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On some forms of e^*-irresoluteness

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Abstract. The main goal of this paper is to introduce and study two new class of functions, called weakly e^*-irresolute functions and strongly e^*-irresolute functions, via the notion of e^*-open set defined by Ekici [7]. We obtain several fundamental properties and characterizations of these functions. Moreover, we investigate not only some of their basic properties but also their relationships with other types of already existing topological functions.

Keywords: e^*-open, e^*-θ-open, weakly e^*-irresolute, strongly e^*-irresolute, strongly e^*-regular space.

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1. Introduction

In 1972, Crossley et al. [3] introduced the notion of irresolute functions in topological spaces. Then the class of α-irresolute functions were introduced by Maheshwari et al. [12]. In the sequel, the class of semi α-irresolute functions [2] (resp. almost α-irresolute functions [1], b-irresolute functions [9], weakly B-irresolute functions [15], β-irresolute functions [13], weakly β-irresolute functions [14], e-irresolute functions [4], a-irresolute functions [4]) were introduced.

In [7], Ekici introduced the notions of e^*-open sets and e^*-continuity in topological spaces. Then Hatır and Noiri [10] defined and investigated δ-β-open sets which are equivalent to e^*-open sets. In [4], Ekici defined an e^*-irresolute function as follows: A

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function $f : X \to Y$ is said to be e^*-irresolute if the inverse image $f^{-1}[V]$ is e^*-open in X for each e^*-open set V of Y. In this paper, we introduce and investigate the concepts of weakly e^*-irresolute functions and strongly e^*-irresolute functions. Also, we obtain several characterizations and study some fundamental properties of these classes of functions.

2. Preliminaries

Throughout the present paper, X and Y always mean topological spaces. Consistent with the content of [5–7, 11, 16], the following definition will be needed in the sequel. Let X be a topological space and A a subset of X. The closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. A subset A is said to be regular open (resp. regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The δ-interior of a subset A of X is the union of all regular open sets of X contained in A and is denoted by $int_\delta(A)$. The subset A is called δ-open if $A = int_\delta(A)$, in other words, a set is δ-open if it is the union of regular open sets. The complement of a δ-open set is called δ-closed. A point $x \in X$ is called a δ-cluster points of A if $A \cap int(cl(V)) \neq \emptyset$ for each open set V containing x. The set of all δ-cluster points of A is called the closure of A and is denoted by $cl_\delta(A)$(or δ-$cl(A)$).

A subset A of a space X is called e^*-open (resp. e-open, a-open) if $A \subseteq cl(int(cl_\delta(A)))$ (resp. $A \subseteq int(cl_\delta(A)) \cup cl(int_\delta(A))$, $A \subseteq int(cl(int_\delta(A)))$). The complement of an e^*-open set is called e^*-closed. The intersection of all e^*-closed sets containing A is called the e^*-closure of A and is denoted by e^*-$cl(A)$. The union of all e^*-open sets of X contained in A is called the e^*-interior of A and is denoted by e^*-$int(A)$.

A subset A of a space X is called e^*-regular if it is e^*-open and e^*-closed. A subset A is said to be e^*-θ-closed if $A = e^*\text{-}cl_\theta(A)$, where $e^*\text{-}cl_\theta(A) := \{x|(\forall U \in e^*O(X,x))(e^*\text{-}cl(U) \cap A \neq \emptyset)\}$. The complement of an e^*-θ-closed set is said to be e^*-θ-open. Equivalently, A is said to be e^*-θ-open if $A = e^*\text{-}int_\theta(A)$, where $e^*\text{-}int_\theta(A) := \{x|(\exists U \in e^*O(X,x))(e^*\text{-}cl(U) \subseteq A)\}$.

The family of all e^*-open (resp. e^*-closed, e^*-regular, e^*-θ-open, e^*-θ-closed, a-open) subsets of X is denoted by $e^*O(X)$ (resp. $e^*C(X)$, $e^*R(X)$, $e^*\theta O(X)$, $e^*\theta C(X)$, $aO(X)$). The family of all e^*-open (e^*-closed, e^*-regular, e^*-θ-open, e^*-θ-closed) sets of X containing a point x of X is denoted by $e^*O(X,x)$ (resp. $e^*C(X,x)$, $e^*R(X,x)$, $e^*\theta O(X,x), e^*\theta C(X,x)$).

We shall use the well-known accepted language almost in the whole of the proofs of theorems in this article.

Theorem 2.1 [11] Let A be a subset of a topological space X. Then the following hold:
(1) $A \in e^*O(X)$ if and only if $e^*\text{-}cl(A) \in e^*R(X)$,
(2) $A \in e^*C(X)$ if and only if $e^*\text{-}int(A) \in e^*R(X)$.

Theorem 2.2 [11] For a subset A of a topological space X, the following are equivalent:
(1) $A \in e^*R(X)$;
(2) $A = e^*\text{-}cl(e^*\text{-}int(A))$;
(3) $A = e^*\text{-}int(e^*\text{-}cl(A))$.

Theorem 2.3 [11] For each subset A of a topological space X, we have

$$e^*\text{-}cl_\theta(A) = \bigcap\{V|A \subseteq V,\ V \in e^*\theta C(X)\} = \bigcap\{V|A \subseteq V,\ V \in e^*R(X)\}.$$

Theorem 2.4 [11] Let A and B be any two subsets of a topological space X. Then the

following properties hold:
(1) $x \in e^*\text{-}cl_\theta(A)$ if and only if $U \cap A \neq \emptyset$ for each $U \in e^*R(X,x)$,
(2) If $A \subseteq B$, then $e^*\text{-}cl_\theta(A) \subseteq e^*\text{-}cl_\theta(B)$,
(3) $e^*\text{-}cl_\theta(e^*\text{-}cl_\theta(A)) = e^*\text{-}cl_\theta(A)$,
(4) If A_λ is e^*-θ-closed in X for each $\lambda \in \triangle$, then $\bigcap_{\lambda\in\triangle} A_\lambda$ is e^*-θ-closed in X.

Corollary 2.5 [11] Let A and $A_\lambda(\lambda \in \triangle)$ be any subsets of topological space X. Then the following properties hold:
(1) A is e^*-θ-open in X if and only if for each $x \in A$ there exists $U \in e^*R(X,x)$ such that $x \in U \subseteq A$,
(2) $e^*\text{-}cl_\theta(A)$ is e^*-θ-closed and $e^*\text{-}int_\theta(A)$ is e^*-θ-open,
(3) If A_λ is e^*-θ-open in X for each $\lambda \in \triangle$, then $\bigcup_{\lambda\in\triangle} A_\lambda$ is e^*-θ-open in X.

Theorem 2.6 [11] For a subset A of a space X, the following properties hold:
(1) If $A \in e^*O(X)$, then $e^*\text{-}cl(A) = e^*\text{-}cl_\theta(A)$,
(2) $A \in e^*R(X)$ if and only if A is e^*-θ-open and e^*-θ-closed.

Lemma 2.7 [7] Let A be a subset of a topological space X. Then the following hold:
(1) $e^*\text{-}int(X \setminus A) = X \setminus e^*\text{-}cl(A)$,
(2) $e^*\text{-}cl(X \setminus A) = X \setminus e^*\text{-}int(A)$,
(3) $e^*\text{-}cl(A) = A \cup int(cl(int_\delta(A)))$.

Definition 2.8 A function $f : X \to Y$ is said to be
(1) e^*-irresolute [4] if $f^{-1}[V] \in e^*O(X)$ for each $V \in e^*O(Y)$,
(2) e^*-continuous [7] if $f^{-1}[V] \in e^*O(X)$ for every open set V of Y,
(3) almost e^*-continuous [8] if for each point $x \in X$ and each open set V containing $f(x)$, there exists $U \in e^*O(X,x)$ such that $f[U] \subseteq int(cl(V))$,
(4) strongly θ-e^*-continuous [11] if for each point $x \in X$ and each open set V containing $f(x)$, there exists $U \in e^*O(X,x)$ such that $f[e^*\text{-}cl(U)] \subseteq V$,
(5) e-continuous [6] if $f^{-1}[V]$ is e-open in X for every open set V of Y,
(6) almost e-continuous [8] (resp. almost a-continuous [8]) if $f^{-1}[V]$ is e-open (resp. a-open [5]) in X for every regular open set V of Y.

3. Characterizations of weakly e^*-irresolute functions

Definition 3.1 A function $f : X \to Y$ is said to be weakly e^*-irresolute (resp. strongly e^*-irresolute) if for each point $x \in X$ and each $V \in e^*O(Y, f(x))$, there exists $U \in e^*O(X,x)$ such that $f[U] \subseteq e^*\text{-}cl(V)$ (resp. $f[e^*\text{-}cl(U)] \subseteq V$).

