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$*$-frames in Hilbert modules over pro-C^*-algebras

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Abstract. In this paper, by using the sequence of multipliers, we introduce frames with algebraic bounds in Hilbert pro-C^*-modules. We investigate the relations between frames and $*$-frames. Some properties of $*$-frames in Hilbert pro-C^*-modules are studied. Also, we show that there exist two differences between $*$-frames in Hilbert pro-C^*-modules and Hilbert C^*-modules. Finally, dual $*$-frames in Hilbert pro-C^*-modules are presented.

Keywords: Hilbert modules over pro-C^*-algebras, standard frame of multipliers, standard $*$-frame of multipliers, $*$-frame operator, pre-$*$-frame operator.

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1. Introduction

Frames in Hilbert spaces introduced by Duffin and Schaffer [5] in 1952, to deal with some problems in the nonharmonic Fourier series. In 1986, Daubechies et al. [4] reintroduced them. By using the sequence of bounded linear operators instead the sequence of element in Hilbert space, many generalizations of frames were presented, e.g. the fusion frames by Casazza et al. [3] and g-frames by Sun [15].

In 2000, Frank and Larson [6, 7] introduced the concept of frames in Hilbert C^*-modules as a generalization of frames in Hilbert spaces. Han et al. [9] and Jing [10]

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continued these generalizations. Later, Zhuraev and Sharipov [16] considered pro-C^*-algebras which the topology is determined by a directed family of C^*-seminormes and introduced Hilbert module over pro-C^*-algebras.

Raeburn and Thompson [14] showed that every Hilbert C^*-module countably generated in the multiplier module admits a frame of multipliers. In 2008, Joita [12] reconsidered ideas Raeburn and Thompson in Hilbert modules over pro-C^*-algebras and proposed frames of multipliers in Hilbert pro-C^*-modules. The notion of $*$-frames in Hilbert C^*-modules was recently presented by Alijani and Dehghan [1].

In this paper, we introduce $*$-frames in Hilbert modules over pro-C^*-algebras and investigate some results for these frames. The paper is organized as follows: in section 2, we recall some facts about pro-C^*-algebras and Hilbert modules over pro-C^*-algebras. In section 3, $*$-frames and examples are introduced. Later, we investigate the pre-$*$-frame operator and the $*$-frame operator for standard $*$-frames, and study some of their important properties. Finally, the dual $*$-frames are surveyed.

Throughout this manuscript, let $\mathcal{A}$ be a unital pro-C^*-algebra with respect to the family of continuous C^*-seminorms $\rho = \{\rho_\lambda\}_{\lambda\in\Lambda}$ and E, F be finitely or countably generated Hilbert $\mathcal{A}$-modules. Also, we use I, J to denote finite or countably infinite index sets.

2. Preliminaries

In this section, we recall some of the basic definitions and properties of pro-C^*-algebras and Hilbert modules over pro-C^*-algebras.

Definition 2.1 [11] A pro-C^*-algebra is a complete Hausdorff complex topological $*$-algebra $\mathcal{A}$ whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_\lambda\}$ converges to 0 iff $\rho(a_\lambda) \longrightarrow 0$ for any continuous C^*-seminorm ρ on $\mathcal{A}$ and we have:

(1) $\rho(ab) \leqslant \rho(a)\rho(b)$,
(2) $\rho(a^*a) = \rho(a)^2$

for all C^*-seminorm ρ on $\mathcal{A}$ and $a, b \in \mathcal{A}$.

***Example* 2.2** [11] Every C^*-algebra is a pro-C^*-algebra.

The set of all continuous C^*-seminorms on $\mathcal{A}$ is denoted by $S(\mathcal{A})$. An element $a \in \mathcal{A}$ is bounded if $\sup\{\rho(a); \rho \in S(A)\} < \infty$. The set of all bounded elements in $\mathcal{A}$ is denoted by $b(\mathcal{A})$. Let $\mathcal{A}$ be a unitary pro-C^*-algebra and $a \in \mathcal{A}$. Then nonzero element a is called strictly nonzero if zero doesn't belong to $\sigma(a)$.

Definition 2.3 [12] Let $\mathcal{A}$ be a pro-C^*-algebra. A pre-Hilbert $\mathcal{A}$-module is a complex vector space E which is also a right $\mathcal{A}$-module, compatible with the complex algebra structure, equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : E \times E \to \mathcal{A}$ which is $\mathbb{C}$-and $\mathcal{A}$-linear in its second variable and satisfies the following conditions:

(i) $\langle x, y\rangle^* = \langle y, x\rangle$;
(ii) $\langle x, x\rangle \geqslant 0$;
(iii) $\langle x, x\rangle = 0$ iff $x = 0$

for every $x, y \in E$.

