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Linear Čech closure spaces

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Abstract. In this paper, we introduce the concept of linear Čech closure spaces and establish the properties of open sets in linear Čech closure spaces (LČCS). Here, we observe that the concept of linearity is preserved by semi-open sets, g-semi open sets, γ-open sets, sgc-dense sets and compact sets in LČCS. We also discuss the concept of relative Čech closure operator, meet and product linear Čech closure operators. Lastly, we describe the Moore class on the LČCS and prove that it is a vector lattice with sufficient properties.

Keywords: Linear Čech closure spaces, semi-open sets, g-semi open sets, γ-open sets, sgc-dense sets, relative Čech closure operator, Moore class, vector lattice.

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1. Introduction

Closure spaces were introduced by Čech [3] and then studied by many authors like Khampakdee [6], Boonpok [2], Roth [8] and etc. Čech closure spaces is a generalization of the concept of topological spaces. Čech described continuity in closure spaces by means of neighborhoods, nets and etc. Roth and Carlson [8] studied a number of separation properties in closure spaces. Thron studied some separation properties in closure spaces. Sunitha [9] studied higher separation properties in closure spaces. Chattopadhyay [4] developed an extension theory of arbitrary closure spaces. The concepts of generalized closed sets and generalized continuous maps of topological spaces were extended to closure spaces by Boonpok and Khampakdee [2].

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In this paper, we introduce and study the notion of linear Čech closure spaces. In section 2, we quote the necessary preliminaries about Čech closure spaces, semi-open sets, g-semi open sets, γ-open sets, sgc-dense sets, relative Čech closure operator, Moore class and etc. Section 3 deals with linear Čech closure spaces (LČCS) together with its characterization. In Section 4, we discuss the linearity of semi-open sets, g-semi open sets, γ-open sets, sgc-dense sets and compact sets in a LČCS . Section 5 describes some operations on LČCS like relative Čech closure operator, meet and product of closure operators. In the last section, we proved the main result that the Moore class in an idempotent T_1 LČCS is a vector lattice.

2. Preliminaries

Definition 2.1 [3] Let X be a set and $\wp(X)$ be its powerset. A function $c : \wp(X) \to \wp(X)$ is called a Čech closure operator for X, if

(1) $c(\phi) = \phi$,
(2) $A \subseteq c(A)$ for all $A \subseteq X$,
(3) $c(A \cup B) = c(A) \cup c(B)$ for all $A, B \subseteq X$.

Then, (X, c) is called Čech closure space or simply closure space.

If in addition $c(c(A)) = c(A)$ for all $A \subseteq X$, then the space (X, c) is called a Kuratowski (topological) space. Further, if for any family of subsets of X such as $\{A_i\}_{(i\in I)}$, $c(\cup_{i\in I} A_i) = \cup_{i\in I} c(A_i)$, then the space is called a total closure space.

Definition 2.2 [3] A function $c : \wp(X) \to \wp(X)$ is called a monotone operator for X if

(1) $c(\phi) = \phi$,
(2) $A \subseteq c(A)$ for all $A \subseteq X$,
(3) $A \subseteq B \Rightarrow c(A) \subseteq c(B)$ for all $A, B \subseteq X$.

Then (X, c) is called monotone space.

Note that a subset A of a closure space (X, c) will be closed, if $c(A) = A$ and open, if its complement is closed, i.e. if $c(X - A) = X - A$. If (X, c) is a closure space, we denote the associated topology on X by t, i.e. $t = \{A^c : c(A) = A\}$.

Definition 2.3 [9] A map $f : (X, c) \to (Y, c')$ is said to be a $c - c'$ morphism or just a morphism, if $f(c(A)) \subseteq c' f(A)$.

Remark 1 *[3]*

(1) A mapping f of a closure space (X, c) onto another one (Y, c') is a $c-c'$ morphism at a point $x \in X$ if and only if the inverse image $f^{-1}(V)$ of each neighborhood V of $f(x)$ is a neighborhood of x.
(2) If f is a $c - c'$ morphism of a space (X, c) into a space (Y, c'), then the inverse image of each open subset of Y is an open subset of X.
(3) If $f : (X, c) \to (Y, c')$ is a morphism, then $f : (X, t) \to (X, t')$ is continuous.

