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Existence of best proximity and fixed points in G_p-metric spaces

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Abstract. In this paper, we establish some best proximity point theorems using new proximal contractive mappings in asymmetric G_p-metric spaces. Our motive is to find an optimal approximate solution of a fixed point equation. We provide best proximity points for cyclic contractive mappings in G_p-metric spaces. As consequences of these results, we deduce fixed point results in G_p-metric spaces. We also provide examples to analyze and support our results.

Keywords: Fixed point, G_p-metric space, orbital admissible mapping.

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1. Introduction

Fixed point theory mainly concerns with the fixed point equation $Tx = x$, where $T : A \to B$ is some nonlinear operator. The solution of this equation is known as fixed point of the operator T. But it is not necessary that the equation has a solution. In that case when T has no fixed point, best approximation results provide an approximate solution to the fixed point equation $Tx = x$. Best proximity point results provide optimal approximate solution of the fixed point equation, in this case we may find an element $x \in A$ which is closest to Tx; that is, the distance between Tx and x is least as compare to other elements of A. Such a point is called the best proximity point of T. Many researchers have directed their attention to this field and proved best proximity point theorems in various settings (see [5, 6, 20, 24, 25, 28]).

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On the other hand, Zand and Nezhad [30] introduced the notion of G_p-metric spaces which are a combination of the notions of partial metric spaces and G-metric spaces (also, see [1]).

Definition 1.1 [30] Let X be a nonempty set. A function $G_p : X \times X \times X \to [0, \infty)$ is called a G_p-metric if the following conditions are satisfied:

$(GP1)$ $x = y = z$ if $G_p(x, y, z) = G_p(z, z, z) = G_p(y, y, y) = G_p(x, x, x)$;
$(GP2)$ $0 \leqslant G_p(x, x, x) \leqslant G_p(x, x, y) \leqslant G_p(x, y, z)$ for all $x, y, z \in X$;
$(GP3)$ $G_p(x, y, z) = G_p(x, z, y) = G_p(y, z, x) = ...$, symmetry in all three variables;
$(GP4)$ $G_p(x, y, z) \leqslant G_p(x, a, a) + G_p(a, y, z) - G_p(a, a, a)$ for any $x, y, z, a \in X$.

Then the pair (X, G_p) is called a G_p-metric space.

A number of authors have published many fixed point results on the setting of generalized metric spaces (see [4–30]). By inspiring this research many authors proved fixed and best proximity point results in G_p-metric spaces. With the $(GP2)$ condition it is easy to see that $G_p(x, x, y) = G_p(x, y, y)$ holds for all $x, y \in X$, this implies the space is symmetric. But then the claim in [30] that each G-metric space is also G_p-metric space is false, since it is well known that the condition of symmetry might not hold in G-metric space. Then to overcome this problem Parvaneh et al. [19] replaced $(GP2)$ by the condition following

$$0 \leqslant G_p(x, x, x) \leqslant G_p(x, x, y) \leqslant G_p(x, y, z) \qquad \forall x, y, z \in X \text{ with } y \neq z.$$

This definition implies that in each case G-metric space is G_p-metric space, but a G_p-metric space might be asymmetric.

***Example* 1.2** [30] Let $X = [0, \infty)$ and $G_p(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$ for all $x, y, z \in X$. Then (X, G_p) is a symmetric G_p-metric space.

***Example* 1.3** [19] Let $X = \{0, 1, 2, 3\}$ and

$$\begin{aligned} A = \{&(1,0,0),(0,1,0),(0,0,1),(2,0,0),(0,2,0),(0,0,2),(3,0,0),(0,3,0),(0,0,3),\\ &(1,2,2),(2,1,2),(2,2,1),(1,3,3,),(3,1,3),(3,3,1),(2,3,3),(3,2,3),(3,3,2)\},\\ B = \{&(0,1,1),(1,0,1),(1,1,0),(0,2,2),(2,0,2),(2,2,0),(0,3,3),(3,0,3),(3,3,0),\\ &(2,1,1),(1,2,1),(1,1,2),(3,1,1),(1,3,1),(1,1,3),(3,2,2),(2,3,2),(2,2,3)\} \end{aligned}$$

Define $G_p : X \times X \times X \to R^+$ by

$$G_p(x, y, z) = \begin{cases} 1, & \text{if } x = y = z \neq 2,\\ 0, & \text{if } x = y = z = 2,\\ 2, & \text{if } (x, y, z) \in A,\\ \frac{5}{2} & \text{if } (x, y, z) \in B,\\ 3 & x \neq y \neq z \neq x. \end{cases}$$

It is easy to see that (X, G_p) is an asymmetric G_p-metric space.

Recently, Ansari et al. [2] introduced a new G-ψ-ϕ-f-proximal contractive type mappings in G-metric spaces. Motivated and inspired by the research we prove certain best proximity point theorems for proximal contractive pair of mappings.

First, we recollect some necessary definitions and fundamental results produced on G_p-metric spaces that we will need in this work.

Proposition 1.4 [30] Every G_p-metric space (X, G_p) defines a metric space (X, d_{G_p}), where $d_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x) - G_p(y,y,y)$ for all $x, y \in X$.