We say that E is a Hilbert $\mathcal{A}$-module (or Hilbert pro-C^*-module over $\mathcal{A}$) if E is complete with respect to the topology determined by the family of seminorms $\bar{\rho}_E(x) = \sqrt{\rho(\langle x, x\rangle)}$ for $x \in E$ and $\rho \in S(\mathcal{A})$.

Let E be a pre-Hilbert $\mathcal{A}$-module. For every $\rho \in S(\mathcal{A})$ and for all $x, y \in E$, the following Cauchy-Schwarz inequality [11] holds

$$\rho(\langle x, y\rangle)^2 \leqslant \rho(\langle x, x\rangle)\rho(\langle y, y\rangle).$$

Definition 2.4 [11] Let E and F be two Hilbert $\mathcal{A}$-modules over pro-C^*-algebra $\mathcal{A}$. An $\mathcal{A}$-module map $T : E \to F$ is called bounded if for all $\rho \in S(\mathcal{A})$, there is $C_\rho > 0$ such that $\bar{\rho}_F(Tx) \leqslant C_\rho \bar{\rho}_E(x)$ for all $x \in E$.

A bounded $\mathcal{A}$-module map from E to F is called an operator from E to F, and it is adjointable if there is a map $T^* : F \to E$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in E,\ y \in F$. Every adjointable map is bounded $\mathcal{A}$-module map. The set of all adjointable maps from E to F is denoted by $L(E, F)$ and we write $L(E)$ for $L(E, E)$ [12].

An element $T \in L(E, F)$ is bounded in $L(E, F)$ if $(\sup\{\overline{\rho}_{L(E,F)}(T); \rho \in S(\mathcal{A})\} < \infty)$. The set of all bounded elements in $L(E, F)$ is denoted by $b(L(E, F))$.

Definition 2.5 [8] Let $\mathcal{A}$ be a pro-C^*-algebra and E, F be two Hilbert $\mathcal{A}$-modules. Then the operator $T : E \to F$ is called uniformly bounded (below), if there exists $C > 0$ such that for each $\rho \in S(\mathcal{A})$,

$$\begin{aligned} &\bar{\rho}_F(Tx) \leqslant C\bar{\rho}_E(x), \quad \textit{for all } \ x \in E, \\ &(\bar{\rho}_F(Tx) \geqslant C\bar{\rho}_E(x), \quad \textit{for all } \ x \in E) \end{aligned} \tag{1}$$

The number C in (1) is called an upper bound for T. Set

$$\begin{aligned} &\|T\|_\infty = \inf\{C : C \ \ \textit{is an upper bound for } T\}, \\ &\hat{\rho}_F(T) = \sup\{\overline{\rho}_F(T(x)) : x \in E, \ \ \overline{\rho}_E(x) \leqslant 1\}. \end{aligned}$$

Clearly, we have $\hat{\rho}(T) \leqslant \|T\|_\infty$ for all $\rho \in S(\mathcal{A})$.

In [12], the Hilbert $M(\mathcal{A})$-module $L(\mathcal{A}, E)$ is called the multiplier module of E and it is denoted by $M(E)$. For all $h \in M(E)$ and $x \in E$, we have $\langle h, x\rangle_{M(E)} = h^*(x)$. Moreover, if $a \in \mathcal{A}$ and $h \in M(E)$, then $h.a$ can be identified by $h(a)$.

A Hilbert $\mathcal{A}$-module E is countably generated if there is a countable set $\{x_n\}_n$ in E such that the submodule of E generated by $\{x_n a;\ a \in \mathcal{A}, n = 1, 2, ...\}$ is dense in E.

The set $H_\mathcal{A}$ of all sequences $(a_n)_n$ with $a_n \in \mathcal{A}$ such that $\sum a_n^* a_n$ convergent in $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with inner product $\langle (a_n)_n, (b_n)_n\rangle_{H_\mathcal{A}} = \sum_n a_n^* b_n$.

Definition 2.6 [12] Let E be a Hilbert pro-C^*-module. The sequence $\{h_n\}_n$ in $M(E)$ is called a standard frame of multipliers in E if for each $x \in E$, the series $\sum\limits_n \langle x, h_n\rangle_{M(E)}\langle h_n, x\rangle_{M(E)}$ converges in $\mathcal{A}$ and there exist two positive constants C and D such that

$$C\langle x, x\rangle_E \leqslant \sum_n \langle x, h_n\rangle_{M(E)}\langle h_n, x\rangle_{M(E)} \leqslant D\langle x, x\rangle_E$$

for all $x \in E$.