Definition 2.4 [9] A homeomorphism is a bijective mapping f such that both f and f^{-1} are morphisms.

Definition 2.5 Let $\{(X_i, c_i) : i \in I\}$ be a family of closure spaces, X be the product of the family $\{X_i\}$ of underlying sets and π_i be the projection of X onto X_i for each $i \in I$. Then the product closure c is the coarsest closure on the product of underlying sets such that all the projections are morphisms.

Lemma 2.6 [3] A Čech closure space is a monotone space.

Definition 2.7 [8] Let (X, c) be a Čech closure space. If $c(A) = A$ for all $A \subseteq X$, then c is called the discrete closure operator on X. If $c(A) = X$ for all $A \subseteq X$, then c is called the trivial operator or indiscrete operator on X.

In a Čech closure space (X, c), c is finitely generated, if $c(A) = \cup\{c(a) : a \in A\}$ for any subset A of X.

Definition 2.8 Let (X, c) be a closure space and A be an arbitrary subset of X. Then the Čech closure operator c_A defined by $c_A(B) = A \cap c(B)$ is called the relative Čech closure operator on A induced by c.

The pair (A, c_A) is said to be a closure subspace of (X, c). It is a closed (open) subspace if A is closed (open) in (X, c).

Definition 2.9 [9] A closure space (X, c) is said to be compact, if every interior cover of X has a finite subcover.

Remark 2 *[9]*

(1) Any image under a c-morphism of a compact space (X, c) is compact.
(2) If (Y, c) is a compact subspace of a Hausdorff closure space (X, c), then Y is closed in (X, c).
(3) Every closed subspace of a compact closure space is compact.
(4) If (X, c) is compact and $Y \subseteq X$, then $c(Y)$ is compact.

3. Linear Čech closure spaces

Definition 3.1 Let V be a vector space and c be a closure operator on V such that

(1) $c(A) + c(B) \subseteq c(A + B), \forall A, B \subset V$,
(2) $\lambda c(A) \subseteq c(\lambda A), \forall A \subset V$ and for all scalars λ,

Then, c is called a Linear Čech Closure Operator(LČCO) and (V, c) is called a linear Čech closure space (LČCS).

***Example* 3.2** The discrete Čech closure space defined on a vector space is a linear Čech closure space. The indiscrete Čech closure space defined on a vector space is a linear Čech closure space.

Proposition 3.3 Let V be a vector space and c be a closure operator on V. Then (V, c) is a linear Čech closure space if and only if $+ : (V \times V, c \times c) \to (V, c)$ and $\lambda\cdot : (V, c) \to (V, c)$ for all $\lambda \in K$ are morphisms, where $(V \times V, c \times c)$ is the product closure space.

Proof. If (V, c) is a linear Čech closure space, it is clear from the axioms of LČCS that the mappings $+ : (V \times V, c \times c) \to (V, c)$ and $\lambda\cdot : (V, c) \to (V, c)$ for all $\lambda \in K$ are morphisms. Conversely, assume that the mappings $+ : (V \times V, c \times c) \to (V, c)$ and $\lambda\cdot : (V, c) \to (V, c)$ for all $\lambda \in K$ are morphisms, where V is a vector space and (V, c) is a closure space. Let $A, B \subseteq V$. Then $A \times B \subseteq V \times V$. Also,

$$\begin{aligned} c(A) + c(B) &= +[c(A) \times c(B)], \qquad \text{by the definition of } + \\ &= +[c \times c(A \times B)], \qquad \text{by the definition of closure product space} \\ &\subseteq c(A + B). \qquad \text{by the definition of morphism} \end{aligned}$$

Similarly, $\lambda c(A) \subseteq c(\lambda A)$. ■

Proposition 3.4 Let (V, c) be a LČCS. Then, the map $T_a : (V, c) \to (V, c)$ given by $T_a(x) = a + x$ and $M_\lambda : (V, c) \to (V, c)$ given by $M_\lambda(x) = \lambda x$ are homeomorphisms for all scalar $\lambda \neq 0$.