Remark 1 *We have the following diagram from the definitions stated above. However, none of these implications is reversible as shown by the following examples.*

$$\begin{array}{ccccc}
\textit{strongly } e^*\textit{-irresolute} & \rightarrow & e^*\textit{-irresolute} & \rightarrow & \textit{weakly } e^*\textit{-irresolute} \\
\downarrow & & \downarrow & & \downarrow \\
\textit{strongly } \theta\textit{-}e^*\textit{-continuity} & \rightarrow & e^*\textit{-continuity} & \rightarrow & \textit{almost } e^*\textit{-continuity} \\
 & \nearrow & & \nearrow & \uparrow \\
e\textit{-continuity} & \rightarrow & \textit{almost } e\textit{-continuity} & \leftarrow & \textit{almost } a\textit{-continuity}
\end{array}$$

Example 3.2 Let $X := \{a, b, c, d\}$ and

$$\tau := \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}\}.$$

It is not difficult to see that $e^*O(X) = e^*\theta O(X) = 2^X \setminus \{\{d\},\{b,d\}\}$ and $e^*R(X) = 2^X \setminus \{\{d\},\{b,d\},\{a,c\},\{a,b,c\}\}$. Define a function $f : X \to X$ such that $f = \{(a,a),(b,c),(c,a),(d,d)\}$. Then f is both strongly θ-e^*-irresolute and almost e^*-continuous but it is neither strongly e^*-irresolute nor weakly e^*-irresolute.

Theorem 3.3 Let $f : X \to Y$ be a function. The following properties are equivalent:
(a) f is weakly e^*-irresolute;
(b) $f^{-1}[V] \subseteq e^*\text{-}int(f^{-1}[e^*\text{-}cl(V)])$ for every $V \in e^*O(Y)$;
(c) $e^*\text{-}cl(f^{-1}[e^*\text{-}int(F)]) \subseteq f^{-1}[F]$ for every $F \in e^*C(Y)$;
(d) $e^*\text{-}cl(f^{-1}[V]) \subseteq f^{-1}[e^*\text{-}cl(V)]$ for every $V \in e^*O(Y)$;
(e) $f^{-1}[e^*\text{-}int(F)] \subseteq e^*\text{-}int(f^{-1}[F])$ for every $F \in e^*C(Y)$.

Proof. $(a) \Rightarrow (b)$: Let $V \in e^*O(Y)$ and $x \in f^{-1}[V]$.

$$\left.\begin{array}{r}(V \in e^*O(Y))(x \in f^{-1}[V]) \Rightarrow V \in e^*O(Y, f(x)) \\ (a)\end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in e^*O(X,x))(f[U] \subseteq e^*\text{-}cl(V))$
$\Rightarrow (\exists U \in e^*O(X,x))(U \subseteq f^{-1}[e^*\text{-}cl(V)])$
$\Rightarrow (\exists U \in e^*O(X,x))(U = e^*\text{-}int(U) \subseteq e^*\text{-}int(f^{-1}[e^*\text{-}cl(V)]))$
$\Rightarrow x \in e^*\text{-}int(f^{-1}[e^*\text{-}cl(V)])$.
$(b) \Rightarrow (c)$: It is obvious from Lemma 2.7.
$(c) \Rightarrow (d)$: Let $V \in e^*O(Y)$.

$$\left.\begin{array}{r}V \in e^*O(Y) \overset{\text{Theorem 2.1}}{\Rightarrow} e^*\text{-}cl(V) \in e^*R(Y) \subseteq e^*C(Y) \\ (c)\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow e^*\text{-}cl(f^{-1}[e^*\text{-}int(e^*\text{-}cl(V))]) = e^*\text{-}cl(f^{-1}[e^*\text{-}cl(V)]) \subseteq f^{-1}[e^*\text{-}cl(V)] \\ e^*\text{-}cl(f^{-1}[V]) \subseteq e^*\text{-}cl(f^{-1}[e^*\text{-}cl(V)])\end{array}\right\} \Rightarrow$$

$\Rightarrow e^*\text{-}cl(f^{-1}[V]) \subseteq f^{-1}[e^*\text{-}cl(V)]$.
$(d) \Rightarrow (e)$: It is obvious from Lemma 2.7.
$(e) \Rightarrow (a)$: Let $x \in X$ and $V \in e^*O(Y, f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in e^*O(Y, f(x))) \Rightarrow Y \setminus V \in e^*C(Y) \\ (e)\end{array}\right\} \Rightarrow$$

$\Rightarrow f^{-1}[e^*\text{-}int(Y \setminus V)] \subseteq e^*\text{-}int\left(f^{-1}[Y \setminus V]\right)$
$\overset{\text{Lemma 2.7}}{\Rightarrow} X \setminus f^{-1}[e^*\text{-}cl(V)] \subseteq X \setminus e^*\text{-}cl\left(f^{-1}[V]\right)$
$\Rightarrow e^*\text{-}cl(f^{-1}[V]) \subseteq f^{-1}[e^*\text{-}cl(V)]$ $(*)$
$V \in e^*O(Y, f(x)) \overset{\text{Theorem 2.1}}{\Rightarrow} e^*\text{-}cl(V) \in e^*R(Y, f(x))$

$$\left.\begin{array}{r}\Rightarrow x \notin f^{-1}[e^*\text{-}cl(Y \setminus e^*\text{-}cl(V))] \\ (*)\end{array}\right\} \Rightarrow x \notin e^*\text{-}cl\left(f^{-1}[Y \setminus e^*\text{-}cl(V)]\right)$$

$\Rightarrow (\exists U \in e^*O(X,x))\left(U \cap f^{-1}[Y \setminus e^*\text{-}cl(V)] = \emptyset\right)$
$\Rightarrow (\exists U \in e^*O(X,x))\left(U \cap \left(f^{-1}[Y] \setminus f^{-1}[e^*\text{-}cl(V)]\right) = \emptyset\right)$
$\Rightarrow (\exists U \in e^*O(X,x))\left(U \cap \left(X \setminus f^{-1}[e^*\text{-}cl(V)]\right) = \emptyset\right)$
$\Rightarrow (\exists U \in e^*O(X,x))\left(f[U] \subseteq f\left[f^{-1}[e^*\text{-}cl(V)]\right] \subseteq e^*\text{-}cl(V)\right)$. ■

Theorem 3.4 Let $f : X \to Y$ be a function. The following properties are equivalent:
(a) f is weakly e^*-irresolute;
(b) $e^*\text{-}cl(f^{-1}[B]) \subseteq f^{-1}[e^*\text{-}cl_\theta(B)]$ for every subset B of Y;
(c) $f[e^*\text{-}cl(A)] \subseteq e^*\text{-}cl_\theta(f[A])$ for every subset A of X;
(d) $f^{-1}[F] \in e^*C(X)$ for every e^*-θ-closed set F of Y;
(e) $f^{-1}[V] \in e^*O(X)$ for every e^*-θ-open set V of Y.

Proof. $(a) \Rightarrow (b)$: Let $B \subseteq Y$ and $x \notin f^{-1}[e^*\text{-}cl_\theta(B)]$.

$$\left.\begin{array}{r} x \notin f^{-1}[e^*\text{-}cl_\theta(B)] \Rightarrow f(x) \notin e^*\text{-}cl_\theta(B) \Rightarrow (\exists V \in e^*O(Y, f(x)))(e^*\text{-}cl(V) \cap B = \emptyset) \\ (a) \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in e^*O(X, x))(f[U] \cap B \subseteq e^*\text{-}cl(V) \cap B = \emptyset)$

$\Rightarrow (\exists U \in e^*O(X, x))(U \cap f^{-1}[B] = \emptyset)$

$\Rightarrow x \notin e^*\text{-}cl(f^{-1}[B])$.

$(b) \Rightarrow (c)$: Let $A \subseteq X$.

$$\left.\begin{array}{r} A \subseteq X \Rightarrow f[A] \subseteq Y \\ (b) \end{array}\right\} \Rightarrow e^*\text{-}cl(A) \subseteq e^*\text{-}cl(f^{-1}[f[A]]) \subseteq f^{-1}[e^*\text{-}cl_\theta(f[A])]$$

$\Rightarrow f[e^*\text{-}cl(A)] \subseteq e^*\text{-}cl_\theta(f[A])$.