3. $*$-Frames

In this section, we introduce standard $*$-frame of multipliers in Hilbert $\mathcal{A}$-module E and investigate examples of standard $*$-frames.

Definition 3.1 Let E be a Hilbert pro-C^*-module. The sequence $\{h_n\}_n$ in $M(E)$ we call a standard $*$-frame of multipliers for E if for each $x \in E$, the series $\sum\limits_n \langle x, h_n\rangle_{M(E)}\langle h_n, x\rangle_{M(E)}$ is convergent in $\mathcal{A}$ and there exist two strictly nonzero elements C and D in $\mathcal{A}$ such that

$$C\langle x, x\rangle_E C^* \leqslant \sum_n \langle x, h_n\rangle_{M(E)}\langle h_n, x\rangle_{M(E)} \leqslant D\langle x, x\rangle_E D^*$$

for all $x \in E$.

If $\lambda = C = D$, then standard $*$-frame $\{h_i\}_{i\in I}$ of multipliers is called a standard λ-tight $*$-frame. If $\{h_i\}_{i\in I}$ possesses an upper $*$-frame bound, but not necessarily a lower $*$-frame bound, we call it standard $*$-Bessel sequence of multipliers for E.

Remark 1 *Every standard frame of multipliers in E with bounds C and D is a standard $*$-frame of multipliers in E with $\mathcal{A}$-valued $*$-frame bounds $(\sqrt{C})1_{\mathcal{A}}$ and $(\sqrt{D})1_{\mathcal{A}}$.*

***Example* 3.2** Let $H_{\mathcal{A}}$ be a Hilbert $\mathcal{A}$-module with the following operations:

$$xy := \{x_i y_i\}_{i\in\mathbb{N}},\ x^* := \{\bar{x}_i\}_{i\in\mathbb{N}},\ \langle\{x_i\}, \{y_i\}\rangle := \sum_{i\in\mathbb{N}} x_i^* y_i,$$

$$\bar{\rho}_{H_{\mathcal{A}}}(x) := (\rho(\langle x, x\rangle_{H_{\mathcal{A}}}))^{\frac{1}{2}}, \quad \forall\ x = \{x_i\}_{i\in\mathbb{N}},\quad y = \{y_i\}_{i\in\mathbb{N}}.$$

Let $J = \mathbb{N}$ and define $\{f_j\}_{j\in J} \in H_{\mathcal{A}}$ by $f_j = \{f_i^j\}_{i\in\mathbb{N}}$ such that

$$f_i^j = \begin{cases} 1_{\mathcal{A}} & i = j \\ 0 & i \neq j \end{cases}, \quad \forall j \in \mathbb{N}.$$

We observe that

$$\langle\{x_i\}, f_j\rangle_{H_{\mathcal{A}}}\langle f_j, \{x_i\}\rangle_{H_{\mathcal{A}}} = \overline{x_j} 1_{\mathcal{A}} \overline{1_{\mathcal{A}}} x_j = \overline{x_j} x_j.$$

Also,

$$\sum_{j\in J} \langle x, f_j\rangle_{H_{\mathcal{A}}}\langle f_j, x\rangle_{H_{\mathcal{A}}} = \sum_{j\in J} \overline{x_j} x_j = \langle x, x\rangle_{H_{\mathcal{A}}}.$$

So $\{f_j\}_{j\in J} \in H_{\mathcal{A}}$ is a standard normalized $*$-frame.

***Example* 3.3** Let $H_{\mathcal{A}}$ be a Hilbert $\mathcal{A}$-module. Then $L(\mathcal{A}, H_{\mathcal{A}})$ is $L(\mathcal{A})$-module with the following operations:

$$uv := \{u_i v_i\}_{i\in\mathbb{N}}, \quad u^* := \{\bar{u}_i\}_{i\in\mathbb{N}}, \quad \langle\{u_i\}, \{v_i\}\rangle := \sum_{i\in\mathbb{N}} u_i v_i^*,$$

$$\bar{\rho}_{H_{\mathcal{A}}}(u) = (\rho(\langle u, u\rangle_{H_{\mathcal{A}}}))^{\frac{1}{2}}, \quad \forall\ u = \{u_i\}_{i\in\mathbb{N}}, \quad v = \{v_i\}_{i\in\mathbb{N}}.$$