Proof. Since $+ : (V \times V, c \times c) \to (V, c)$ is a morphism, T_a is a morphism and T_{-a} is the inverse morphism for T_a, hence T_a is a homeomorphism. Similarly M_λ is a morphism if $\lambda \neq 0$ and $M_{1/\lambda}$ is the inverse morphism for M_λ. ■

Proposition 3.5 Let (V, c) be a LČCS. If $A \subseteq V$, then $a + A$ is open for all $a \in V$ if and only if A is open.

Proof. By the above proposition T_a is a homeomorphism for all $a \in V$. So if A is open, the inverse image of A under T_{-a} (i.e. $a + A$) is open. Again, if $a + A$ is open, then A is open (since $T_a(A) = a + A$). ■

Proposition 3.6 Let (V, c) be a LČCS. Then

(1) for every neighborhood W of 0 (the identity element of V), there exists neighborhoods V_1 and V_2 of 0 such that $V_1 + V_2 \subseteq W$;
(2) for every neighborhood W of x, there exist a neighborhood V_1 of 0 and V_2 of x such that $V_1 + V_2 \subseteq W$, which further imply that $V_1 + x \subseteq W$.

Proof. 1. Since $0 + 0 = 0$ and $+$ is a morphism, there exists neighborhoods V_1 and V_2 of 0, such that $V_1 + V_2 \subseteq W$.
2. Proof follows directly from 1. ■

4. Linearity of certain subsets of LČCS

Proposition 4.1 If A is a subset of V and B is an open subset of V, then $A + B$ is open. Also, λA for all scalar $\lambda \neq 0$ is open if and only if A is open.

Proof. Since B is open, $a + B$ is open for all $a \in A$. $A + B = \cup\{a + B : a \in A\}$ is open being arbitrary union of open sets. By the homeomorphism M_λ, A is open if and only if λA is open. ■

Proposition 4.2 Compact sets in a LČCS preserve linearity (i.e. if A and B are compact sets in a LČCS, then $A + B$ and λA are compact sets).

Proof. Since A and B are compact, $A \times B$ is compact in $(V \times V, c \times c)$. Then $A + B$ is compact, being the image of a compact set under a morphism. Similarly, λA is also compact. ■

Proposition 4.3 If H is a subspace of a vector space V and (V, c) is a LČCS, then $c(H)$ is a subspace of V.

Proof. Since H is a subspace of V, $H + H = H$ and $\lambda H = H$. Now, $c(H) + c(H) \subseteq c(H + H) = c(H)$ and $\lambda c(H) \subseteq c(\lambda H) = c(H)$. Thus, $c(H)$ is a subspace of V. ∎

Definition 4.4 [6] Let (X, c) be a closure space. A subset A of X is called a semi open set, if there exists an open set G in (X, c) such that $G \subseteq A \subseteq c(G)$. A subset $A \subseteq X$ is called a semi-closed set if its complement is semi-open.

Definition 4.5 [6] A subset B of a closure space (V, c) is called generalized semi-open or g-semiopen, if there exists a semi-open subset A of (V, c) such that $A \subseteq B \subseteq c(A)$.

Proposition 4.6 Let (V, c) be a LČCS. If A is any subset of V and B is semi-open (g-semiopen) in V, then $A + B$ is semi-open (g-semiopen).

Proof. Since B is semi-open, there exists an open set G in V such that, $G \subseteq B \subseteq c(G)$. By Proposition 4.1, $A + G$ is open and

$$\begin{aligned} A + G \subseteq A + B &\subseteq A + c(G) && (\text{ since } B \subseteq c(G)) \\ &\subseteq c(A) + c(G) && (\text{ since } A \subseteq c(A)) \\ &\subseteq c(A + G) && (\text{ since } c \text{is a LČCO }). \end{aligned}$$

Hence, $A + B$ is semi-open. The proof in the case of g-semiopen follows similarly. ∎

Corollary 4.7 Linearity is preserved by semi-open sets and g-semiopen sets in a LČCS.