Proposition 1.5 [30] Let (X, G_p) be a G_p-metric space. Then for any x, y, z and $a \in X$, it follows that

(i) $G_p(x,y,z) \leqslant G_p(x,x,y) + G_p(x,x,z) - G_p(x,x,x)$;
(ii) $G_p(x,y,y) \leqslant 2G_p(x,x,y) - G_p(x,x,x)$;
(iii) $G_p(x,y,z) \leqslant G_p(x,a,a) + G_p(y,a,a) + G_p(z,a,a) - 2G_p(a,a,a)$;
(iv) $G_p(x,y,z) \leqslant G_p(x,a,z) + G_p(a,y,z) - G_p(a,a,a)$.

Definition 1.6 [21] Let $T : X \to X$ be a map and $\alpha : X \times X \to R$ be a function. Then T is said to be α-orbital admissible if $\alpha(x, Tx) \geqslant 1$ implies $\alpha(Tx, T^2x) \geqslant 1$.

Definition 1.7 [21] Let $T : X \to X$ be a map and $\alpha : X \times X \to R$ be a function. Then T is said to be triangular α-orbital admissible if T is α-orbital admissible, and $\alpha(x,y) \geqslant 1$ and $\alpha(y, Ty) \geqslant 1$ imply $\alpha(x, Ty) \geqslant 1$.

Definition 1.8 [9] A function $\phi : [0,\infty) \to [0,\infty)$ is called upper semi-continuous from the right if for each $t \geqslant 0$ and each sequence $\{t_n\}_{n\in N}$ such that $t_n \geqslant t$ and $\lim_{n\to\infty} t_n = t$, then equality holds $\limsup_{n\to\infty} \phi(t_n) \leqslant \phi(t)$.

Definition 1.9 [9] Let (X, G_p) be a G_p-metric space and $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of sequence $\{x_n\}$ if

$$\lim_{m,n\to\infty} G_p(x, x_m, x_n) = G_p(x,x,x).$$

Definition 1.10 [9] Let (X, G_p) be a G_p-metric space. A sequence $\{x_n\}$ is called a G_p-Cauchy if and only if $\lim_{m,n,r\to\infty} G_p(x_n, x_m, x_r)$ exists and finite.

Definition 1.11 [9] A G_p-metric space (X, G_p) is said to be G_p-complete if and only if every G_p-Cauchy sequence in X is G_p-convergent to $x \in X$ such that

$$\lim_{m,n,r\to\infty} G_p(x_n, x_m, x_r) = G_p(x,x,x).$$

Definition 1.12 [3] Let (X, G_p) be a complete G_p-metric space, $\alpha : X \times X \to R$ be a function and let $T : X \to X$ be a map. We say that the sequence $\{x_n\}$ is α-regular, if the following condition is satisfied:
If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$, for all n and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geqslant 1$ for all k.

Definition 1.13 [13] Let $T : A \to B$ be a map and $\alpha : X \times X \to [0,\infty)$ be a function. The mapping T is said to be α-proximal admissible if

$$\left.\begin{array}{r}\alpha(x,y) \geqslant 1\\ d(u,Tx) = d(A,B)\\ d(v,Ty) = d(A,B)\end{array}\right\} \Longrightarrow \alpha(u,v) \geqslant 1$$

for all $x, y, u, v \in A$.

Lemma 1.14 [21] Let $T : X \to X$ be a triangular α-orbital admissible mapping. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geqslant 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geqslant 1$ for all $m, n \in N$ with $n < m$.

Remark 1 *([26]) Let (X, d) be a metric space. Then for given nonempty subsets A and B, we define A_0 and B_0 as follows:*

$$d(A,B) = inf\{d(x,y) : x \in A \;\; and \;\; y \in B\},$$
$$A_0 = \{x \in A : d(x,y) = d(A,B) \;\; for\ some \;\; y \in B\},$$
$$B_0 = \{y \in B : d(x,y) = d(A,B) \;\; for\ some \;\; x \in A\}.$$

If $A \cap B \neq \phi$, then A_0 and B_0 are nonempty. It is also interesting to note that if A and B are closed subsets of normed linear space such that $d(A, B) \succ 0$ then A_0 and B_0 are contained in the boundaries of A and B respectively.

2. Main results

Before proving our main result, firstly we introduce the following definition of G_p-ϕ-proximal contraction.

Definition 2.1 Let A and B be two nonempty subsets of a G_p-metric space (X, G_p) and $T : A \to B$ be a non-self mapping. We say that T is a G_p-ϕ-proximal contraction mapping if for $x, y, u, v \in A$

$$\left.\begin{array}{r} d_{Gp}(u,Tx) = d_{Gp}(A,B) \\ d_{Gp}(u^*,Tu) = d_{Gp}(A,B) \\ d_{Gp}(v,Ty) = d_{Gp}(A,B) \end{array}\right\} \Longrightarrow G_p(u,u^*,v) \leqslant \phi(G_p(x,u,y)), \tag{1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function from the right such that $\phi(t) < t$ for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$.

Theorem 2.2 Let A and B be two non-empty subsets of G_p-metric space (X, G_p) such that (A, G_p) is complete G_p-metric space, A_0 is non-empty and B is approximatively compact with respect to A. Assume that $T : A \to B$ is G_p-ϕ-proximal contraction mapping such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.