Let $J = \mathbb{N}$ and define $h_j \in L(\mathcal{A}, H_{\mathcal{A}})$ by $h_j = \{h_i^j\}_{i\in\mathbb{N}}$ such that

$$h_i^j(a) = \begin{cases} \langle a, C1_{\mathcal{A}}\rangle & i = j \\ 0 & i \neq j \end{cases}, \quad \forall j \in \mathbb{N},$$

where C is constant.

$$\rho(\sum_j h_i^j(a)\overline{h_i^j(a)}) = \rho(h_i^i(a)\overline{h_i^i(a)}) = \rho(\langle a, C1_{\mathcal{A}}\rangle\, \overline{\langle a, C1_{\mathcal{A}}\rangle}) = \rho(\langle a, C1_{\mathcal{A}}\rangle)^2 < \infty,$$

which implies that h_j is well-defined and adjointable. $h_j^* \in L(H_{\mathcal{A}}, \mathcal{A})$ is obtained by $h_j^* = \{h_i^{j^*}\}_{i\in\mathbb{N}}$, as $h_i^{j^*}(\{x_i\}) = C1_{\mathcal{A}}x_j$, and we have

$$\langle \{x_i\}, h_j\rangle_{M(H_{\mathcal{A}})}\langle h_j, \{x_i\}\rangle_{M(H_{\mathcal{A}})} = \overline{h_j^*(\{x_i\})}h_j^*(\{x_i\}) = \overline{C1_{\mathcal{A}}}\overline{x}_j C1_{\mathcal{A}}x_j.$$

So,

$$\begin{aligned}\sum_{j\in J}\langle x, h_j\rangle_{M(H_{\mathcal{A}})}\langle h_j, x\rangle_{M(H_{\mathcal{A}})} &= \sum_{j\in J}\left\langle \{x_i\}_{i\in\mathbb{N}}, h_j\right\rangle_{M(H_{\mathcal{A}})}\left\langle h_j, \{x_i\}_{i\in\mathbb{N}}\right\rangle_{M(H_{\mathcal{A}})}\\ &= \sum_{j\in J}\overline{C1_{\mathcal{A}}}\overline{x}_j x_j C1_{\mathcal{A}}\\ &= C1_{\mathcal{A}}\sum_{j\in J}\overline{x}_j x_j C1_{\mathcal{A}} = C1_{\mathcal{A}}\,\langle x, x\rangle_{H_{\mathcal{A}}}\, C1_{\mathcal{A}}.\end{aligned}$$

Consequently, $\{h_j\}_{j\in I}$ in $M(H_{\mathcal{A}})$ is a standard $C1_{\mathcal{A}}$-tight $*$- frame of multipliers in $H_{\mathcal{A}}$.

4. $*$-Frames and their properties

In this section, we investigate the pre-$*$-frame operator and the $*$-frame operator for standard $*$-frames. Then we study some properties of them.

Proposition 4.1 Let the sequence $\{h_i\}_{i\in I}$ be a standard $*$-frame of multipliers in E. Then $\{\langle h_i, x\rangle_{M(E)}\}_{i\in I} \in H_{\mathcal{A}}$.

Definition 4.2 Let the sequence $\{h_i\}_{i\in I}$ be a standard $*$-frame of multipliers in E, thus we can define a linear map $T : E \to H_{\mathcal{A}}$ by $T(x) = \{\langle h_i, x\rangle_{M(E)}\}_{i\in I}$ is called the pre-$*$-frame operator or $*$-frame transform for $\{h_i\}_{i\in I}$.

Theorem 4.3 Let $\{h_i\}_{i\in I}$ be a standard $*$-frame of multipliers in E with lower and upper $*$-frame bounds C and D, respectively. Then the pre-$*$-frame operator T is invertible and $\hat{\rho}_{E,H_{\mathcal{A}}}(T) \leqslant \rho(D)$.

Proof. Let J be an arbitrary finite subset of I. Using the Cauchy-Schwarz inequality, for any $\rho \in S(\mathcal{A})$ and $\{y_i\}_{i\in I} \in H_{\mathcal{A}}$, we have