Proof. By the above theorem, sum of two semi-open (g-semiopen) sets in a LČCS is again semi-open (g-semiopen). Also, if A is semi-open in a LČCS, then there exists an open set G in V such that $G \subseteq A \subseteq c(G)$, which implies $\lambda G \subseteq \lambda A \subseteq \lambda c(G)$. Since c is a LČCO, $\lambda c(G) \subseteq c(\lambda G)$. Thus, $\lambda G \subseteq \lambda A \subseteq c(\lambda G)$, where λG is open by Proposition 4.1. Hence, λA is also semi-open. Proof in the case of g-semi-open sets is similar to that of semi-open sets. ∎

Definition 4.8 [6] A set B in a closure space (V, c) is said to be γ-open if there exists an open subset G of V such that $G \subseteq B$ and $c(G) = c(B)$. A subset B of V is γ-closed if its complement is γ-open.

Proposition 4.9 Let (L, c) be an idempotent LČCS. If A and B are γ-open subsets of (L, c), then $A + B$ and λA are also γ-open.

Proof. Since A and B are γ-open there exists two open sets G_1 and G_2 such that $G_1 \subseteq A$, $c(A) = c(G_1)$, $G_2 \subseteq B$ and $c(B) = c(G_2)$. We have $A \subseteq c(A)$ and $B \subseteq c(B)$. Thus, $G_1 \subseteq A \subseteq c(A) = c(G_1)$ and $G_2 \subseteq B \subseteq c(B) = c(G_2)$. Then $G_1 + G_2 \subseteq A + B \subseteq c(A) + c(B) = c(G_1) + c(G_2)$. Hence, $A + B \subseteq c(G_1) + c(G_2) \subseteq c(G_1 + G_2)$ and $c(A + B) \subseteq c[c(G_1 + G_2)] = c(G_1 + G_2)$. Since c is an idempotent closure operator, we have $cc(A) = c(A), \forall A \subseteq G$. Now, $G_1 \subseteq c(A)$ and $G_2 \subseteq c(B)$ implies $G_1 + G_2 \subseteq c(A) + c(B) \subseteq c(A + B)$. Then, $c(G_1 + G_2) \subseteq c(c(A + B)) = c(A + B)$, which implies that $c(G_1 + G_2) = c(A + B)$. Thus, $G_1 + G_2 \subseteq A + B \subseteq c(A + B) = c(G_1 + G_2)$, where $(G_1 + G_2)$ is open, showing that $A + B$ is γ-open. Also, since $G_1 \subseteq A \subseteq c(A) = c(G_1)$, $\lambda G_1 \subseteq \lambda A \subseteq \lambda c(A) = \lambda c(G_1)$. But $\lambda c(G_1) \subseteq c(\lambda G_1)$. Thus, $\lambda A \subseteq c(\lambda G_1)$ and so, $c(\lambda A) \subseteq c(c(\lambda G_1)) = c(\lambda G_1)$. On the other hand, $\lambda G_1 \subseteq \lambda c(A) \subseteq c(\lambda A)$. Since c is an idempotent linear Čech closure operator, $c(\lambda G_1) \subseteq c(c(\lambda A)) = c(\lambda A)$. Hence, $c(\lambda G_1) = c(\lambda A)$. Thus, λA is γ-open. ∎

Definition 4.10 [4] A non-empty subset D of V will be called sgc-dense in (V, c) if $c(D) = X$.

Proposition 4.11 Let (V, c) be a linear Čech closure space, A be a sgc-dense subset of V, B an arbitrary subset of V. Then, $A + B$ and $\lambda A(\lambda \neq 0)$ are sgc-dense in V.

