),\end{aligned}$$

which implies

$$\mathcal{S}(u_m, u_m, u) \leqslant \mathcal{S}(Tu_m, Tu_m, u). \tag{6}$$

From the inequalities (5) and (6), we obtain

$$\mathcal{S}(u_m, u_m, u) \leqslant \mathcal{S}(Tu_m, Tu_m, u) \leqslant \frac{1}{3}\mathcal{S}(T^2u_m, T^2u_m, Tu_m). \tag{7}$$

Also using the hypothesis and Lemma 2.3, we have

$$\begin{aligned}\frac{1}{3}\mathcal{S}(Tu_m, Tu_m, u_m) &= \frac{1}{3}\mathcal{S}(u_m, u_m, u_{m+1})\\ &< \mathcal{S}(u_m, u_m, u_{m+1})\\ &= \mathcal{S}(Tu_m, Tu_m, u_m)\end{aligned}$$

and

$$\begin{aligned}\tau_1 + F(\mathcal{S}(T^2u_m, T^2u_m, Tu_m)) &= \tau_1 + F(\mathcal{S}(Tu_m, Tu_m, T^2u_m))\\ &\leqslant F(\mathcal{S}(u_m, u_m, Tu_m))\\ &\quad + \tau_2 \min\left\{ \begin{array}{c} \mathcal{S}(u_m, u_m, Tu_m), \mathcal{S}(u_m, u_m, T^2u_m),\\ \mathcal{S}(Tu_m, Tu_m, Tu_m) \end{array} \right\}\\ &= F(\mathcal{S}(Tu_m, Tu_m, u_m)),\end{aligned}$$

which implies

$$\tau_1 + F(\mathcal{S}(T^2u_m, T^2u_m, Tu_m)) \leqslant F(\mathcal{S}(Tu_m, Tu_m, u_m)). \tag{8}$$

From the inequality (8), then we have

$$F(\mathcal{S}(T^2u_m, T^2u_m, Tu_m)) < F(\mathcal{S}(Tu_m, Tu_m, u_m)),$$

and therefore, using strictly increasing property of F we obtain that

$$\mathcal{S}(T^2u_m, T^2u_m, Tu_m) < \mathcal{S}(Tu_m, Tu_m, u_m) \tag{9}$$

Using the inequalities (5), (7) and (9), we obtain

$$\begin{aligned}\mathcal{S}(T^2u_m, T^2u_m, Tu_m) &< \mathcal{S}(Tu_m, Tu_m, u_m)\\ &\leqslant 2\mathcal{S}(Tu_m, Tu_m, u) + \mathcal{S}(u_m, u_m, u)\\ &< \frac{2}{3}\mathcal{S}(T^2u_m, T^2u_m, Tu_m) + \frac{1}{3}\mathcal{S}(T^2u_m, T^2u_m, Tu_m)\\ &= \mathcal{S}(T^2u_m, T^2u_m, Tu_m),\end{aligned}$$

which is a contradiction. Therefore, the inequalities given in (4) are satisfied. So using Lemma 2.3, for each $n \in \mathbb{N}$, we get

$$\tau_1 + F(\mathcal{S}(Tu_n, Tu_n, Tu)) \leqslant F(\mathcal{S}(u_n, u_n, u)) + \tau_2 \min\left\{ \begin{array}{c} \mathcal{S}(u_n, u_n, Tu_n), \mathcal{S}(u_n, u_n, Tu),\\ \mathcal{S}(u, u, Tu_n) \end{array} \right\},$$

which implies

$$\tau_1 + F(\mathcal{S}(Tu_n, Tu_n, Tu)) \leqslant F(\mathcal{S}(u_n, u_n, u)) + \tau_2 \min \left\{ \begin{array}{c} \mathcal{S}(u_n, u_n, u_{n+1}), \mathcal{S}(u_n, u_n, Tu), \\ \mathcal{S}(u, u, u_{n+1}) \end{array} \right\}. \tag{10}$$