Proof. Since the subset A_0 is non-empty, we take $x_0 \in A_0$. Taking $Tx_0 \in T(A_0) \subseteq B_0$ in account, we can find $x_1 \in A_0$ such that $d_{Gp}(x_1, Tx_0) = d_{Gp}(A, B)$. Further since $Tx_1 \in T(A_0) \subseteq B_0$, it follows that there is an element $x_2 \in A_0$ such that $d_{Gp}(x_2, Tx_1) = d_{Gp}(A, B)$. Repeatedly, we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$d_{Gp}(x_{n+1}, Tx_n) = d_{Gp}(A,B)$$

for all $n \in N \cup \{0\}$. In (1), set $x = x_{n-1}, u = x_n, u* = x_{n+1}, y = x_n$ and $v = x_{n+1}$. Then we have $G_p(x_n, x_{n+1}, x_{n+1}) \leqslant \phi(G_p(x_{n-1}, x_n, x_n))$. Hence,

$$G_p(x_n, x_{n+1}, x_{n+1}) \leqslant \phi(G_p(x_{n-1}, x_n, x_n)) < G_p(x_{n-1}, x_n, x_n). \tag{2}$$

So, the sequence $\{G_p(x_n, x_{n+1}, x_{n+1})\}$ is decreasing sequence in R^+ and it is convergent to $t \in R^+$. We claim that $t = 0$. Suppose, to the contrary, that $t > 0$. Taking limit as

$n \to \infty$ in (2), we have

$$\lim_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) \leqslant \phi(\lim_{n\to\infty} G_p(x_{n-1}, x_n, x_n));$$

that is, $t \leqslant \phi(t)$ which is contradiction, since $\phi(t) < t$. Thus, $t = 0$ and

$$\lim_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) = 0. \tag{3}$$

Next, we claim that the sequence $\{x_n\}$ is G_p-Cauchy sequence. Suppose, to the contrary, that there exists $\epsilon < 0$, and a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$G_p(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) \geqslant \epsilon \tag{4}$$

with $n(k) \geqslant m(k) > k$. Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer satisfying (4). Hence,

$$G_p(x_{m(k)}, x_{m(k)+1}, x_{n(k)-1}) < \epsilon.$$

Now, we have

$$\begin{aligned}
\epsilon &\leqslant G_p(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) \\
&= G_p(x_{n(k)}, x_{m(k)}, x_{m(k)+1}) \\
&\leqslant G_p(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G_p(x_{n(k)-1}, x_{m(k)}, x_{m(k)+1}) - G_p(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)-1}) \\
&\leqslant G_p(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G_p(x_{n(k)-1}, x_{m(k)}, x_{m(k)+1}) \\
&< G_p(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + \epsilon. \qquad (5)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
G_p(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) &= G_p(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)}) \\
&\leqslant G_p(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G_p(x_{n(k)}, x_{n(k)-1}, x_{n(k)}) \\
&\quad - G_p(x_{n(k)}, x_{n(k)}, x_{n(k)}) \\
&< G_p(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G_p(x_{n(k)}, x_{n(k)-1}, x_{n(k)}) \\
&= G_p(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G_p(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) \\
&= 2G_p(x_{n(k)-1}, x_{n(k)}, x_{n(k)}). \qquad (6)
\end{aligned}$$

By putting (6) in (5), we have

$$\begin{aligned}
\epsilon &\leqslant G_p(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) \\
&< G_p(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + \epsilon \\
&< 2G_p(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + \epsilon. \qquad (7)
\end{aligned}$$

Taking limit $k \to \infty$ and using (3), we get

$$\lim_{k\to\infty} G_p(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) = \epsilon. \tag{8}$$

Also,

$$\begin{aligned} G_p(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) &\leqslant G_p(x_{m(k)-1}, x_{m(k)+1}, x_{n(k)}) - G_p(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)-1}) \\ &\leqslant G_p(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G_p(x_{n(k)}, x_{m(k)-1}, x_{m(k)+1}) \\ &\leqslant G_p(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G_p(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) \\ &\quad + G_p(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}) - G_p(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)-1}) \\ &\leqslant G_p(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G_p(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) \\ &\quad + G_p(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}) \end{aligned} \tag{9}$$

and

$$\begin{aligned} G_p(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}) &\leqslant G_p(x_{n(k)}, x_{m(k)-1}, x_{m(k)+1}) - G_p(x_{n(k)}, x_{n(k)}, x_{n(k)}) \\ &\leqslant G_p(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G_p(x_{m(k)-1}, x_{m(k)+1}, x_{n(k)}) \\ &\leqslant G_p(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G_p(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) \\ &\quad + G_p(x_{m(k)}, x_{m(k)+1}, x_{n(k)}). \end{aligned} \tag{10}$$

Taking limit $k \to \infty$ and applying (3), (6) and (8), we get

$$\lim_{k\to\infty} G_p(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}) = \epsilon.$$

In the similar way, we can prove that

$$\lim_{k\to\infty} G_p(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}) = \epsilon.$$

For equation (1) with $x = x_{n(k)-1}, u = x_{m(k)}, u* = x_{m(k)+1}, y = x_{n(k)-1}, v = x_{n(k)}$, we have

$$G_p(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) \leqslant \phi(G_p(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1})).$$