$$\begin{aligned}(\overline{\rho}_{H_{\mathcal{A}}}(Tx))^2 &= (\overline{\rho}_{H_{\mathcal{A}}}(\{\langle h_i, x\rangle_{M(E)}\}_{i\in J}))^2\\ &= \sup\left\{\left(\rho(\left\langle \{\langle h_i, x\rangle_{M(E)}\}_i, \{y_i\}_i\right\rangle)\right)^2 : \overline{\rho}_{H_{\mathcal{A}}}(\{y_i\}_i) \leqslant 1\right\}\\ &\leqslant \sup_{\overline{\rho}_{H_{\mathcal{A}}}(\{y_i\}_i)\leqslant 1}\rho\left(\left\langle \{\langle h_i, x\rangle_{M(E)}\}_i, \{\langle h_i, x\rangle_{M(E)}\}_i\right\rangle\right) \times \sup_{\overline{\rho}_{H_{\mathcal{A}}}(\{y_i\}_i)\leqslant 1}\rho(\langle \{y_i\}, \{y_i\}\rangle)\\ &\leqslant \rho(\textstyle\sum_{i\in J}\overline{\langle h_i, x\rangle_{M(E)}}\,\langle h_i, x\rangle_{M(E)})\\ &\leqslant \rho(D)\rho(\langle x, x\rangle_E)\rho(D^*)\\ &= (\rho(D))^2(\overline{\rho}_E(x))^2.\end{aligned}$$

This shows $\overline{\rho}_{H_{\mathcal{A}}}(Tx) \leqslant \rho(D)\overline{\rho}_E(x)$ for all $x \in E$, so T is well-defined, bounded and $\hat{\rho}_{E,H_{\mathcal{A}}}(T) \leqslant \rho(D)$. Since $\langle Tx, Tx\rangle_E = \sum_n \langle x, h_n\rangle_{M(E)}\langle h_n, x\rangle_{M(E)}$ for each $x \in E$, we observe that

$$C\langle x, x\rangle_E C^* \leqslant \langle Tx, Tx\rangle_E \leqslant D\langle x, x\rangle_E D^*.$$

Suppose that $x \in E$ and $Tx = 0$. Thus, $x = 0$ and T is invertible. ■

Proposition 4.4 Let $\{h_i\}_{i\in I}$ be a sequence in $M(E)$. Suppose that $Q : x \to \{\langle h_i, x\rangle_{M(E)}\}_{i\in I}$ is an invertible element in $b(L(E, H_{\mathcal{A}}))$. Then $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers in E.

Proof. Let the sequence $\{a_i\}_{i\in I}$ be in $H_{\mathcal{A}}$. We can write

$$\begin{aligned}\left\langle\{a_i\}_{i\in I}, Q(x)\right\rangle_{H_{\mathcal{A}}} &= \left\langle\{a_i\}_{i\in I}, \{\langle h_i, x\rangle_{M(E)}\}_{i\in I}\right\rangle_{H_{\mathcal{A}}}\\ &= \sum_{i\in I}\overline{a_i}\langle h_i, x\rangle_{M(E)}\\ &= \left\langle\sum_{i\in I} h_i a_i, x\right\rangle_E.\end{aligned}$$

This shows $Q^*(\{a_i\}_{i\in I}) = \sum_{i\in I} h_i a_i$. Moreover, Q^* is an invertible element in $b(L(H_{\mathcal{A}}, E))$. Define $U := Q^*Q$. Hence, U and $U^{\frac{1}{2}}$ are positive and invertible elements in $b(L(E))$. As see in the [11], we have

$$\left\|U^{\frac{-1}{2}}\right\|_\infty^{-2}\langle x, x\rangle_E \leqslant \left\langle U^{\frac{1}{2}}x, U^{\frac{1}{2}}x\right\rangle_E \leqslant \left\|U^{\frac{1}{2}}\right\|_\infty^{2}\langle x, x\rangle_E.$$

Since $\langle Qx, Qx\rangle_E = \sum_{i\in I}\langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)}$, series is convergent in $\mathcal{A}$ and

$$\begin{aligned}(\left\|U^{\frac{-1}{2}}\right\|_\infty^{-1} 1_{\mathcal{A}})\langle x, x\rangle_E(\left\|U^{-\frac{1}{2}}\right\|_\infty^{-1} 1_{\mathcal{A}})^* &\leqslant \sum_{i\in I}\langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)}\\ &\leqslant (\left\|U^{\frac{1}{2}}\right\|_\infty 1_{\mathcal{A}})\langle x, x\rangle_E(\left\|U^{\frac{1}{2}}\right\|_\infty 1_{\mathcal{A}})^*.\end{aligned}$$

So $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers in E. ■

We define the synthesis operator for $*$-frame $\{h_i\}_{i\in I}$ as follows:

$$T^* : H_{\mathcal{A}} \to E\ ,\ T^*(\{a_i\}) = \sum_{i\in I} h_i a_i.$$

Definition 4.5 Let the sequence $\{h_i\}_{i\in I}$ in $M(E)$ be a standard $*$-frame of multipliers in E with pre-$*$-frame operator T. The $*$-frame operator $S : E \to E$ is defined by $Sx = T^*Tx = \sum_{i\in I} h_i\langle h_i, x\rangle_{M(E)}$.