Using (10), the condition (F_2') and Lemma 3.1, we obtain $\lim_{n\to\infty} F(\mathcal{S}(Tu_n, Tu_n, Tu)) = -\infty$ and $\lim_{n\to\infty} \mathcal{S}(Tu_n, Tu_n, Tu) = 0$. Hence, we have

$$\mathcal{S}(u, u, Tu) = \lim_{n\to\infty} \mathcal{S}(u_{n+1}, u_{n+1}, Tu) = \lim_{n\to\infty} \mathcal{S}(Tu_n, Tu_n, Tu) = 0$$

and so u is a fixed point of T. Finally, we show that u is a unique fixed point of T. On the contrary, v is another fixed point of T such that $u \neq v$. Then we have $\mathcal{S}(Tu, Tu, Tv) = \mathcal{S}(u, u, v) > 0$ and

$$\frac{1}{3}\mathcal{S}(Tu, Tu, u) = 0 < \mathcal{S}(u, u, v).$$

Using the hypothesis, we obtain

$$\begin{aligned} F(\mathcal{S}(u, u, v)) &= F(\mathcal{S}(Tu, Tu, Tv)) \\ &< \tau_1 + F(\mathcal{S}(Tu, Tu, Tv)) \\ &\leqslant F(\mathcal{S}(u, u, v)) + \tau_2 \min \left\{ \begin{array}{c} F(\mathcal{S}(u, u, Tu)), F(\mathcal{S}(u, u, Tv)), \\ F(\mathcal{S}(v, v, Tu)) \end{array} \right\}, \end{aligned}$$

which implies $F(\mathcal{S}(u, u, v)) < F(\mathcal{S}(u, u, v))$. Now, using the strictly increasing property of F, we get $\mathcal{S}(u, u, v) < \mathcal{S}(u, u, v)$, which is a contradiction. Therefore, u is a unique fixed point of T. ■

Now we give the following illustrative example.

***Example* 3.4** Let us consider the sequence $\{A_n\}$ defined as $A_n = 2 + 4 + \cdots + 2n = n(n+1)$. Let $X = \{A_n : n \in \mathbb{N}\}$ and the function $\mathcal{S} : X \times X \times X \to [0, \infty)$ be defined as in Example 2.4. Then $(X, \mathcal{S})$ is a complete S-metric space and the S-metric is not generated by any metric. Let us consider the self-mapping $T : X \to X$ defined by

$$Tu = \left\{ \begin{array}{ll} A_1 & u = A_1 \\ A_{n-1} & u = A_n \text{ for } (n > 1) \end{array} \right.$$

for all $u \in X$. If we take the mapping $F(t) = -\frac{1}{t} + t$, $\tau_1 = 4$ and $\tau_2 = 0$, then T is a Suzuki-Berinde type F_S-contraction on X. Indeed, we show this under the following cases:
Case 1. Let $1 = n < m$. Then we have

$$\mathcal{S}(TA_m, TA_m, TA_1) = 2\,|TA_m - TA_1| = 2\,|A_{m-1} - A_1| = 2\,[4 + 6 + \cdots + 2(m-1)]$$

and

$$\mathcal{S}(A_m, A_m, A_1) = 2\,|A_m - A_1| = 2\,[4 + 6 + \cdots + 2m]\,.$$

Since $m > 1$, so we get

$$4 - \frac{1}{2\left[4+6+\cdots+2(m-1)\right]} + 2\left[4+6+\cdots+2(m-1)\right]$$
$$< -\frac{1}{2\left[4+6+\cdots+2m\right]} + 2\left[4+6+\cdots+2(m-1)+2m\right].$$

Hence, we have

$$4 - \frac{1}{2\left|TA_m - TA_1\right|} + 2\left|TA_m - TA_1\right| < -\frac{1}{2\left|A_m - A_1\right|} + 2\left|A_m - A_1\right|.$$

Case 2. By the similar arguments used in above, we get

$$4 - \frac{1}{2\left|TA_m - TA_1\right|} + 2\left|TA_m - TA_1\right| < -\frac{1}{2\left|A_m - A_1\right|} + 2\left|A_m - A_1\right|$$

for $1 \leqslant m < n$.
Case 3. Let $1 < n < m$. Then we get

$$\begin{aligned}\mathcal{S}(TA_m, TA_m, TA_n) &= 2\left|TA_m - TA_n\right| = 2\left|A_{m-1} - A_{n-1}\right| \\ &= 2\left[2n + 2(n+1) + \cdots + 2(m-1)\right]\end{aligned}$$

and

$$\mathcal{S}(A_m, A_m, A_n) = 2\left|A_m - A_n\right| = 2\left[2(n+1) + 2(n+2) + \cdots + 2m\right].$$