Now, let $k \to \infty$ in above relation. Then $\epsilon \leqslant \phi(\epsilon)$, which is contradiction. Thus,

$$\lim_{m,n\to\infty} G_p(x_m, x_{m+1}, x_n) = 0;$$

that is, $\{x_n\}$ is a Cauchy sequence. Since (A, G_p) is a complete G_p-metric space, there exists $z \in A$ such that $x_n \to z$ as $n \to \infty$. On the other hand, for all $n \in N$, we can write

$$\begin{aligned} d_{Gp}(z, B) &\leqslant d_{Gp}(z, Tx_n) \\ &\leqslant d_{Gp}(z, x_{n+1}) + d_{Gp}(x_{n+1}, Tx_n) \\ &= d_{Gp}(z, x_{n+1}) + d_{Gp}(A, B). \end{aligned}$$

Taking the limits $n \to \infty$, we get $\lim_{n\to\infty} d_{Gp}(z, Tx_n) = d_{Gp}(z, B) = d_{Gp}(A, B)$. Since B is approximatively compact w.r.t. A. So the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n(k)}\}$ that converges to some $y^* \in B$. Hence,

$$d_{Gp}(z, y^*) = \lim_{n\to\infty} d_{Gp}(x_{n(k)+1}, Tx_{n(k)}) = d_{Gp}(A, B).$$

So, $z \in A_0$. Since $Tz \in T(A_0) \subseteq B_0$, there exists $w \in A_0$ such that $d_{Gp}(w, Tz) = d_{Gp}(A, B)$. Consider (1) with $u = x_{n+1}, u* = x_{n+2}, v = w, x = z, y = z$, we have

$$G_p(x_{n+1}, x_{n+2}, w) \leqslant \phi(G_p(x_n, x_{n+1}, z)).$$

Now, taking limit $n \to \infty$, we obtain

$$G_p(z, z, w) \leqslant \phi(G_p(z, z, z)) < G_p(z, z, z) < G_p(z, z, w) \qquad \forall z \neq w,$$

which is contradiction. Thus $z = w$. ■

Definition 2.3 Let A and B be two nonempty subsets of G_p-metric space (X, G_p) and $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : A \to B$ is said to be $G_p - \alpha - \phi$ proximal contraction if, for all $x, y, u, v \in A$,

$$\left.\begin{array}{r} d_{Gp}(u, Tx) = d_{Gp}(A, B) \\ d_{Gp}(v, Ty) = d_{Gp}(A, B) \end{array}\right\} \Longrightarrow \alpha(x, y) G_p(u, v, v) \leqslant \phi(G_p(x, y, y)),$$

where $\phi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function from the right such that $\phi(t) < t$ for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$.

Theorem 2.4 Let A and B be two nonempty subsets of G_p-metric space (X, G_p) such that (A, G_p) is complete G_p-metric space and $\alpha : X \times X \to [0, \infty)$ be a function. Also, let $T : A \to B$ be a mapping and the pair (A, B) has P-property. Suppose that the following conditions are satisfied:

(1) T is G_p-α-ϕ-proximal contraction;
(2) T is α-proximal admissible and $T(A_0) \subseteq B_0$;
(3) if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geqslant 1$ for all k;
(4) there exist $x_0, x_1 \in A$ such that $d_{Gp}(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geqslant 1$.

Then there exists an elements $x^* \in A$ such that $d_{Gp}(x^*, Tx^*) = d_{Gp}(A, B)$.

Proof. Let x_0, x_1 be two elements in A such that $d_{Gp}(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geqslant 1$. Thus, $x_1 \in A_0$. As $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. Since T is α-proximal admissible, it follows $\alpha(x_1, x_2) \geqslant 1$,. Continuing in this way, we can construct a sequence $\{x_n\}$ in A_0 such that $d_{Gp}(x_n, Tx_{n-1}) = d_{Gp}(A, B)$ and $\alpha(x_n, x_{n-1}) \geqslant 1$ for all $n \in N$. This implies that

$$d_{Gp}(x_n, Tx_{n-1}) = d_{Gp}(A, B),$$
$$d_{Gp}(x_{n+1}, Tx_n) = d_{Gp}(A, B).$$

Using lemma 1.14, since T is G_p-α-proximal contraction, we have

$$\begin{aligned} G_p(x_n, x_{n+1}, x_{n+1}) &\leqslant \alpha(x_{n-1}, x_n)G_p(x_n, x_{n+1}, x_{n+1}) \\ &\leqslant \phi(G_p(x_{n-1}, x_n, x_n)) \\ &\leqslant \phi(G_p(x_{n-1}, x_n, x_n)) \\ &< G_p(x_{n-1}, x_n, x_n), \end{aligned}$$

which implies that $G_p(x_n, x_{n+1}, x_{n+1}) < G_p(x_{n-1}, x_n, x_n)$. Therefore, the sequence $\{G_p(x_n, x_{n+1}, x_{n+1})\}$ is decreasing sequence in R^+ and it is convergent to $t \in R^+$. We claim that $t = 0$. Suppose, on the contrary that $t > 0$. Taking limit $n \to \infty$, we have $t \leqslant \phi(t)$, which is contradiction. Hence, $t = 0$; that is, $\lim_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) = 0$. Now, we will show that $\{x_n\}$ is a G_p-Cauchy sequence. Suppose, to the contrary, that there exists $\epsilon > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$G_p(x_{m_k}, x_{n_k}, x_{n_k}) \geqslant \epsilon \tag{11}$$

with $n_k \geqslant m_k > k$. Further corresponding to m_k, we can choose n_k in such a way that it is the smallest integer and satisfying (11). Hence,

$$G_p(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) < \epsilon. \tag{12}$$