Remark 2 *Let $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers in E with lower and upper $*$-frame bounds C and D, respectively. Then*

$$C\langle x, x\rangle_E C^* \leqslant \langle Sx, x\rangle_E \leqslant D\langle x, x\rangle_E D^*$$

for all $x \in E$.

Theorem 4.6 Let the sequence $\{h_i\}_{i\in I}$ in $M(E)$ be a standard $*$-frame of multipliers in E with $*$-frame operator S, lower and upper $*$-frame bounds C and D, respectively. Then the following holds:

i. S is invertible, positive and self-adjoint operator.
ii. $(\rho(C^{-1}))^{-2} \leqslant \hat{\rho}_E(S) \leqslant (\rho(D))^2$ and $C^*CI_E \leqslant S \leqslant D^*DI_E$.

Proof. i. Suppose that $Sx = 0$ for any x in E. By Remark *2* we observe that $\langle x, x\rangle_E = 0$, which implies S is invertible. Since $S = T^*T$ and $\langle Sx, x\rangle_E = \langle Tx, Tx\rangle_E$, S is positive and self-adjoint operator.

ii. Let J be an arbitrary finite subset of I. Using Cauchy-Schwarz inequality and for any $\rho \in S(\mathcal{A})$, we have

$$\begin{aligned}\left(\bar{\rho}_E(\sum_{j\in J} h_j\langle h_j, x\rangle_{M(E)})\right)^2 &= \left\{\sup_{\bar{\rho}_E(y)\leqslant 1} \rho(\left\langle \sum_{j\in J} h_j\langle h_j, x\rangle_{M(E)}, y\right\rangle)\right\}^2 \\ &= \left\{\sup_{\bar{\rho}_E(y)\leqslant 1} \rho(\sum_{j\in J}\langle x, h_j\rangle_{M(E)}\langle h_j, y\rangle_{M(E)})\right\}^2 \\ &\leqslant \left(\rho(\sum_{j\in J}\langle x, h_j\rangle_{M(E)}\langle h_j, x\rangle_{M(E)})\right) \\ &\times \sup_{\bar{\rho}_E(y)\leqslant 1}\left(\rho(\sum_{j\in J}\langle y, h_j\rangle_{M(E)}\langle h_j, y\rangle_{M(E)})\right) \\ &\leqslant \sup_{\bar{\rho}_E(y)\leqslant 1} (\rho(D\langle x, x\rangle_E D^*)\rho(D\langle y, y\rangle_E D^*)) \\ &\leqslant (\rho(D))^4(\bar{\rho}_E(x))^2.\end{aligned}$$

So $\{\sum_{j=1}^{n} h_j\, \langle h_j, x\rangle_{M(E)}\}_n$ is a cauchy sequence in Hilbert pro-C^*-module E and the series $\sum_{j\in J} h_j\, \langle h_j, x\rangle_{M(E)}$ is $\bar{\rho}$-convergent in E, which means S is well-define. The above proof for $J = \mathbb{N}$ shows that

$$(\overline{\rho}_E(S(x)))^2 = \left(\overline{\rho}_E(\sum\nolimits_{j\in J} h_j\langle h_j, x\rangle_{M(E)})\right)^2 \leqslant (\rho(D))^4(\overline{\rho}_E(x))^2.$$

Furthermore, $\overline{\rho}_E(x)(\rho(C^{-1}))^{-2} \leqslant \overline{\rho}_E(S(x))$. Hence, $(\rho(C^{-1}))^{-2} \leqslant \hat{\rho}_E(S) \leqslant (\rho(D))^2$. By Remark *2*, $CC^*I_E \leqslant S \leqslant DD^*I_E$. ∎

In [1], the authors showed that every $*$-frame in Hilbert C^*-modules can be studied as frame with different bounds. But, in Hilbert pro-C^*-modules, we have the following different result.

Proposition 4.7 Let the sequence $\{h_i\}_{i\in I}$ in $M(E)$ be a standard $*$-frame of multipliers for E with $*$-frame bounds in $b(\mathcal{A})$. Then $\{h_i\}_{i\in I}$ is a standard frame of multipliers in E

with lower and upper frame bounds $\left\|S^{\frac{-1}{2}}\right\|_\infty^{-2}$ and $\left\|S^{\frac{1}{2}}\right\|_\infty^{2}$.