Proof. Since A is a sgc-dense subset of V, $c(A) = V$. Now, $c(A) + c(B) \subseteq c(A + B)$, i.e. $V + c(B) \subseteq c(A+B)$, i.e., $V \subseteq c(A+B)$. Since $A, B \subseteq V$, $A+B \subseteq V$ and $c(A+B) \subseteq V$. Thus, $c(A + B) = V$; that is, $A + B$ is sgc-dense. Also, $\lambda c(A) \subseteq c(\lambda A)$, i.e., $\lambda V \subseteq c(\lambda A)$ with $\lambda \neq 0$. Since $V \subseteq c(\lambda A)$, $V = c(\lambda A)$. Hence, λA is sgc-dense. Thus sgc-dense sets preserves linearity in a LČCS. ■

5. Operations on Linear Čech Closure Operators

Proposition 5.1 The composition of two linear Čech closure operators is again a linear Čech closure operator.

Proof. Let c_1 and c_2 be two linear Čech closure operators on a vector space V. Then $(c_1 \circ c_2)(\phi) = c_1(c_2(\phi)) = c_1(\phi) = \phi$. Also, $A \subseteq c_2(A) \subseteq c_1(c_2(A)) = (c_1 \circ c_2)(A)$. Moreover, $(c_1 \circ c_2)(A \cup B) = c_1(c_2(A \cup B)) = c_1(c_2(A) \cup c_2(B)) = c_1(c_2(A)) \cup c_1(c_2(B)) = (c_1 \circ c_2)(A) \cup (c_1 \circ c_2)(B)$. Also, since $(c_1 \circ c_2)(A) + (c_1 \circ c_2)(B) = c_1(c_2(A)) + c_1(c_2(B)) \subseteq c_1(c_2(A) + c_2(B))$ and $c_2(A) + c_2(B) \subseteq c_2(A + B)$ implies that $c_1(c_2(A) + c_2(B)) \subseteq c_1(c_2(A+B))$, we have $(c_1 \circ c_2)(A) + (c_1 \circ c_2)(B) \subseteq (c_1 \circ c_2)(A+B)$. Further, $\lambda(c_1 \circ c_2)(A) = \lambda(c_1(c_2(A)) \subseteq c_1(\lambda c_2(A)) \subseteq c_1(c_2(\lambda A)) = (c_1 \circ c_2)(\lambda A)$. Thus, the composition of two linear Čech closure operators is a linear Čech closure operator. ■

Proposition 5.2 Let (V, c) be a LČCS and A is a subspace of the vector space V, then the closure operator c_A is a linear closure operator on A.

Proof. Let $H, K \subset A$ and $x + y \in c_A(H) + c_A(K)$. Then $x \in c_A(H)$ and $y \in c_A(K)$, and i.e., $x \in A \cap c(H)$ and $y \in A \cap c(K)$. Thus, $x + y \in c(H) + c(K)$ and $x + y \in A$. Since $c(H) + c(K) \subseteq c(H + K)$, so $x + y \in A \cap c(H + K) = c_A(H + K)$. Hence, $c_A(H) + c_A(K) \subseteq c_A(H + K)$. Similarly, $\lambda c_A(H) \subseteq c_A(\lambda H)$. Hence, c_A is a LČCO on A. ■

Proposition 5.3 Let (X_1, c_1) and (X_2, c_2) be two disjoint closure spaces and $X = X_1 \times X_2$. Then the product closure operator $c = c_1 \otimes c_2$ on X defined by $c(A) = c_1(p_1(A)) \times c_2(p_2(A))$ is a LČC operator, where $p_1(A)$ and $p_2(A)$ are the projections of A onto X_1 and X_2, respectively.