$(c) \Rightarrow (d)$: Let $F \in e^*\theta C(Y)$.

$$\left.\begin{array}{r} F \in e^*\theta C(Y) \Rightarrow (e^*\text{-}cl_\theta(F) = F)(f^{-1}[F] \subseteq X) \\ (c) \end{array}\right\} \Rightarrow$$

$\Rightarrow f[e^*\text{-}cl(f^{-1}[F])] \subseteq e^*\text{-}cl_\theta(f[f^{-1}[F]]) \subseteq e^*\text{-}cl_\theta(F) = F$

$$\left.\begin{array}{r} \Rightarrow e^*\text{-}cl(f^{-1}[F]) \subseteq f^{-1}[F] \\ f^{-1}[F] \subseteq e^*\text{-}cl(f^{-1}[F]) \end{array}\right\} \Rightarrow f^{-1}[F] = e^*\text{-}cl(f^{-1}[F]) \Rightarrow f^{-1}[F] \in e^*C(X).$$

$(d) \Rightarrow (e)$: Obvious.

$(e) \Rightarrow (a)$: Let $x \in X$ and $V \in e^*O(Y, f(x))$.

$$\left.\begin{array}{r} V \in e^*O(Y, f(x)) \Rightarrow e^*\text{-}cl(V) \in e^*R(Y) \Rightarrow e^*\text{-}cl(V) \in e^*\theta O(Y) \\ (e) \end{array}\right\} \Rightarrow$$

$\Rightarrow (U := f^{-1}[e^*\text{-}cl(V)] \in e^*O(X, x))(f[U] \subseteq e^*\text{-}cl(V))$. ■

Theorem 3.5 Let $f : X \to Y$ be a function. The following properties are equivalent:
(a) f is weakly e^*-irresolute;
(b) For each $x \in X$ and each $V \in e^*O(Y, f(x))$, there exists $U \in e^*O(X, x)$ such that $f[e^*\text{-}cl(U)] \subseteq e^*\text{-}cl(V)$;
(c) $f^{-1}[F] \in e^*R(X)$ for every $F \in e^*R(Y)$.

Proof. $(a) \Rightarrow (b)$: Let $x \in X$ and $V \in e^*O(Y, f(x))$.

$$\left.\begin{array}{r} V \in e^*O(Y, f(x)) \Rightarrow e^*\text{-}cl(V) \in e^*R(Y, f(x)) \Rightarrow e^*\text{-}cl(V) \in e^*\theta O(Y, f(x)) \cap e^*\theta C(Y, f(x)) \\ \text{Theorem 3.4 (d),(e)} \end{array}\right\} \Rightarrow$$

$\Rightarrow (U := f^{-1}[e^*\text{-}cl(V)] \in e^*R(X, x))(f[e^*\text{-}cl(U)] \subseteq e^*\text{-}cl(V))$.

$(b) \Rightarrow (c)$: Let $F \in e^*R(Y)$ and $x \in f^{-1}[F]$.

$$\left.\begin{array}{r} (F \in e^*R(Y))(x \in f^{-1}[F]) \Rightarrow F \in e^*R(Y, f(x)) \\ (b) \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in e^*O(X, x))(f[e^*\text{-}cl(U)] \subseteq e^*\text{-}cl(F) = F)$

$\Rightarrow (\exists U \in e^*O(X, x))(U \subseteq e^*\text{-}cl(U) \subseteq f^{-1}[F])$

$\Rightarrow x \in e^*\text{-}int(f^{-1}[F])$

Then we have $f^{-1}[F] \in e^*O(X)$ $(*)$

$$\left.\begin{array}{r} F \in e^*R(Y) \Rightarrow Y \setminus F \in e^*R(Y) \Rightarrow f^{-1}[Y \setminus F] \in e^*O(X) \\ f^{-1}[Y \setminus F] = X \setminus f^{-1}[F] \end{array}\right\} \Rightarrow f^{-1}[F] \in e^*C(X) \ \ (**)$$

$(*), (**) \Rightarrow f^{-1}[F] \in e^*R(X)$.

$(c) \Rightarrow (a)$: Let $x \in X$ and $V \in e^*O(Y, f(x))$.

$$\left.\begin{array}{r} V \in e^*O(Y, f(x)) \Rightarrow e^*\text{-}cl(V) \in e^*R(Y, f(x)) \\ (c) \end{array}\right\} \Rightarrow$$

$\Rightarrow (U := f^{-1}[e^*\text{-}cl(V)] \in e^*R(X, x))(f[U] \subseteq e^*\text{-}cl(V))$. ■

Theorem 3.6 Let $f : X \to Y$ be a function. The following properties are equivalent:
(a) f is weakly e^*-irresolute;
(b) $f^{-1}[V] \subseteq e^*\text{-}int_\theta(f^{-1}[e^*\text{-}cl_\theta(V)])$ for every $V \in e^*O(Y)$;
(c) $e^*\text{-}cl_\theta(f^{-1}[V]) \subseteq f^{-1}[e^*\text{-}cl_\theta(V)]$ for every $V \in e^*O(Y)$.

Proof. The proof is similar to Theorems 3.4 and 3.5. Hence, it is omitted. ∎

Theorem 3.7 Let $f : X \to Y$ be a function. The following properties are equivalent:
(a) f is weakly e^*-irresolute;
(b) $e^*\text{-}cl_\theta(f^{-1}[B]) \subseteq f^{-1}[e^*\text{-}cl_\theta(B)]$ for every subset B of Y;
(c) $f[e^*\text{-}cl_\theta(A)] \subseteq e^*\text{-}cl_\theta(f[A])$ for every subset A of X;
(d) $f^{-1}[F]$ is e^*-θ-closed in X for every e^*-θ-closed set F of Y;
(e) $f^{-1}[V]$ is e^*-θ-open in X for every e^*-θ-open set V of Y.

Proof. The proof is similar to Theorems 3.4 and 3.5. Hence, it is omitted. ∎

4. Some properties of weakly e^*-irresolute functions

In this section, we investigate some fundamental properties of weakly e^*-irresolute functions.

Definition 4.1 A topological space X is said to be strongly e^*-regular if for each $F \in e^*C(X)$ and each $x \notin F$, there exist disjoint e^*-open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 4.2 Let X be a topological space. Then the following properties are equivalent:
(a) X is strongly e^*-regular;
(b) For each $U \in e^*O(X)$ and each $x \in U$, there exists $V \in e^*O(X, x)$ such that $e^*\text{-}cl(V) \subseteq U$;
(c) For each $U \in e^*O(X)$ and each $x \in U$, there exists $V \in e^*R(X, x)$ such that $V \subseteq U$.

Proof. $(a) \Rightarrow (b)$: Let $U \in e^*O(X)$ and $x \in U$.

$$\left.\begin{array}{r} U \in e^*O(X,x) \Rightarrow x \notin X \setminus U \in e^*C(X) \\ (a) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists V \in e^*O(X,x))(\exists W \in e^*O(X))(X \setminus U \subseteq W)(V \cap W = \emptyset)$$
$$\Rightarrow (\exists V \in e^*O(X,x))(V \subseteq e^*\text{-}cl(V) \subseteq X \setminus W \subseteq U).$$

$(b) \Rightarrow (c)$: Let $U \in e^*O(X)$ and $x \in U$.

$$\left.\begin{array}{r} U \in e^*O(X,x) \\ (b) \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\exists W \in e^*O(X,x))(e^*\text{-}cl(W) \subseteq U) \\ V := e^*\text{-}cl(W) \end{array}\right\} \Rightarrow (V \in e^*R(X,x))(V \subseteq U).$$

$(c) \Rightarrow (a)$: Let $F \in e^*C(X)$ and $x \notin F$.

$$\left.\begin{array}{r} x \notin F \in e^*C(X) \Rightarrow X \setminus F \in e^*O(X,x) \\ (c) \end{array}\right\} \Rightarrow (\exists V \in e^*R(X,x))(V \subseteq X \setminus F)$$
$$\Rightarrow (\exists V \in e^*O(X,x))(\exists W := X \setminus V \in e^*O(X))(F \subseteq W)(V \cap W = \emptyset).$$
∎

Theorem 4.3 Let Y be a strongly e^*-regular space. Then a function $f : X \to Y$ is weakly e^*-irresolute if and only if it is e^*-irresolute.