Proof. Since $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for E, $\sum_{i\in I}\langle x,h_i\rangle_{M(E)}\langle h_i,x\rangle_{M(E)}$ is convergent in $\mathcal{A}$ for all $x\in E$. By Theorem 4.6, $S^{\frac{1}{2}}$ is invertible and positive operator and there are C,D in $b(\mathcal{A})$ such that

$$C\langle x,x\rangle C^* \leqslant \left\langle S^{\frac{1}{2}}x, S^{\frac{1}{2}}x\right\rangle \leqslant D\langle x,x\rangle D^*.$$

So, for each x in E, $\overline{\rho}_{H_\mathcal{A}}(S^{\frac{1}{2}}) \leqslant \rho(D)\overline{\rho}_E(x)$ and $\widehat{\rho}(S^{\frac{1}{2}}) \leqslant \rho(D)$. Since $D\in b(\mathcal{A})$, $S^{\frac{1}{2}}\in b(L(E))$. According to [12], we have

$$\left\|S^{\frac{-1}{2}}\right\|_\infty^{-2}\langle x,x\rangle_E \leqslant \sum_{i\in I}\langle x,h_i\rangle_{M(E)}\langle h_i,x\rangle_{M(E)} \leqslant \left\|S^{\frac{1}{2}}\right\|_\infty^{2}\langle x,x\rangle_E.$$

■

In [1], the authors showed that for every $*$-frame in Hilbert C^*-modules, pre-$*$-frame operator T is closed rang and T^* is surjective. In Hilbert pro-C^*-modules, by Proposition 2.13 in [8], and Proposition 2.2 and Theorem 2.5 in [2], we have the following useful result that this is different by similar Theorem in Hilbert C^*-modules.

Remark 3 *Let E be a Hilbert pro-C^*-module. If $\{h_i\}_{i\in I}$ is a standard $*$-frame with lower and upper $*$-frame bounds C and D in $b(\mathcal{A})$, then pre-$*$-frame operator T is closed rang and T^* is surjective.*

Proposition 4.8 Let the sequence $\{h_i\}_{i\in I}$ be a standard $*$-frame in $M(E)$. If there exists an invertible map $V\in b(L(E,F))$, then $\{Voh_i\}_{i\in I}$ is a standard $*$-frame of multipliers for F.

Proof. Let J be an arbitrary finite subset of I. Since $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for E with $*$-frame bounds C and D for any $y\in F$, we have:

$$\sum_{i\in J}\langle y,Vh_i\rangle_{M(F)}\langle Vh_i,y\rangle_{M(F)} = \sum_{i\in J}\langle V^*y,h_i\rangle_{M(E)}\langle h_i,V^*y\rangle_{M(E)}.$$

So $\sum_{i\in I}\langle y,Vh_i\rangle_{M(F)}\langle Vh_i,y\rangle_{M(F)}$ is convergent in $\mathcal{A}$. We have

$$\begin{aligned}\langle V^*y,V^*y\rangle_E &\leqslant C^{-1}\sum_{i\in I}\langle V^*y,h_i\rangle_{M(E)}\langle h_i,V^*y\rangle_{M(E)}C^{*^{-1}}\\ &= C^{-1}\sum_{i\in I}\langle y,Vh_i\rangle_{M(F)}\langle Vh_i,y\rangle_{M(F)}C^{*^{-1}}.\end{aligned}$$

similarly,

$$\sum_{i\in I}\langle y,Vh_i\rangle_{M(F)}\langle Vh_i,y\rangle_{M(F)} \leqslant D\langle V^*y,V^*y\rangle_E D^*.$$

Since V^* is an invertible element in $b(L(F,E))$ by [[13], 2.8], then

$$\left\|V^{*^{-1}}\right\|_\infty^{-2}\langle y,y\rangle \leqslant \langle V^*y,V^*y\rangle \leqslant \|V^*\|_\infty^2\langle y,y\rangle.$$

We conclude that

$$\left\|V^{*^{-1}}\right\|_{\infty}^{-1} C\langle y,y\rangle_F(\left\|V^{*^{-1}}\right\|_{\infty}^{-1} C)^* \leqslant \sum_{i\in I}\langle y,Vh_i\rangle_{M(F)}\langle Vh_i,y\rangle_{M(F)}$$

$$\leqslant \|V^*\|_\infty D\langle y,y\rangle_F(\|V^*\|_\infty D)^*.$$

■

Theorem 4.9 Suppose that sequence $\{h_i\}_{i\in I}$ in $M(E)$ is a standard $*$-frame of multipliers in E with $*$-frame bounds C and D respectively. Then the following holds:

i. The sequence $\{S^{-1}h_i\}_{i\in I}$ is a standard $*$-frame of multipliers in E with $*$-frame bounds D^{-1} and C^{-1}, respectively. Also $(D^*)^{-1}D^{-1}I_E \leqslant S^{-1} \leqslant (C^*)^{-1}C^{-1}I_E$.