Since $1 < n < m$ and $4n + 4 \leqslant 4m$, we have

$$4 - \frac{1}{2\left[2n + 2(n+1) + \cdots + 2(m-1)\right]} + 2\left[2n + 2(n+1) + \cdots + 2(m-1)\right]$$
$$< -\frac{1}{2\left[2(n+1) + 2(n+2) + \cdots + 2m\right]} + 2\left[2(n+1) + 2(n+2) + \cdots + 2(m-1) + 2m\right].$$

So we get

$$4 - \frac{1}{2\left|TA_m - TA_n\right|} + 2\left|TA_m - TA_n\right| < -\frac{1}{2\left|A_m - A_n\right|} + 2\left|A_m - A_n\right|.$$

Therefore, T is a Suzuki-Berinde type F_S-contraction and $TA_1 = A_1$; that is, A_1 is a unique fixed point of T.

If we take $\tau_2 = 0$ then we get the following corollaries.

Corollary 3.5 Let $(X, \mathcal{S})$ be a complete S-metric space and $T : X \to X$ be a self-mapping. If there exist $\tau_1 > 0$ and $F \in \mathcal{F}$ such that for each $u, v \in X$ with $Tu \neq Tv$, we have $\frac{1}{3}\mathcal{S}(Tu, Tu, u) < \mathcal{S}(u, u, v)$ implies $\tau_1 + F(\mathcal{S}(Tu, Tu, Tv)) \leqslant F(\mathcal{S}(u, u, v))$, then T has a unique fixed point $u \in X$ and the sequence $\{T^n u_0\}$ converges to u for every $u_0 \in X$.

Corollary 3.6 Let $(X, \mathcal{S})$ be a complete S-metric space and $T : X \to X$ be a self-mapping. If there exist $\tau_1 > 0$ and $F \in \mathcal{F}$ such that for each $u, v \in X$ with $Tu \neq Tv$, we have $\tau_1 + F(\mathcal{S}(Tu, Tu, Tv)) \leqslant F(\mathcal{S}(u, u, v))$, then T has a unique fixed point $u \in X$ and the sequence $\{T^n u_0\}$ converges to u for every $u_0 \in X$.

If we consider Theorem 2.6, then we get the Suzuki-Berinde type fixed-point theorem on b-metric spaces.

Theorem 3.7 Let (X, d^S) be a complete b-metric space and $T : X \to X$ be a self-mapping. If there exist $F \in \mathcal{F}$, $\tau_1 > 0$ and $\tau_2 \geqslant 0$ such that for each $u, v \in X$ with $Tu \neq Tv$, we have $\dfrac{1}{3} d^S(Tu, u) < d^S(u, v)$ implies

$$\tau_1 + F(d^S(Tu, Tv)) \leqslant F(d^S(u, v)) + \tau_2 \min \left\{ d^S(Tu, u), d^S(Tv, u), d^S(Tu, v) \right\},$$

then T has a unique fixed point $u \in X$ and the sequence $\{T^n u_0\}$ converges to u for every $u_0 \in X$.

Proof. By the similar arguments used in the proof of Theorem 3.3, it is clear. ■

In [11] and [23], a circle and a disc are defined on a S-metric space as follows, respectively:

$$C^S_{u_0,r} = \{u \in X : \mathcal{S}(u, u, u_0) = r\} \text{ and } D^S_{u_0,r} = \{x \in X : \mathcal{S}(u, u, u_0) \leqslant r\}.$$

Definition 3.8 [11] Let $(X, \mathcal{S})$ be a S-metric space, $C^S_{u_0,r}$ be a circle and $T : X \to X$ be a self-mapping. If $Tu = u$ for every $u \in C^S_{u_0,r}$ then the circle $C^S_{u_0,r}$ is called as the fixed circle of T.

We introduce the notion of Suzuki-Berinde type F^S_C-contraction on S-metric spaces.