Proof. Necessity. Let $V \in e^*O(Y)$ and $x \in f^{-1}[V]$.

$$\left.\begin{array}{r} (V \in e^*O(Y))(x \in f^{-1}[V]) \Rightarrow V \in e^*O(Y, f(x)) \\ Y \text{ is strongly } e^*\text{-regular} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\exists W \in e^*O(Y, f(x)))(e^*\text{-}cl(W) \subseteq V) \\ f \text{ is weakly } e^*\text{-irresolute} \end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in e^*O(X,x))(f[U] \subseteq e^*\text{-}cl(W) \subseteq V)$$
$$\Rightarrow (\exists U \in e^*O(X,x))(U \subseteq f^{-1}[V])$$
$$\Rightarrow x \in e^*\text{-}int(f^{-1}[V])$$

Then we have $f^{-1}[V] \in e^*O(X)$. Therefore f is e^*-irresolute.
Sufficiency. It is obvious. ∎

Lemma 4.4 For subsets A and B of topological spaces X and Y, respectively, the following properties hold:
(a) $int_\delta(A \times B) = int_\delta(A) \times int_\delta(B)$,
(b) $cl_\delta(A \times B) = cl_\delta(A) \times cl_\delta(B)$,
(c) $e^*\text{-}cl(X \times B) = X \times e^*\text{-}cl(B)$,
(d) If $A \in e^*O(X)$ and $B \in e^*O(Y)$, then $A \times B \in e^*O(X \times Y)$.

Proof. *(a)* Let $(x, y) \in int_\delta(A \times B)$.
$(x, y) \in int_\delta(A \times B) \Rightarrow (\exists U \in \mathcal{U}(x, y))(int(cl(U)) \subseteq A \times B)$
$\Rightarrow (\exists \mathcal{A}_1 \subseteq \tau_1)(\exists \mathcal{A}_2 \subseteq \tau_2) \left(U = \bigcup_{(A_1 \in \mathcal{A}_1)(A_2 \in \mathcal{A}_2)} (A_1 \times A_2)\right) (int(cl(U)) \subseteq A \times B)$
$\Rightarrow (\exists \mathcal{A}_1 \subseteq \tau_1)(\exists \mathcal{A}_2 \subseteq \tau_2) \left(int\left(cl\left(\bigcup_{(A_1 \in \mathcal{A}_1)(A_2 \in \mathcal{A}_2)} (A_1 \times A_2)\right)\right) \subseteq A \times B\right)$
$\Rightarrow (\exists \mathcal{A}_1 \subseteq \tau_1)(\exists \mathcal{A}_2 \subseteq \tau_2) \left(int\left(\bigcup_{(A_1 \in \mathcal{A}_1)(A_2 \in \mathcal{A}_2)} cl(A_1 \times A_2)\right) \subseteq A \times B\right)$
$\Rightarrow (\exists \mathcal{A}_1 \subseteq \tau_1)(\exists \mathcal{A}_2 \subseteq \tau_2) \left(\bigcup_{(A_1 \in \mathcal{A}_1)(A_2 \in \mathcal{A}_2)} int(cl(A_1 \times A_2)) \subseteq A \times B\right)$
$\Rightarrow (\exists \mathcal{A}_1 \subseteq \tau_1)(\exists \mathcal{A}_2 \subseteq \tau_2) \left(\bigcup_{(A_1 \in \mathcal{A}_1)(A_2 \in \mathcal{A}_2)} [int(cl(A_1)) \times int(cl(A_2))] \subseteq A \times B\right)$
$\Rightarrow (\exists U_1 \in \mathcal{U}(x))(\exists U_2 \in \mathcal{U}(y)) ([int(cl(U_1)) \times int(cl(U_2))] \subseteq A \times B)$
$\Rightarrow (\exists U_1 \in \mathcal{U}(x))(int(cl(U_1)) \subseteq A)(\exists U_2 \in \mathcal{U}(y))(int(cl(U_2)) \subseteq B)$
$\Rightarrow (x \in int_\delta(A))(y \in int_\delta(B))$
$\Rightarrow (x, y) \in int_\delta(A) \times int_\delta(B)$
Then we have

$$int_\delta(A \times B) \subseteq int_\delta(A) \times int_\delta(B) \tag{1}$$

Let $(x, y) \in int_\delta(A) \times int_\delta(B)$.
$(x, y) \in int_\delta(A) \times int_\delta(B) \Rightarrow (x \in int_\delta(A) \wedge y \in int_\delta(B))$
$\Rightarrow (\exists U \in \mathcal{U}(x))(int(cl(U)) \subseteq A) \wedge (\exists V \in \mathcal{U}(y))(int(cl(V)) \subseteq B)$
$\Rightarrow (U \times V \in \mathcal{U}(x, y))(int(cl(U)) \times int(cl(V)) \subseteq A \times B)$
$\Rightarrow (U \times V \in \mathcal{U}(x, y))(int(cl(U)) \times int(cl(V)) = int(cl(U \times V)) \subseteq A \times B)$
$\Rightarrow (x, y) \in int_\delta(A \times B)$
Then we have

$$int_\delta(A) \times int_\delta(B) \subseteq int_\delta(A \times B) \tag{2}$$

It follows from (1) and (2) that $int_\delta(A) \times int_\delta(B) = int_\delta(A \times B)$.
(b) Let $(x, y) \notin cl_\delta(A \times B)$.
$(x, y) \notin cl_\delta(A \times B) \Rightarrow (\exists U \in \mathcal{U}(x, y))(int(cl(U)) \cap (A \times B) = \emptyset)$
$\Rightarrow (\exists U_1 \in \mathcal{U}(x))(\exists U_2 \in \mathcal{U}(y))(int(cl(U_1 \times U_2)) \cap (A \times B) \subseteq int(cl(U)) \cap (A \times B) = \emptyset)$
$\Rightarrow (\exists U_1 \in \mathcal{U}(x))(\exists U_2 \in \mathcal{U}(y))([int(cl(U_1)) \times int(cl(U_2))] \cap (A \times B) = \emptyset)$
$\Rightarrow (\exists U_1 \in \mathcal{U}(x))(\exists U_2 \in \mathcal{U}(y))([int(cl(U_1)) \cap A] \times [int(cl(U_2)) \cap B] = \emptyset)$
$\Rightarrow (\exists U_1 \in \mathcal{U}(x))(\exists U_2 \in \mathcal{U}(y))(int(cl(U_1)) \cap A = \emptyset \vee int(cl(U_2)) \cap B = \emptyset)$
$\Rightarrow (\exists U_1 \in \mathcal{U}(x))(int(cl(U_1)) \cap A = \emptyset) \vee (\exists U_2 \in \mathcal{U}(y))(int(cl(U_2)) \cap B = \emptyset)$
$\Rightarrow x \notin cl_\delta(A) \vee y \notin cl_\delta(B)$
$\Rightarrow (x, y) \notin cl_\delta(A) \times cl_\delta(B)$
Then we have

$$cl_\delta(A) \times cl_\delta(B) \subseteq cl_\delta(A \times B) \tag{3}$$

Let $(x, y) \notin cl_\delta(A) \times cl_\delta(B)$.

$(x,y)\notin cl_\delta(A)\times cl_\delta(B)\Rightarrow (x\notin cl_\delta(A)\ \vee\ y\notin cl_\delta(B))$
$\Rightarrow (\exists U_1\in\mathcal{U}(x))(int(cl(U_1))\cap A=\emptyset)\ \vee\ (\exists U_2\in\mathcal{U}(y))(int(cl(U_2))\cap B=\emptyset)$
$\Rightarrow (\exists U_1\in\mathcal{U}(x))(\exists U_2\in\mathcal{U}(y))(int(cl(U_1))\cap A=\emptyset\ \vee\ int(cl(U_2))\cap B=\emptyset)$
$\Rightarrow (\exists U_1\in\mathcal{U}(x))(\exists U_2\in\mathcal{U}(y))([int(cl(U_1))\cap A]\times[int(cl(U_2))\cap B]=\emptyset)$
$\Rightarrow (\exists U_1\in\mathcal{U}(x))(\exists U_2\in\mathcal{U}(y))([int(cl(U_1))\times int(cl(U_2))]\cap(A\times B)=\emptyset)$
$\Rightarrow (U_1\times U_2\in\mathcal{U}(x,y))(int(cl(U_1\times U_2))\cap(A\times B)=\emptyset)$
$\Rightarrow (x,y)\notin cl_\delta(A\times B)$
Then we have

$$cl_\delta(A\times B)\subseteq cl_\delta(A)\times cl_\delta(B) \tag{4}$$

It follows from (3) and (4) that $cl_\delta(A\times B)=cl_\delta(A)\times cl_\delta(B)$.
(c) Let $B\subseteq Y$.