ii. The equality $x=\sum_{i=1}^{\infty} h_i\left\langle S^{-1}h_i,x\right\rangle_{M(E)}=\sum_{i=1}^{\infty} S^{-1}h_i\left\langle h_i,x\right\rangle_{M(E)}$ is valid for every $x\in E$ and S is the unique operator with this property.

Proof. i. Let x be an arbitrary element in E. Then

$$\left\langle S^{-1}x,x\right\rangle_E=\left\langle S^{-1}x,SS^{-1}x\right\rangle_E=\sum_{i\in I}\left\langle S^{-1}x,h_i\right\rangle_{M(E)}\left\langle h_i,S^{-1}x\right\rangle_{M(E)}.$$

Similarly, $\left\langle x,S^{-1}x\right\rangle_E=\sum_{i\in I}\left\langle S^{-1}x,h_i\right\rangle_{M(E)}\left\langle h_i,S^{-1}x\right\rangle_{M(E)}$. Hence, S^{-1} is self-adjoint. Since $\{h_i\}_{i\in I}$ is standard $*$-frame of multipliers, we have

$$C\left\langle S^{-1}x,S^{-1}x\right\rangle_E C^* \leqslant \sum_{i\in I}\left\langle S^{-1}x,h_i\right\rangle_{M(E)}\left\langle h_i,S^{-1}x\right\rangle_{M(E)}=\left\langle S^{-1}x,x\right\rangle_E.$$

This shows that S^{-1} is positive operator and

$$\left\langle x,S^{-1}x\right\rangle_E=\sum_{i\in I}\left\langle S^{-1}x,h_i\right\rangle_{M(E)}\left\langle h_i,S^{-1}x\right\rangle_{M(E)}\leqslant D\left\langle S^{-1}x,S^{-1}x\right\rangle_E D^*,$$

which implies for all $x\in E$

$$D^{-1}\left\langle S^{-1}x,x\right\rangle_E(D^*)^{-1}\leqslant\left\langle S^{-1}x,S^{-1}x\right\rangle_E\leqslant C^{-1}\left\langle S^{-1}x,x\right\rangle_E(C^*)^{-1}.$$

So $D^{-1}(D^*)^{-1}S^{-1} \leqslant (S^{-1})^2 \leqslant C^{-1}(C^*)^{-1}S^{-1}$. Since S is positive operator, $D^{-1}(D^*)^{-1}I_E\leqslant S^{-1}\leqslant C^{-1}(C^*)^{-1}I_E$. For all $x\in E$, we have

$$D^{-1}\left\langle x,x\right\rangle_E(D^*)^{-1}\leqslant\sum_{i\in I}\left\langle S^{-1}x,h_i\right\rangle_{M(E)}\left\langle h_i,S^{-1}x\right\rangle_{M(E)}\leqslant C^{-1}\left\langle x,x\right\rangle_E(C^*)^{-1}.$$

Consequently $\{S^{-1}h_i\}_{i\in I}$ is a standard $*$-frame of multipliers in E.

ii. For every $x\in E$, we have:

$$x=S(S^{-1}x)=\sum_{i\in I}h_i\left\langle S^{-1}h_i,x\right\rangle_{M(E)},$$

$$x=S^{-1}(Sx)=\sum_{i\in I}S^{-1}h_i\langle h_i,x\rangle_{M(E)}.$$

Suppose that $\theta \in L(E)$ is an invertible positive operator such that for every $x \in E$, $x = \sum_{i\in I} h_i \langle \theta^{-1} h_i, x\rangle_{M(E)}$. Then

$$x = \sum_{i\in I} h_i \langle \theta^{-1} h_i, x\rangle_{M(E)} = \sum_{i=1}^{\infty} h_i \langle h_i, (\theta^{-1})^* x\rangle_{M(E)} = S(\theta^{-1})^* x,$$

which implies that $S(\theta^{-1})^* = I_E$. By tacking adjoints on both sides, we get $\theta^{-1}S = I_E$, and hence $\theta = S$. ■

The sequence $\{S^{-1}h_i\}_{i\in I}$ is called the canonical standard $*$-frame of multipliers of $\{h_i\}_{i\in I}$.

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