On the other hand,

$$\begin{aligned} \epsilon &\leqslant G_p(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ &= G_p(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G_p(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) - G_p(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)-1}) \\ &\leqslant G_p(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G_p(x_{n(k)-1}, x_{n(k)}, x_{n(k)}). \end{aligned}$$

By taking limit, we have $\epsilon \leqslant \lim_{k\to\infty} G_p(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leqslant \epsilon$, which implies that

$$\lim_{k\to\infty} G_p(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \epsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence. Since (A, G_p) is complete G_P-metric space, there exists $z \in A$ such that $x_n \to z$ as $n \to \infty$. On the other hand, for all $n \in N$, we can write

$$\begin{aligned} d_{Gp}(z, B) &\leqslant d_{Gp}(z, Tx_n) \\ &\leqslant d_{Gp}(z, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &= d_{Gp}(z, x_{n+1}) + d_{Gp}(A, B). \end{aligned} \tag{13}$$

Taking the limit from (13) as $n \to \infty$, we get

$$\lim_{n\to\infty} d_{Gp}(z, Tx_n) = d_{Gp}(z, B) = d_{Gp}(A, B).$$

So $z \in A_0$. Since $Tz \in T(A_0) \subseteq B_0$, there exists $w \in A_0$ such that $d_{Gp}(w, Tz) = d(A, B)$ and $d_{Gp}(x_{n+1}, Tx_n) = d(A, B)$. Consider

$$G_p(x_{n+1}, w, w) \leqslant \alpha(x_n, z)G_p(x_{n+1}, w, w) \leqslant \phi(G_p(x_n, z, z)).$$

Taking limit $n \to \infty$, we have

$$G_p(w,z,z) \leqslant \phi(G_p(z,z,z)) < G_p(z,z,z) \leqslant G_p(z,z,w)$$

for $w \neq z$. This implies $G_p(w,z,z) < G_p(z,z,w)$ for $w \neq z$, which is contradiction. Thus $z = w$ and $d_{Gp}(w,Tw) = d_{Gp}(A,B)$. Hence T has a best proximity point. ■

Definition 2.5 Let $T : X \to X$ and $\eta : X \times X \times X \to [0,\infty)$. We say that T is η-orbital admissible if for all $x, y, z \in X$, $\eta(x,Ty,Ty) \geqslant 1$ implies $\eta(Tx,T^2y,T^2z) \geqslant 1$.

Definition 2.6 Let $T : X \to X$ and $\eta : X \times X \times X \to [0,\infty)$ be two functions, then T is said to be triangular η-orbital admissible if T is η-orbital admissible and $\eta(x,y,y) \geqslant 1, \eta(y,Ty,Ty) \geqslant 1$ implies $\eta(x,Ty,Ty) \geqslant 1$.

Lemma 2.7 [21] Let $T : X \to X$ be a triangular η-orbital admissible mapping. Assume that there exists $x_1 \in X$ such that $\eta(x_1,Tx_1,Tx_1) \geqslant 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\eta(x_n,x_m,x_m) \geqslant 1$ for all $m, n \in N$ with $n < m$.

Proof. Since T is η-orbital admissible and $\eta(x_1,Tx_1,Tx_1) \geqslant 1$; that is, $\eta(x_1,x_2,x_2) \geqslant 1$ we deduce that $\eta(Tx_1,Tx_2,Tx_2) = \eta(x_2,x_3,x_3) \geqslant 1$. By continuing this process, we get $\eta(x_n,x_{n+1},x_{n+1}) \geqslant 1$ for all $n \geqslant 1$. Suppose that $\eta(x_n,x_m,x_m) \geqslant 1$ and prove that $\eta(x_n,x_{m+1},x_{m+1}) \geqslant 1$ where $m > n$. Since T is triangular η -orbital admissible and $\eta(x_m,x_{m+1},x_{m+1}) \geqslant 1$ we get that $\eta(x_n,x_{m+1},x_{m+1}) \geqslant 1$. Hence we have proved that $\eta(x_n,x_m,x_m) \geqslant 1$ for all $n, m \in N$ with $m > n$. ■

Definition 2.8 Let A and B be two nonempty subsets of a G_p metric space (X,G_p). Let $T : A \cup B \to A \cup B$ be a non-self mapping such that $T(A) \subset B, T(B) \subset A$. T is said to be G_p-η-proximal cyclic weak contraction if for $x, u, u* \in A$, $v, y \in B$

$$\left.\begin{aligned} d_{Gp}(u,Tu*) = d_{Gp}(A,B) \\ d_{Gp}(u*,Tx) = d_{Gp}(A,B) \\ d_{Gp}(v,Ty) = d_{Gp}(A,B) \end{aligned}\right\} \Longrightarrow \eta(u*,u,v)G_p(u*,u,v) \leqslant \phi(M(x,v,y)), \tag{14}$$

where $M(x,v,y) = \max\{G_p(x,v,y), G_p(x,Tx,Tx), G_p(y,Ty,Ty)\}$.