Definition 3.9 Let $(X, \mathcal{S})$ be a S-metric space and $T : X \to X$ be a self-mapping. T is called a Suzuki-Berinde type F^S_C-contraction on X if there exist $F \in \mathcal{F}$, $\tau_1 > 0$, $\tau_2 \geqslant 0$ and $u_0 \in X$ such that for each $u \in X$ with $Tu \neq u$, we have $\frac{1}{3}\mathcal{S}(u, u, u_0) < \mathcal{S}(Tu, Tu, u)$ implies

$$\begin{aligned} \tau_1 + F(\mathcal{S}(Tu, Tu, u)) \leqslant F(\mathcal{S}(u, u, u_0)) + \tau_2 \min\{&\mathcal{S}(Tu_0, Tu_0, u_0), \\ &\mathcal{S}(Tu_0, Tu_0, u), \mathcal{S}(Tu, Tu, u_0)\}. \end{aligned}$$

Using Definition 3.9, we obtain the following proposition.

Proposition 3.10 Let $(X, \mathcal{S})$ be a S-metric space and $T : X \to X$ be a self-mapping. If T is a Suzuki-Berinde type F^S_C-contraction with $u_0 \in X$ then we have $Tu_0 = u_0$.

Proof. Assume that $Tu_0 \neq u_0$. From the definition of the Suzuki-Berinde type F^S_C-contraction, we get $\dfrac{1}{3}\mathcal{S}(u_0, u_0, u_0) < \mathcal{S}(Tu_0, Tu_0, u_0)$ and so

$$\begin{aligned} \tau_1 + F(\mathcal{S}(Tu_0, Tu_0, u_0)) &\leqslant F(\mathcal{S}(u_0, u_0, u_0)) + \tau_2 \min \left\{ \begin{array}{c} \mathcal{S}(Tu_0, Tu_0, u_0), \mathcal{S}(Tu_0, Tu_0, u_0), \\ \mathcal{S}(Tu_0, Tu_0, u_0) \end{array} \right\} \\ &= F(0) + \tau_2 \mathcal{S}(Tu_0, Tu_0, u_0), \end{aligned}$$

which is a contradiction with the definition of F. Therefore, we obtain $Tu_0 = u_0$. ■

Now we prove the fixed-circle theorem.

Theorem 3.11 Let $(X, \mathcal{S})$ be a S-metric space, T be a self-mapping on X satisfying the Suzuki-Berinde type F_C^S-contractive condition with $u_0 \in X$ and $r = \min\{\mathcal{S}(Tu, Tu, u) : Tu \neq u\}$. If $\mathcal{S}(Tu, Tu, u_0) = r$ for all $u \in C_{u_0,r}^S$ then $C_{u_0,r}^S$ is a fixed circle of T. Especially, T fixes every circle $C_{u_0,\rho}^S$ with $\rho < r$.

Proof. Let $u \in C_{u_0,r}^S$ and $Tu \neq u$. By the definition of r, we have $\frac{1}{3}\mathcal{S}(u, u, u_0) = \frac{r}{3} < \mathcal{S}(Tu, Tu, u)$. Now, using the Suzuki-Berinde type F_C^S-contractive property, Proposition 3.10, Lemma 2.3 and the strictly increasing property of F, we obtain

$$\begin{aligned} F(\mathcal{S}(Tu, Tu, u)) &\leqslant F(\mathcal{S}(u, u, u_0)) - \tau_1 + \tau_2 \min \left\{ \begin{array}{c} \mathcal{S}(Tu_0, Tu_0, u_0), \mathcal{S}(u, u, u_0), \\ \mathcal{S}(Tu, Tu, u_0) \end{array} \right\} \\ &= F(r) - \tau_1 \\ &< F(r) \\ &\leqslant F(\mathcal{S}(Tu, Tu, u)), \end{aligned}$$

which is a contradiction. Therefore, we find $Tu = u$ and so $C_{u_0,r}^S$ is a fixed circle of T.

Finally, we show that T also fixes any circle $C_{u_0,\rho}^S$ with $\rho < r$. Let $u \in C_{u_0,\rho}^S$ and suppose that $Tu \neq u$. By the Suzuki-Berinde type F_C^S-contractive property, we have

$$F(\mathcal{S}(Tu, Tu, u)) \leqslant F(\mathcal{S}(u, u, u_0)) - \tau_1 < F(\rho) \leqslant F(\mathcal{S}(Tu, Tu, u)),$$

which is a contradiction. Hence we get $Tu = u$. Thus, $C_{u_0,\rho}^S$ is a fixed circle of T. ■

As an immediate result of Theorem 3.11, we obtain the following corollary.