Proof. We have

$$
\begin{aligned}
c(A) + c(B) &= \prod_{i=1}^{2} c_i(p_i(A)) + \prod_{i=1}^{2} c_i(p_i(B)) \\
&= \prod_{i=1}^{2} [c_i(p_i(A)) + c_i(p_i(B))] \\
&\subseteq \prod_{i=1}^{2} c_i(p_i(A) + p_i(B)) \\
&= \prod_{i=1}^{2} c_i(p_i(A + B)) = c(A + B).
\end{aligned}
$$

Similarly, $\lambda c(A) \subseteq c(\lambda A)$. ■

Proposition 5.4 Let V be a vector space and c_1 and c_2 be two linear Čech closure operators on V. Then the meet closure operator $c : \wp(X) \to \wp(X)$ defined by $c(A) = c_1(A) \cap c_2(A)$ is a linear Čech closure operator.

Proof. If $A, B \subseteq V$, then

$$\begin{aligned} c(A) + c(B) &= c_1(A) \cap c_2(A) + c_1(B) \cap c_2(B) \\ &\subseteq (c_1(A) + c_1(B)) \cap (c_2(A) + c_2(B)) \\ &\subseteq c_1(A + B) \cap c_2(A + B) = c(A + B). \end{aligned}$$

Similarly, $\lambda c(A) \subseteq c(\lambda A)$. ■

6. Moore class in a LČCS

Proposition 6.1 Let (V, c) be a T_1 LČCS. The Moore class of closed sets in V, i.e., $M_c = \{A \subseteq V \mid A = c(A)\}$ is a vector space.

Proof. Since addition and scalar multiplication are homeomorphisms, if A and B are closed, then $A + B$ and λA are closed for any scalar $\lambda \neq 0$. Since all singletons are closed in a T_1 LČCS, then $\{0\}$ is closed and it is the zero element in M_c. Now, Let $A \in M_c$. Then $A = c(A)$. Also, $-c(-A) \subseteq c(A) = A$ implies that $c(-A) \subseteq -A$. Thus, $-A = c(-A)$. Hence, $-A \in M_c$. Thus M_c is a vector space. ■

Definition 6.2 An ordered vector space is a real vector space E which is also an ordered space with the linear and order structures connected by the implications

(1) If $x, y, z \in E$ and $x \leqslant y$ then $x + z \leqslant y + z$,
(2) If $x, y \in E$, $x \leqslant y$ and $0 \leqslant \alpha \in R$ then $\alpha x \leqslant \alpha y$.

The set $E_+ = \{x \in E : x \geqslant 0\}$ is termed the positive cone in E and its elements are termed positive (rather than non-negative). An ordered vector space which is also a lattice is a vector lattice or Riesz space.

Proposition 6.3 The Moore class M_c in an idempotent T_1 LČCS is a vector lattice with inclusion order.

Proof. Venkateswarlu et el. [11] has already proved that in any idempotent closure space M_c is a complete lattice. Now to prove it is a vector lattice, let $A, B \in M_c$ and $A \subseteq B$. Then, clearly, $A + C \subseteq B + C$ for all $C \in M_c$. Also, if $0 \leqslant \alpha$ and $\alpha A \subseteq \alpha B$, then M_c is a vector lattice. ■

Remark 3

(1) The positive cone of M_c is $M_{c+} = \{B \subseteq X \mid \{0\} \subseteq B\}$, i.e., the set of all closed sets containing the element 0.
(2) The set of all closed subspaces belong to the positive cone.
(3) If $B \in M_c$, the positive part of B is

$$B^+ = \begin{cases} B & \textit{if } 0 \in B \\ B \cup \{0\} & \textit{if } 0 \notin B \end{cases}$$

and the negative part of B is

$$B^{-} = \begin{cases} -B & \text{if } 0 \in B \\ -B \cup \{0\} & \text{if } 0 \notin B \end{cases}$$

Also, $\mid B \mid = B \cup -B$.

(4) *The set of all closed subspaces form a meet lattice as the intersection of two closed subspaces is again a closed subspace.*

7. Conclusion

The notions of closure system and closure operator are very useful tools in several areas of mathematics. They play an important role in the study of topological spaces, Boolean algebras and convex sets. Also, the theory of generalized closure spaces has been found very important and useful in the study of image analysis. So, linear Čech closure spaces is a relevant concept in Čech closure spaces which possess significant properties that are used in several fields of applications.

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