Theorem 2.9 Let A and B be two nonempty subsets of a G_p-metric space (X,G_p), (A,G_p) and (B,G_p) be complete G_p-metric spaces, A_0 be nonempty set and B_0 be approximatively compact w.r.t. A. Assume that $T : A \cup B \to A \cup B$ is $G_p - \eta$-proximal cyclic weak contraction such that $T(A) \subset B$, $T(B) \subset A$ and $T(A_0) \subseteq B_0$ and T is triangular η-proximal admissible mapping such that $\eta(Tx_1,Tx_1,x_0) \geqslant 1$. Then T has a best proximity point.

Proof. If $x_0 \in A_0$, then $x_1 = Tx_0 \in T(A_0) \subseteq B$. thus, $d_{Gp}(x_0,Tx_0) = d_{Gp}(x_0,x_1) = d_{Gp}(A,B)$. Further since $x_2 = Tx_1 \in T(B_0) \subseteq A$, it follows that $d_{Gp}(x_1,Tx_1) = d_{Gp}(x_1,x_2) = d_{Gp}(A,B)$. Recursively, we obtain sequence $\{x_n\}$ in $A \cup B$ satisfying $d_{Gp}(x_n,x_{n+1}) = d_{Gp}(A,B)$ for all $n \in N \cup \{0\}$. In (14), set $x = x_{n-1}, u = x_{n+1}, u* = x_{n+1}, y = x_n$ and $v = x_n$. Then we get

$$\eta(x_{n+1},x_{n+1},x_n)G_p(x_{n+1},x_{n+1},x_n) \leqslant \phi(M(x_{n-1},x_n,x_n)), \tag{15}$$

where

$$\begin{aligned} M(x_{n-1}, x_n, x_n) &= \max\{G_p(x_{n-1}, x_n, x_n), G_p(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G_p(x_n, Tx_n, Tx_n)\} \\ &= \max\{G_p(x_{n-1}, x_n, x_n), G_p(x_{n-1}, x_n, x_n), G_p(x_n, x_{n+1}, x_{n+1})\} \\ &= \max\{G_p(x_{n-1}, x_n, x_n), G_p(x_n, x_{n+1}, x_{n+1})\}. \end{aligned}$$

Let $M(x_{n-1}, x_n, x_n) = G_p(x_n, x_{n+1}, x_{n+1})$. Then we have $\phi(M(x_{n-1}, x_n, x_n)) = \phi(G_p(x_n, x_{n+1}, x_{n+1}))$. Hence,

$$\begin{aligned} G_p(x_n, x_{n+1}, x_{n+1}) &= G_p(x_{n+1}, x_{n+1}, x_n) \\ &\leqslant \eta(x_{n+1}, x_{n+1}, x_n) G_p(x_{n+1}, x_{n+1}, x_n) \\ &= \eta(x_{n+1}, x_{n+1}, x_n) G_p(x_n, x_{n+1}, x_{n+1}) \\ &\leqslant \phi(M(x_{n-1}, x_n, x_n)) \\ &= \phi(G_p(x_n, x_{n+1}, x_{n+1})). \end{aligned}$$

So, $G_p(x_n, x_{n+1}, x_{n+1}) = 0$; that is, $x_n = x_{n+1}$ and each x_n is fixed point, which is contradiction. Hence, $M(x_{n-1}, x_n, x_n) = G_p(x_{n-1}, x_n, x_n)$ and $G_p(x_n, x_{n+1}, x_{n+1}) \leqslant \phi(G_p(x_{n-1}, x_n, x_n)) < G_p(x_{n-1}, x_n, x_n)$, which implies that $G_p(x_n, x_{n+1}, x_{n+1}) < G_p(x_{n-1}, x_n, x_n)$. Thus, the sequence $\{G_p(x_n, x_{n+1}, x_{n+1})\}$ is decreasing sequence in R^+. So, it is convergent to $t \in R^+$. We claim that $t = 0$. Suppose on the contrary that $t > 0$. Taking limit $n \to \infty$, we have

$$\lim_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) \leqslant \lim_{n\to\infty} \phi(G_p(x_{n-1}, x_n, x_n)).$$

Thus, $t \leqslant \phi(t)$, which is contradiction. Hence, $t = 0$.

$$\lim_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) = 0. \tag{16}$$

Now, we will show that $\{x_n\}$ is an G_p-Cauchy sequence. Suppose on the contrary that, there exists $\epsilon > 0$ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$G_p(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geqslant \epsilon \tag{17}$$

with $n(k) \geqslant m(k) > k$. Further, corresponding to value of $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer satisfying inequality (17). Hence,

$$G_p(x_{m(k)}, G_p(x_{(n(k)-1)}, G_p(x_{n(k)-1})) < \epsilon.$$

By (16) and (17), we have

$$\begin{aligned} &\epsilon \leqslant G_p(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ &\quad = G_p(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G_p(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) - G_p(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)-1}) \\ &\quad \leqslant G_p(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G_p(x_{n(k)-1}, x_{n(k)}, x_{n(k)}). \end{aligned}$$

By taking limit, $\lim_{k\to\infty} G_p(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \epsilon$. Now, let $G_p(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) < \epsilon$ and $G_p(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) < \epsilon$ for all $k \geqslant k_0$ with a $k_0 \in N$. Then, for all $k \geqslant k_0$,

$$G_p(x_{m(k)}, x_{n(k)}, x_{n(k)}) = M(x_{m(k)}, x_{n(k)}, x_{n(k)}).$$

So $\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \epsilon$, and for all $k \in N$, $M(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geqslant \epsilon$ (by (17)). Since ϕ is upper semi-continuous from the right, we deduce that

$$\limsup_{k\to\infty} \phi(M(x_{m(k)}, x_{n(k)}, x_{n(k)})) \leqslant \phi(\epsilon).$$