Corollary 3.12 Let $(X, \mathcal{S})$ be a S-metric space, T be a self-mapping on X satisfying the Suzuki-Berinde type F_C^S-contractive condition with $u_0 \in X$ and $r = \min\{\mathcal{S}(Tu, Tu, u) : Tu \neq u\}$. If $\mathcal{S}(Tu, Tu, u_0) = r$ for all $u \in C_{u_0,r}^S$ then T fixes the disc $D_{u_0,r}^S$.

***Example* 3.13** Let $X = \left\{1, 2, \frac{5}{2}, e - \frac{1}{2}, e, e + \frac{1}{2}\right\}$ and the S-metric be defined as in Example 2.4. Then $(X, \mathcal{S})$ is a S-metric space. Let us define the self-mapping $T : X \to X$ as

$$Tu = \begin{cases} \frac{5}{2} & u = 2 \\ u & \text{otherwise} \end{cases}$$

for all $u \in X$. Then the self-mapping T is a Suzuki-Berinde type F_C^S-contraction with $F = \ln u$, $u_0 = e$, $\tau_1 = 0.5$ and $\tau_2 \geqslant 0$. Indeed, for $u = 2$, we get

$$\begin{aligned} \frac{1}{3}\mathcal{S}(u, u, u_0) &= \frac{2e - 4}{3} < \mathcal{S}(Tu, Tu, u) = 1 \\ &\Rightarrow \mathcal{S}(Tu, Tu, u) = 1 < \mathcal{S}(u, u, u_0) = 2e - 4 \\ &\Rightarrow \ln(1) < \ln(2e - 4) = \ln(2(e - 4)) \\ &\Rightarrow 0.5 < \ln 2 + \ln(e - 4) \\ &\Rightarrow \tau_1 + F(\mathcal{S}(Tu, Tu, u)) \leqslant F(\mathcal{S}(u, u, u_0)) \\ &\quad + \tau_2 \min\{\mathcal{S}(Tu_0, Tu_0, u_0), \mathcal{S}(Tu_0, Tu_0, u), \mathcal{S}(Tu, Tu, u_0)\}. \end{aligned}$$

Using Theorem 3.11, we get $r = \min\{\mathcal{S}(Tu, Tu, u) : Tu \neq u\} = 1$. It is clear that T fixes the circle $C_{e,1}^S = \left\{e - \frac{1}{2}, e + \frac{1}{2}\right\}$ and the disc $D_{e,1}^S = \left\{\frac{5}{2}, e - \frac{1}{2}, e, e + \frac{1}{2}\right\}$.

If we consider Theorem 2.6, then we get the Suzuki-Berinde type fixed-circle theorem on b-metric spaces.

Theorem 3.14 Let (X, d^S) be a b-metric space, $T : X \to X$ be a self-mapping and $r = \min\{d^S(Tu, u) : Tu \neq u\}$. If there exist $F \in \mathcal{F}$, $\tau_1 > 0$, $\tau_2 \geqslant 0$ and $u_0 \in X$ such that for each $u \in X$ with $Tu \neq u$, we have $\frac{1}{3}d^S(u, u_0) < d^S(Tu, u)$ implies

$$\tau_1 + F(d^S(Tu, u)) \leqslant F(d^S(u, u_0)) + \tau_2 \min\{d^S(Tu_0, u_0), d^S(Tu_0, u), d^S(Tu, u_0)\},$$

then $C^{d^S}_{u_0,r} = \{u \in X : d^S(u, u_0) = r\}$ is a fixed circle of T with the condition $d^S(Tu, u_0) = r$. Especially, T fixes every circle $C^{d^S}_{u_0,\rho}$ with $\rho < r$, that is, T fixes the disc $D^{d^S}_{u_0,r} = \{u \in X : d^S(u, u_0) \leqslant r\}$.

4. Conclusion

In this section, we prove a Suzuki-Berinde fixed-point theorem and a Suzuki-Berinde fixed-circle theorem using the Suzuki-Berinde and Wardowski's techniques on S-metric spaces. Similarly, new fixed-point or fixed-circle results can be obtained using or modified the known techniques used in some fixed-point theorems on metric and some generalized metric spaces.

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