$$\begin{array}{rcl} e^*\text{-}cl(X\times B) & \overset{\text{Lemma 2.7}}{=} & (X\times B)\cup int(cl(int_\delta(X\times B)))\\ & \overset{(a)}{=} & (X\times B)\cup int(cl(int_\delta(X)\times int_\delta(B)))\\ & = & (X\times B)\cup int(cl(X\times int_\delta(B)))\\ & = & (X\times B)\cup int(cl(X)\times cl(int_\delta(B)))\\ & = & (X\times B)\cup int(X\times cl(int_\delta(B)))\\ & = & (X\times B)\cup (X\times int(cl(int_\delta(B))))\\ & = & (X\cup X)\times(B\cup int(cl(int_\delta(B))))\\ & \overset{\text{Lemma 2.7}}{=} & X\times e^*\text{-}cl(B). \end{array}$$

(d) Let $A\in e^*O(X)$ and $B\in e^*O(Y)$.

$$\begin{array}{rcl} cl(int(cl_\delta(A\times B))) & \overset{(b)}{=} & cl(int[cl_\delta(A)\times cl_\delta(B)])\\ & = & cl[int(cl_\delta(A))\times int(cl_\delta(B))]\\ & = & cl(int(cl_\delta(A)))\times cl(int(cl_\delta(B)))\\ & \overset{\text{Hypothesis}}{\supseteq} & A\times B \end{array}$$

Then we have $A\times B\in e^*O(X\times Y)$. ■

Theorem 4.5 A function $f:X\to Y$ is weakly e^*-irresolute if the graph function defined by $g(x)=(x,f(x))$ for each $x\in X$ is weakly e^*-irresolute.

Proof. Let $x\in X$ and $V\in e^*O(Y,f(x))$.

$$\left.\begin{array}{r} V\in e^*O(Y,f(x))\overset{\text{Lemma 4.4}(d)}{\Rightarrow} X\times V\in e^*O(X\times Y,g(x))\\ g \text{ is weakly } e^*\text{-irresolute}\end{array}\right\}\overset{\text{Lemma 4.4}(c)}{\Rightarrow}$$
$\Rightarrow (\exists U\in e^*O(X,x))(g[U]\subseteq e^*\text{-}cl(X\times V)=X\times e^*\text{-}cl(V))$
$\Rightarrow (\exists U\in e^*O(X,x))(f[U]\subseteq e^*\text{-}cl(V))$. ■

Definition 4.6 [4] A topological space X is said to be e^*-T_2 if for each pair of distinct points $x,y\in X$, there exist $U\in e^*O(X,x)$ and $V\in e^*O(X,y)$ such that $U\cap V=\emptyset$.

Lemma 4.7 A topological space X is e^*-T_2 if and only if for each pair of distinct points $x,y\in X$, there exist $U\in e^*O(X,x)$ and $V\in e^*O(X,y)$ such that $e^*\text{-}cl(U)\cap e^*\text{-}cl(V)=\emptyset$.

Proof. Necessity. Let $x,y\in X$ and $x\neq y$.

$$\left.\begin{array}{r}(x,y\in X)(x\neq y)\\ X \text{ is } e^*\text{-}T_2\end{array}\right\}\Rightarrow (\exists U\in e^*O(X,x))(\exists V\in e^*O(X,y))(U\cap V=\emptyset)$$

$\Rightarrow (\exists U \in e^*O(X,x))(\exists V \in e^*O(X,y))(U \subseteq X \setminus V)$
$\Rightarrow (\exists U \in e^*O(X,x))(\exists V \in e^*O(X,y))(e^*\text{-}cl(U) \subseteq e^*\text{-}cl(X \setminus V) = X \setminus V)$
$\Rightarrow (\exists U \in e^*O(X,x))(\exists V \in e^*O(X,y))(V \subseteq X \setminus e^*\text{-}cl(U))$
$\overset{\text{Theorem 2.1}}{\Rightarrow} (\exists U \in e^*O(X,x))(\exists V \in e^*O(X,y))(e^*\text{-}cl(V) \subseteq X \setminus e^*\text{-}cl(U))$
$\Rightarrow (\exists U \in e^*O(X,x))(\exists V \in e^*O(X,y))(e^*\text{-}cl(U) \cap e^*\text{-}cl(V) = \emptyset)$.
Sufficiency. It is obvious. ■

Theorem 4.8 If Y is an e^*-T_2 space and $f : X \to Y$ is weakly e^*-irresolute injection, then X is e^*-T_2.

Proof. Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}\left.\begin{array}{r}(x,y \in X)(x \neq y)\\ f \text{ is injective}\end{array}\right\} \Rightarrow f(x) \neq f(y)\\ Y \text{ is } e^*\text{-}T_2\end{array}\right\} \overset{\text{Lemma 4.7}}{\Rightarrow}$$

$$\left.\begin{array}{r}\Rightarrow (\exists V \in e^*O(Y,f(x)))(\exists W \in e^*O(Y,f(y)))(e^*\text{-}cl(V) \cap e^*\text{-}cl(W) = \emptyset)\\ f \text{ is weakly } e^*\text{-irresolute}\end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists G \in e^*O(X,x))(\exists H \in e^*O(X,y))(f[G \cap H] \subseteq f[G] \cap f[H] \subseteq e^*\text{-}cl(V) \cap e^*\text{-}cl(W) = \emptyset)$
$\Rightarrow (\exists G \in e^*O(X,x))(\exists H \in e^*O(X,y))(f[G \cap H] = \emptyset)$
$\Rightarrow (\exists G \in e^*O(X,x))(\exists H \in e^*O(X,y))(G \cap H = \emptyset)$. ■

Definition 4.9 A function $f : X \to Y$ is said to have a strongly e^*-closed graph if for each $(x,y) \notin G(f)$, there exist $U \in e^*O(X,x)$ and $V \in e^*O(Y,y)$ such that $[e^*\text{-}cl(U) \times e^*\text{-}cl(V)] \cap G(f) = \emptyset$.

Theorem 4.10 If Y is an e^*-T_2 space and $f : X \to Y$ is weakly e^*-irresolute, then f has a strongly e^*-closed graph.

Proof. Let $(x,y) \notin G(f)$.

$$\left.\begin{array}{r}(x,y) \notin G(f) \Rightarrow y \neq f(x)\\ Y \text{ is } e^*\text{-}T_2\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\overset{\text{Lemma 4.7}}{\Rightarrow} (\exists V \in e^*O(Y,f(x)))(\exists W \in e^*O(Y,y))(e^*\text{-}cl(V) \cap e^*\text{-}cl(W) = \emptyset)\\ f \text{ is weakly } e^*\text{-irresolute}\end{array}\right\} \Rightarrow$$

$\overset{\text{Theorem 3.5}}{\Rightarrow} (\exists U \in e^*O(X,x))(\exists W \in e^*O(Y,y))(f[e^*\text{-}cl(U)] \cap e^*\text{-}cl(W) = \emptyset)$
$\Rightarrow (\exists U \in e^*O(X,x))(\exists W \in e^*O(Y,y))([e^*\text{-}cl(U) \times e^*\text{-}cl(W)] \cap G(f) = \emptyset)$. ■

Theorem 4.11 If a function $f : X \to Y$ is weakly e^*-irresolute injection and f has a strongly e^*-closed graph, then X is e^*-T_2.

Proof. Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}\left.\begin{array}{r}(x,y \in X)(x \neq y)\\ f \text{ is injective}\end{array}\right\} \Rightarrow f(x) \neq f(y) \Rightarrow (x, f(y)) \notin G(f)\\ f \text{ has a strongly } e^*\text{-closed graph}\end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists G \in e^*O(X,x))(\exists V \in e^*O(Y,f(y)))([e^*\text{-}cl(G) \times e^*\text{-}cl(V)] \cap G(f) = \emptyset)$

$$\left.\begin{array}{r}\Rightarrow (\exists G \in e^*O(X,x))(\exists V \in e^*O(Y,f(y)))(f[e^*\text{-}cl(G)] \cap e^*\text{-}cl(V) = \emptyset)\\ f \text{ is weakly } e^*\text{-irresolute}\end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists G \in e^*O(X,x))(\exists H \in e^*O(X,y))(f[G] \cap f[H] \subseteq f[e^*\text{-}cl(G)] \cap f[H] \subseteq f[e^*\text{-}cl(G)] \cap e^*\text{-}cl(V) = \emptyset)$
$\Rightarrow (\exists G \in e^*O(X,x))(\exists H \in e^*O(X,y))(f[G \cap H] \subseteq f[G] \cap f[H] = \emptyset)$
$\Rightarrow (\exists G \in e^*O(X,x))(\exists H \in e^*O(X,y))(G \cap H = \emptyset)$. ■

Definition 4.12 [4] A topological space X is said to be e^*-connected if it can not be written as the union of two nonempty disjoint e^*-open sets.

Theorem 4.13 If a function $f : X \to Y$ is weakly e^*-irresolute surjection and X is e^*-connected, then Y is e^*-connected.