Also,

$$\begin{aligned}
\epsilon &\leqslant G_p(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\
&= G_p(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\
&\leqslant G_p(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G_p(x_{m(k)}, x_{n(k)}, x_{m(k)}) - G_p(x_{m(k)}, x_{m(k)}, x_{m(k)}) \\
&= G_p(x_{m(k)}, x_{m(k)}, x_{n(k)}) + G_p(x_{m(k)}, x_{n(k)}, x_{m(k)}) - G_p(x_{m(k)}, x_{m(k)}, x_{m(k)}) \\
&\leqslant \eta(x_{m(k)}, x_{m(k)}, x_{n(k)}) G_p(x_{m(k)}, x_{m(k)}, x_{n(k)}) + G_p(x_{m(k)}, x_{n(k)}, x_{m(k)}) \\
&\quad - G_p(x_{m(k)}, x_{m(k)}, x_{m(k)}) \\
&\leqslant \phi(M(x_{m(k)}, x_{n(k)}, x_{n(k)})) + G_p(x_{m(k)}, x_{n(k)}, x_{m(k)}) - G_p(x_{m(k)}, x_{m(k)}, x_{m(k)}) \\
&\leqslant \phi(M(x_{m(k)}, x_{n(k)}, x_{n(k)})) + G_p(x_{m(k)}, x_{n(k)}, x_{n(k)}) + G_p(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\
&\quad - G_p(x_{n(k)}, x_{n(k)}, x_{n(k)}) - G_p(x_{m(k)}, x_{m(k)}, x_{m(k)}).
\end{aligned}$$

Taking limit $k \to \infty$, then $\epsilon \leqslant \phi(\epsilon) < \epsilon$. Consequently $\lim_{m,n\to\infty} G_p(x_m, x_n, x_n) = 0$ and $\{x_n\}$ is a Cauchy sequence in G_p-complete G_p-metric space (X, G_p). Since A and B are complete, there exists $z \in A \subset A \cup B$ such that $x_n \to z$ as $n \to \infty$. On the other hand,

$$\begin{aligned}
d_{Gp}(z, B) &\leqslant d_{Gp}(z, Tx_n) \\
&= d_{Gp}(z, x_{n+1}) \\
&\leqslant d_{Gp}(z, x_n) + d_{Gp}(x_n, x_{n+1}) \\
&\leqslant d_{Gp}(z, x_n) + d_{Gp}(x_n, x_{n+1}) \\
&\leqslant d_{Gp}(z, x_n) + d_{Gp}(A, B) \\
&\leqslant d_{Gp}(z, x_n) + d_{Gp}(z, B)
\end{aligned}$$

for each $n \in N$. Taking limit $n \to \infty$ in above inequality, we get

$$\begin{aligned}
d_{Gp}(z, B) &\leqslant \lim_{n\to\infty} d_{Gp}(z, Tx_n) \\
&= d_{Gp}(z, B) \\
&= d_{Gp}(A, B).
\end{aligned}$$

Since B is approximatively compact w.r.t. A, so the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n(k)}\}$ that converges to some $y* \in B \subset A \cup B$. Hence,

$$d_{Gp}(z, y*) = \lim_{n\to\infty} d_{Gp}(x_{n(k)}, Tx_{n(k)}) = d_{Gp}(A, B).$$

So, $z \in A_0$. Now, since $Tz \in T(A_0) \subseteq B_0$, there exists $w \in A_0$ such that $d_{Gp}(w, Tz) = d_{Gp}(A, B)$. From given condition (14) with $x = x_{n-1}$, $u = w$, $u* = z$, $y = x_n$ and $v = x_n$, we get

$$G_p(z, w, x_n) \leqslant \eta(z, w, x_n)G_p(z, w, x_n) \leqslant \phi(M(x_{n-1}, x_n, x_n)), \tag{18}$$

where

$$\begin{aligned} M(x_{n-1}, x_n, x_n) &= \max\{G_p(x_{n-1}), x_n, x_n), G_p(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G_p(x_n, Tx_n, Tx_n)\} \\ &= \max\{G_p(x_{n-1}, x_n, x_n), G_p(x_n, x_{n+1}, x_{n+1})\}. \end{aligned}$$

Using (18), we have

$$G_p(z, w, x_n) \leqslant \eta(z, w, x_n)G_p(z, w, x_n) \leqslant \phi(\max\{G_p(x_{n-1}, x_n, x_n), G_p(x_n, x_{n+1}, x_{n+1})\}).$$

Taking limit $n \to \infty$, we get $G_p(z, w, z) \leqslant \phi(0) = 0$ and so, $G_p(z, z, w) = 0$. This implies $z = w$. Thus, $d_{Gp}(z, Tz) = d_{Gp}(A, B)$ and T has a best proximity point. ■

If we consider the above theorem with $\eta(u*, u, v) = 1$ and $\phi(t) = t$, then we get the following corollary.

Corollary 2.10 Let A, B be two non-empty subsets of a G_p-metric space (X, G_p) such that $(A, G_p), (B, G_p)$ are complete G_p-metric spaces, A_0 is non-empty and B_0 is approximatively compact w.r.t. A. Assume that $T : A \cup B \to A \cup B$ such that

$$\left.\begin{aligned} d_{Gp}(u, Tu*) = d_{Gp}(A, B) \\ d_{Gp}(u*, Tu) = d_{Gp}(A, B) \\ d_{Gp}(v, Ty) = d_{Gp}(A, B) \end{aligned}\right\} \Longrightarrow G_p(u*, u, v) \leqslant M(x, v, y),$$

where $M(x, v, y) = \max\{G_p(x, v, y), G_p(x, Tx, Tx), G_p(y, Ty, Ty)\}$, $T(A) \subset B$, $T(B) \subset A$ and $T(A_0) \subset B_0$. Then T has a best proximity point.

***Example* 2.11** Let $X = \{0, 1, 2, 3\}$ and

$$\begin{aligned} A = \{&(1,0,0),(0,1,0),(0,0,1),(2,0,0),(0,2,0),(0,0,2),(3,0,0),(0,3,0),(0,0,3), \\ &(1,2,2),(2,1,2),(2,2,1),(1,3,3,),(3,1,3),(3,3,1),(2,3,3),(3,2,3),(3,3,2)\}, \\ B = \{&(0,1,1),(1,0,1),(1,1,0),(0,2,2),(2,0,2),(2,2,0),(0,3,3),(3,0,3),(3,3,0), \\ &(2,1,1),(1,2,1),(1,1,2),(3,1,1),(1,3,1),(1,1,3),(3,2,2),(2,3,2),(2,2,3)\}. \end{aligned}$$

Define $G_p : X \times X \times X \to R^+$ by

$$G_p(x, y, z) = \begin{cases} 1, & \text{if } x = y = z \neq 2, \\ 0, & \text{if } x = y = z = 2, \\ 2, & \text{if } (x, y, z) \in A, \\ \frac{5}{2} & \text{if } (x, y, z) \in B, \\ 3 & x \neq y \neq z \neq x. \end{cases}$$

Define the mappings $T : A \cup B \to A \cup B$ for $A = \{0, 2\}$ and $B = \{1, 3\}$ by

$$T(x) = \begin{cases} 0 & \text{if } x = 3 \\ x+1 & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(x,y,z) = \begin{cases} 1 & \text{if } x \in A \cup B \\ 0 & \text{otherwise} \end{cases}$$

Also, consider $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = \frac{9t}{10}$. Clearly $d_{Gp}(A,B) = 1$, $T(A) \subset B$, $T(B) \subset A$ and T is a G_p-η-cyclic weak contraction for $u = u^* = 2, x = 0 \in A$ and $v = 1, y = 3 \in B$. Thus, we have

$$G_p(u^*, u, v) = G_p(2,2,1) = 2$$

and

$$\begin{aligned} M(x,v,y) &= \max\{G_p(x,v,y), G_p(x,Tx,Tx), G_p(y,Ty,Ty)\} \\ &= \max\{G_p(0,1,3), G_p(0,1,1), G_p(3,1,1)\} \\ &= \max\{3, \frac{5}{2}, \frac{5}{2}\} \\ &= 3. \end{aligned}$$

Hence,

$$\eta(u^*, u, v) G_p(u^*, u, y) \leqslant \phi(M(x,v,y)).$$

Thus,

$$\left.\begin{aligned} d_{Gp}(u, Tu^*) = d_{Gp}(A,B) \\ d_{Gp}(u^*, Tu) = d_{Gp}(A,B) \\ d_{Gp}(v, Ty) = d_{Gp}(A,B) \end{aligned}\right\} \Longrightarrow \eta(u^*, u, v) G_p(u^*, u, v) \leqslant M(x,v,y).$$

Hence, T is G_p-η-cyclic weak contraction mapping. All conditions of above theorem holds and T has a best proximity point. Here, $z = 0$ is best proximity point of T.

As an application to our best proximity point results we here derive fixed point theorem as in the form of the following. As in definition, we have

$$\left.\begin{aligned} d_{Gp}(u, Tu^*) = d_{Gp}(A,B) \\ d_{Gp}(u^*, Tu) = d_{Gp}(A,B) \\ d_{Gp}(v, Ty) = d_{Gp}(A,B) \end{aligned}\right\} \Longrightarrow \eta(u^*, u, v) G_p(u^*, u, v) \leqslant \phi(M(x,v,y)). \tag{19}$$

If we consider $A = B = X$, then

$$\left.\begin{aligned} u = Tu^* \\ u* = Tx \\ v = Ty \end{aligned}\right\} \Longrightarrow u = T^2(x).$$

Then condition (19) becomes $\eta(Tx, T^2x, Ty) G_p(Tx, T^2x, Ty) \leqslant \phi(M(x, Ty, y))$. Now, we have the following fixed point theorem.

Theorem 2.12 Let (X, G_p) be a complete G_p-metric space and $T : X \to X$ be a mapping satisfying the following condition

$$\eta(Tx, T^2x, Ty) G_p(Tx, T^2x, Ty) \leqslant \phi(M(x, Ty, y))$$

for all $x, y \in X$ and $\phi \in \Phi$. Then T has a fixed point.

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