Proof. Suppose that Y is not e^*-connected.

$$Y \text{ is not } e^*\text{-connected} \Rightarrow (\exists V, W \in e^*O(Y) \setminus \{\emptyset\})(V \cap W = \emptyset)(V \cup W = Y)$$
$$\Rightarrow (V, W \in e^*R(Y) \setminus \{\emptyset\})(f^{-1}[V \cap W] = f^{-1}[\emptyset])(f^{-1}[V \cup W] = f^{-1}[Y])$$
$$\left.\begin{array}{r} \Rightarrow (V, W \in e^*R(Y) \setminus \{\emptyset\})(f^{-1}[V] \cap f^{-1}[W] = \emptyset)(f^{-1}[V] \cup f^{-1}[W] = X) \\ f \text{ is weakly } e^*\text{-irresolute surjection} \end{array}\right\} \overset{\text{Theorem 3.5}}{\Rightarrow}$$
$$\Rightarrow (f^{-1}[V], f^{-1}[W] \in e^*R(X) \setminus \{\emptyset\})(f^{-1}[V] \cap f^{-1}[W] = \emptyset)(f^{-1}[V] \cup f^{-1}[W] = X)$$

Then X is not e^*-connected. ■

5. Strongly e^*-irresolute functions and their some fundamental properties

Theorem 5.1 Let $f : X \to Y$ be a function. The following properties are equivalent:
(a) f is strongly e^*-irresolute;
(b) For each $x \in X$ and each $V \in e^*O(Y, f(x))$, there exists $U \in e^*O(X, x)$ such that $f[e^*\text{-}cl_\theta(U)] \subseteq V$;
(c) For each $x \in X$ and each $V \in e^*O(Y, f(x))$, there exists $U \in e^*R(X, x)$ such that $f[U] \subseteq V$;
(d) For each $x \in X$ and each $V \in e^*O(Y, f(x))$, there exists $U \in e^*\theta O(X, x)$ such that $f[U] \subseteq V$;
(e) $f^{-1}[V]$ is e^*-θ-open in X for every $V \in e^*O(Y)$;
(f) $f^{-1}[F]$ is e^*-θ-closed in X for every $F \in e^*C(Y)$;
(g) $f[e^*\text{-}cl_\theta(A)] \subseteq e^*\text{-}cl(f[A])$ for every subset A of X;
(h) $e^*\text{-}cl_\theta(f^{-1}[B]) \subseteq f^{-1}[e^*\text{-}cl(B)]$ for every subset B of Y.

Proof. $(a) \Rightarrow (b)$, $(b) \Rightarrow (c)$, $(c) \Rightarrow (d)$ and $(e) \Rightarrow (f)$ are clear.

$(d) \Rightarrow (e)$: Let $V \in e^*O(Y)$ and $x \in f^{-1}[V]$.

$$\left.\begin{array}{r} (V \in e^*O(Y))(x \in f^{-1}[V]) \Rightarrow V \in e^*O(Y, f(x)) \\ (d) \end{array}\right\} \Rightarrow (\exists U \in e^*\theta O(X, x))(f[U] \subseteq V)$$
$$\Rightarrow (\exists U \in e^*\theta O(X, x))(U \subseteq f^{-1}[V]) \Rightarrow x \in e^*\text{-}int_\theta(f^{-1}[V])$$

Then $f^{-1}[V] \in e^*\theta O(X)$.

$(f) \Rightarrow (g)$: Let $A \subseteq X$.

$$\left.\begin{array}{r} A \subseteq X \Rightarrow e^*\text{-}cl(f[A]) \in e^*C(Y) \\ (f) \end{array}\right\} \Rightarrow f^{-1}[e^*\text{-}cl(f[A])] \in e^*\theta C(X) \quad (*)$$
$$A \subseteq f^{-1}[f[A]] \Rightarrow e^*\text{-}cl_\theta(A) \subseteq e^*\text{-}cl_\theta(f^{-1}[f[A]]) \subseteq e^*\text{-}cl_\theta(f^{-1}[e^*\text{-}cl(f[A])]) \quad (**)$$
$$(*), (**) \Rightarrow e^*\text{-}cl_\theta(A) \subseteq f^{-1}[e^*\text{-}cl(f[A])] \Rightarrow f[e^*\text{-}cl_\theta(A)] \subseteq e^*\text{-}cl(f[A]).$$

$(g) \Rightarrow (h)$: Let $B \subseteq Y$.

$$\left.\begin{array}{r} B \subseteq Y \Rightarrow f^{-1}[B] \subseteq X \\ (g) \end{array}\right\} \Rightarrow f[e^*\text{-}cl_\theta(f^{-1}[B])] \subseteq e^*\text{-}cl(f[f^{-1}[B]]) \subseteq e^*\text{-}cl(B)$$
$$\Rightarrow e^*\text{-}cl_\theta(f^{-1}[B]) \subseteq f^{-1}[e^*\text{-}cl(B)].$$

$(h) \Rightarrow (a)$: Let $x \in X$ and $V \in e^*O(Y, f(x))$.

$$\left.\begin{array}{r} V \in e^*O(Y, f(x)) \Rightarrow f(x) \notin Y \setminus V \in e^*C(Y) \\ (h) \end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r} \Rightarrow e^*\text{-}cl_\theta(f^{-1}[Y \setminus V]) \subseteq f^{-1}[e^*\text{-}cl(Y \setminus V)] = f^{-1}[Y \setminus V] \\ f^{-1}[Y \setminus V] \subseteq e^*\text{-}cl_\theta(f^{-1}[Y \setminus V]) \end{array}\right\} \Rightarrow$$
$$\Rightarrow x \notin f^{-1}[Y \setminus V] = e^*\text{-}cl_\theta(f^{-1}[Y \setminus V])$$
$$\Rightarrow x \notin f^{-1}[Y \setminus V] \in e^*\theta C(X)$$

$\Rightarrow x \in f^{-1}[V] \in e^*\theta O(X)$
$\Rightarrow (\exists U \in e^*O(X,x))(e^*\text{-}cl(U) \subseteq f^{-1}[V])$
$\Rightarrow (\exists U \in e^*O(X,x))(f[e^*\text{-}cl(U)] \subseteq V)$. ■

Theorem 5.2 An e^*-irresolute function $f : X \to Y$ is strongly e^*-irresolute if and only if X is strongly e^*-regular.

Proof. Necessity. Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r} f : X \to X, f(x) = x \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (f \text{ is } e^*\text{-irresolute})(f \text{ is strongly } e^*\text{-irresolute}) \\ V \in e^*O(X,x) \Rightarrow f(x) = x \in V \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in e^*O(X,x))(f[e^*\text{-}cl(U)] \subseteq V)$
$\Rightarrow (\exists U \in e^*O(X,x))(U \subseteq e^*\text{-}cl(U) \subseteq V)$.
Then X is strongly e^*-regular from Lemma 4.2.
Sufficiency. Let $V \in e^*O(Y, f(x))$.

$$\left.\begin{array}{r} f \text{ is } e^*\text{-irresolute} \\ V \in e^*O(Y, f(x)) \end{array}\right\} \Rightarrow \left.\begin{array}{r} f^{-1}[V] \in e^*O(X,x) \\ X \text{ is strongly } e^*\text{-regular} \end{array}\right\} \Rightarrow (\exists U \in e^*O(X,x))(U \subseteq e^*\text{-}cl(U) \subseteq f^{-1}[V])$$

$\Rightarrow (\exists U \in e^*O(X,x))(f[e^*\text{-}cl(U)] \subseteq V)$
Then f is strongly e^*-irresolute. ■

Corollary 5.3 Let X be a strongly e^*-regular space. Then $f : X \to Y$ is strongly e^*-irresolute if and only if f is e^*-irresolute.

Theorem 5.4 Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f. If g is strongly e^*-irresolute, then f is strongly e^*-irresolute and X is strongly e^*-regular.

Proof. Let $x \in X$ and $V \in e^*O(Y, f(x))$.

$$(x \in X)(V \in e^*O(Y, f(x))) \overset{\text{Lemma 4.4}(d)}{\Rightarrow} \left.\begin{array}{r} X \times V \in e^*O(X \times Y, g(x)) \\ g \text{ is strongly } e^*\text{-irresolute} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in e^*O(X,x))(g[e^*\text{-}cl(U)] \subseteq X \times V)$
$\Rightarrow (\exists U \in e^*O(X,x))(f[e^*\text{-}cl(U)] \subseteq V)$
This shows that f is strongly e^*-irresolute. Let $U \in e^*O(X,x)$.

$$U \in e^*O(X,x) \overset{\text{Lemma 4.4}(d)}{\Rightarrow} \left.\begin{array}{r} U \times Y \in e^*O(X \times Y, g(x)) \\ g \text{ is strongly } e^*\text{-irresolute} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists G \in e^*O(X,x))(g[e^*\text{-}cl(G)] \subseteq U \times Y)$
$\Rightarrow (\exists G \in e^*O(X,x))(f[e^*\text{-}cl(G)] \subseteq U)$
This shows that X is strongly e^*-regular. ■

Lemma 5.5 Let (X,τ) be a topological space and $A \subseteq Y \subseteq X$. If (X,τ) is a regular space, then $\delta\text{-}cl_Y(A) = \delta\text{-}cl(A) \cap Y$ where $\delta\text{-}cl_Y(A)$ denotes the δ-closure of A in the subspace Y.

Proof. Let (X,τ) be a regular space.

$$\left.\begin{array}{r} (X,\tau) \text{ is regular} \\ Y \subseteq X \end{array}\right\} \Rightarrow \left.\begin{array}{r} (Y,\tau_Y) \text{ is regular} \\ A \subseteq Y \end{array}\right\} \Rightarrow \delta\text{-}cl_Y(A) = cl_Y(A) = cl(A) \cap Y = \delta\text{-}cl(A) \cap Y.$$

■

Corollary 5.6 Let (X,τ) be a topological space and $A \subseteq Y \subseteq X$. If (X,τ) is a compact Hausdorff space, then $\delta\text{-}cl_Y(A) = \delta\text{-}cl(A) \cap Y$.

Proof. It follows from the fact that every compact Hausdorff space is regular. ■

Remark 2 *As shown by the following examples, the equality which is given Lemma 5.5 can not be true when topological space is not a regular space.*

***Example* 5.7** Let $X := \{a, b, c, d\}$ and

$$\tau := \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}.$$

If $Y = \{b, c, d\} \subseteq X$, then $\tau_Y = \{T \cap Y | T \in \tau\} = \{\emptyset, Y, \{b\}, \{c\}, \{b, c\}, \{b, d\}\}$ where τ_Y is relative topology on Y. For subset $\{c\}$ of Y, $\delta\text{-}cl_Y(\{c\}) = \{c\} \subseteq \delta\text{-}cl(\{c\}) \cap Y = \{c, d\}$.

***Example* 5.8** Let $X := \{a, b, c, d\}$ and

$$\tau := \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}.$$

If $Y = \{b, c, d\} \subseteq X$, then $\tau_Y = \{T \cap Y | T \in \tau\} = \{\emptyset, Y, \{b\}, \{c\}, \{b, c\}, \{b, d\}\}$ where τ_Y is relative topology on Y. For subset $\{d\}$ of Y, $\delta\text{-}cl_Y(\{d\}) = \{b, d\} \supseteq \delta\text{-}cl(\{d\}) \cap Y = \{d\}$.

Lemma 5.9 Let (X, τ) be a regular topological space and $A, Y \subseteq X$. If $A \in e^*O(X)$ and $Y \in aO(X)$, then $A \cap Y \in e^*O(Y)$.

Proof. Let (X, τ) be a regular space, $A \in e^*O(X)$ and $Y \in aO(X)$. Then

$$\left.\begin{array}{r} A \in e^*O(X) \Rightarrow A \subseteq cl(int(cl_\delta(A))) \\ Y \in aO(X) \Rightarrow Y \subseteq int(cl(int_\delta(Y))) \end{array}\right\} \Rightarrow Y \cap A \subseteq int(cl(int_\delta(Y))) \cap cl(int(cl_\delta(A)))$$

$$\begin{aligned} \Rightarrow Y \cap A &\subseteq int(cl(int_\delta(Y))) \cap cl(int(cl_\delta(A))) \\ &\subseteq cl\left[int(cl(int_\delta(Y))) \cap int(cl_\delta(A))\right] \\ &\subseteq cl\left[cl(int_\delta(Y)) \cap int(cl_\delta(A))\right] \\ &\subseteq cl\left[cl[int_\delta(Y) \cap int(cl_\delta(A))]\right] \\ &= cl[int_\delta(Y) \cap int(cl_\delta(A))] \end{aligned}$$

$$\Rightarrow Y \cap A \subseteq cl[int_\delta(Y) \cap int(cl_\delta(A))] \cap Y = cl_Y[int_\delta(Y) \cap int(cl_\delta(A))] \quad (*)$$

Also,

$$\left.\begin{array}{l} (A \subseteq X)(Y \subseteq X) \Rightarrow int_\delta(Y) \cap int(cl_\delta(A)) \subseteq Y \\ (A \subseteq X)(Y \subseteq X) \Rightarrow int_\delta(Y) \cap int(cl_\delta(A)) \in \tau \end{array}\right\} \Rightarrow int_\delta(Y) \cap int(cl_\delta(A)) \in \tau_Y$$

$$\Rightarrow int_Y\left[int_\delta(Y) \cap int(cl_\delta(A))\right] = int_\delta(Y) \cap int(cl_\delta(A)) \quad (**)$$

Thus,

$$\begin{array}{llcl} (*), (**) \Rightarrow & Y \cap A & \subseteq & cl_Y\left[int_Y\left[int_\delta(Y) \cap int(cl_\delta(A))\right]\right] \\ & & = & cl_Y\left[int_Y\left[int_\delta(Y) \cap int(cl_\delta(A))\right] \cap int_\delta(Y)\right] \\ & & = & cl_Y\left[int_Y\left[int_\delta(Y) \cap int(cl_\delta(A)) \cap Y\right]\right] \\ & & \subseteq & cl_Y\left[int_Y\left[int_\delta(Y) \cap cl_\delta(A) \cap Y\right]\right] \\ & & \subseteq & cl_Y\left[int_Y\left[cl_\delta(int_\delta(Y) \cap A) \cap Y\right]\right] \\ & & \subseteq & cl_Y\left[int_Y\left[cl_\delta(Y \cap A) \cap Y\right]\right] \\ & & \overset{\text{Lemma 5.5}}{=} & cl_Y\left[int_Y\left[\delta\text{-}cl_Y(Y \cap A)\right]\right]. \end{array}$$

■

Lemma 5.10 Let (X, τ) be a regular space and $A \subseteq Y \subseteq X$. If $A \in e^*O(Y)$ and $Y \in e^*O(X)$, then $A \in e^*O(X)$.

Proof. Let (X, τ) be a regular space, $A \in e^*O(Y)$ and $Y \in e^*O(X)$. Then

$$\left.\begin{array}{r} A \in e^*O(Y) \Rightarrow A \subseteq cl_Y\left(int_Y\left(\delta\text{-}cl_Y(A)\right)\right) = cl_Y\left(int_Y\left[\delta\text{-}cl(A) \cap Y\right]\right) \subseteq cl_Y\left(int_Y\left(\delta\text{-}cl(A)\right)\right) \\ int_Y\left(\delta\text{-}cl(A)\right) \in \tau_Y \Rightarrow (\exists U \in \tau)(int_Y(\delta\text{-}cl(A)) = U \cap Y \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow cl_Y\left(int_Y\left(\delta\text{-}cl(A)\right)\right) = cl_Y\left(U \cap Y\right) \\ Y \in e^*O(X) \Rightarrow Y \subseteq cl(int(\delta\text{-}cl(Y))) \end{array}\right\} \Rightarrow$$

$$\begin{array}{rcl}\Rightarrow A\subseteq cl_Y\left[U\cap cl(int(\delta\text{-}cl(Y)))\right] & \subseteq & cl_Y\left[cl\left[U\cap int(\delta\text{-}cl(Y))\right]\right] \\ & = & cl_Y\left[cl\left[int(U)\cap int(\delta\text{-}cl(Y))\right]\right] \\ & = & cl_Y\left[cl\left[int[U\cap \delta\text{-}cl(Y)]\right]\right] \\ & \overset{\text{Lemma 5.5}}{\subseteq} & cl_Y\left[cl\left[int[\delta\text{-}cl(U\cap Y)]\right]\right] \\ & = & cl\left[cl\left[int[\delta\text{-}cl(U\cap Y)]\right]\right]\cap Y \\ & \subseteq & cl\left[cl\left[int[\delta\text{-}cl(U\cap Y)]\right]\right] \\ & = & cl\left[int[\delta\text{-}cl(U\cap Y)]\right]\end{array}$$

$\Rightarrow A\subseteq cl\left[int[\delta\text{-}cl(U\cap Y)]\right]\subseteq cl\left(int(\delta\text{-}cl(int_Y(\delta\text{-}cl(A))))\right)\subseteq cl\left(int(\delta\text{-}cl(\delta\text{-}cl(A)))\right) = cl\left(int(\delta\text{-}cl(A))\right).$ ■

Lemma 5.11 Let X be a regular topological space and $A\subseteq Y\subseteq X$ and Y is a-open in X. Then the following properties hold:
(a) $A\in e^*O(Y)$ if and only if $A\in e^*O(X)$,
(b) $e^*\text{-}cl(A)\cap Y = e^*\text{-}cl_Y(A)$, where $e^*\text{-}cl_Y(A)$ denotes the e^*-closure of A in the subspace Y.

Proof. *(a) Necessity.* Let $A\in e^*O(Y)$.

$$\left.\begin{array}{r} A\in e^*O(Y)\\ Y\in aO(X)\Rightarrow Y\in e^*O(X)\end{array}\right\}\overset{\text{Lemma 5.10}}{\Rightarrow} A\in e^*O(X).$$

Sufficiency. Let $A\in e^*O(X)$.

$$\left.\begin{array}{r} A\in e^*O(X)\\ Y\in aO(X)\end{array}\right\}\overset{\text{Lemma 5.9}}{\Rightarrow}\left.\begin{array}{r} A\cap Y\in e^*O(Y)\\ A\subseteq Y\Rightarrow A=A\cap Y\end{array}\right\}\Rightarrow A\in e^*O(Y).$$

(b) Let $x\in e^*\text{-}cl(A)\cap Y$ and $V\in e^*O(Y,x)$.

$$\left.\begin{array}{l} V\in e^*O(Y,x)\overset{(a)}{\Rightarrow} V\in e^*O(X,x)\\ x\in e^*\text{-}cl(A)\cap Y\Rightarrow x\in e^*\text{-}cl(A)\end{array}\right\}\Rightarrow V\cap A\neq\emptyset$$

Then we have $x\in e^*\text{-}cl_Y(A)$.
Let $x\in e^*\text{-}cl_Y(A)$ and $V\in e^*O(X,x)$.

$$\left.\begin{array}{r} V\in e^*O(X,x)\overset{\text{Lemma 5.9}}{\Rightarrow} V\cap Y\in e^*O(Y,x)\\ x\in e^*\text{-}cl_Y(A)\end{array}\right\}\Rightarrow\emptyset\neq A\cap(V\cap Y)\subseteq A\cap V$$

Then we have $x\in e^*\text{-}cl(A)\cap Y$. ■

Theorem 5.12 Let X be a regular space. If $f:X\to Y$ is strongly e^*-irresolute and X_0 is an a-open subset of X, then the restriction $f|_{X_0}:X_0\to Y$ is strongly e^*-irresolute.

Proof. Let $x\in X_0$ and $V\in e^*O(Y,f(x))$.

$$\left.\begin{array}{r}(x\in X_0)(V\in e^*O(Y,f(x)))\\ f\text{ is strongly } e^*\text{-irresolute}\end{array}\right\}\Rightarrow\left.\begin{array}{r}(\exists U\in e^*O(X,x))(f[e^*\text{-}cl(U)]\subseteq V)\\ (U_0:=U\cap X_0)(X_0\in aO(X))\end{array}\right\}\overset{\text{Lemma 5.9 and 5.11}(b)}{\Rightarrow}$$

$\Rightarrow (U_0\in e^*O(X_0,x))(f|_{X_0}[e^*\text{-}cl_{X_0}(U_0)] = f[e^*\text{-}cl_{X_0}(U_0)]\subseteq f[e^*\text{-}cl(U_0)]\subseteq f[e^*\text{-}cl(U)]\subseteq V).$
This shows that $f|_{X_0}$ is strongly e^*-irresolute. ■

Definition 5.13 A function $f:X\to Y$ is said to be e^*-open [4] if $f[U]\in e^*O(Y)$ for each $U\in e^*O(X)$.

Lemma 5.14 If $f:X\to Y$ is e^*-irresolute and V is e^*-θ-open in Y, then $f^{-1}[V]$ is e^*-θ-open in X.

Proof. Let $V\in e^*\theta O(Y)$ and $x\in f^{-1}[V]$.
$(V\in e^*\theta O(Y))(x\in f^{-1}[V])\Rightarrow$

$$V\in e^*\theta O(Y,f(x))\Rightarrow\left.\begin{array}{r}(\exists W\in e^*O(Y,f(x))(W\subseteq e^*\text{-}cl(W)\subseteq V)\\ f\text{ is } e^*\text{-irresolute}\end{array}\right\}\Rightarrow$$

$\Rightarrow (f^{-1}[W] \in e^*O(X,x))(f^{-1}[W] \subseteq e^*\text{-}cl(f^{-1}[W]) \subseteq f^{-1}[e^*\text{-}cl(W)] \subseteq f^{-1}[V])$
$\Rightarrow f^{-1}[V] \in e^*\theta O(X)$. ■

Theorem 5.15 Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the following properties hold:
(a) If f is strongly e^*-irresolute and g is e^*-irresolute, then the composition $g \circ f : X \to Z$ is strongly e^*-irresolute.
(b) If f is e^*-irresolute and g is e^*-irresolute, then $g \circ f$ is strongly e^*-irresolute.
(c) If $f : X \to Y$ is e^*-open bijection and $g \circ f : X \to Z$ is strongly e^*-irresolute, then g is strongly e^*-irresolute.

Proof. *(a)* Let $V \in e^*O(Z)$.

$$\left.\begin{array}{r} V \in e^*O(Z) \\ g \text{ is } e^*\text{-irresolute} \end{array}\right\} \Rightarrow \left.\begin{array}{r} g^{-1}[V] \in e^*O(Y) \\ f \text{ is st. } e^*\text{-irresolute} \end{array}\right\} \Rightarrow$$

$\Rightarrow f^{-1}[g^{-1}[V]] = (g \circ f)^{-1}[V] \in e^*\theta O(X) \subseteq e^*O(X)$.
(b) Let $V \in e^*O(Z)$.

$$\left.\begin{array}{r} V \in e^*O(Z) \\ g \text{ is strongly } e^*\text{-irresolute} \end{array}\right\} \Rightarrow g^{-1}[V] \in e^*\theta O(Y) \Rightarrow \left.\begin{array}{r} g^{-1}[V] \in e^*O(Y) \\ f \text{ is } e^*\text{-irresolute} \end{array}\right\} \Rightarrow$$

$\Rightarrow f^{-1}[g^{-1}[V]] = (g \circ f)^{-1}[V] \in e^*O(X)$.
(c) Let $W \in e^*O(Z)$.

$$\left.\begin{array}{r} W \in e^*O(Z) \\ g \circ f \text{ is strongly } e^*\text{-irresolute} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (g \circ f)^{-1}[W] = f^{-1}[g^{-1}[W]] \in e^*\theta O(X) \\ f \text{ is } e^*\text{-open bijection} \Rightarrow f^{-1} \text{ is } e^*\text{-irresolute} \end{array}\right\} \Rightarrow$$

$\Rightarrow f[f^{-1}[g^{-1}[W]]] = g^{-1}[W] \in e^*\theta O(Y)$. ■

Theorem 5.16 If $f : X \to Y$ is strongly e^*-irresolute and Y is e^*-T_2, then the subset $E = \{(x,y) | f(x) = f(y)\}$ is e^*-θ-closed in $X \times X$.

Proof. Let $(x,y) \notin E$.

$$\left.\begin{array}{r} (x,y) \notin E \Rightarrow f(x) \neq f(y) \\ Y \text{ is } e^*\text{-}T_2 \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\exists V \in e^*O(Y, f(x)))(\exists W \in e^*O(Y, f(y)))(U \cap W = \emptyset) \\ f \text{ is strongly } e^*\text{-irresolute} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in e^*O(X,x))(\exists G \in e^*O(X,y))(f[e^*\text{-}cl(U)] \cap f[e^*\text{-}cl(G)] \subseteq V \cap W = \emptyset)$
$\Rightarrow (U \times G \in e^*O(X \times X, (x,y)))(e^*\text{-}cl(U \times G) \cap E \subseteq [e^*\text{-}cl(U) \times e^*\text{-}cl(G)] \cap E = \emptyset)$
$\Rightarrow (U \times G \in e^*O(X \times X, (x,y)))(e^*\text{-}cl(U \times G) \subseteq \backslash E)$
$\Rightarrow (x,y) \in e^*\text{-}int_\theta(\backslash E)$
Then $\backslash E$ is e^*-θ-open in $X \times X$. Therefore E is e^*-θ-closed in $X \times X$. ■

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