

## Spectral triples of weighted groups

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**Abstract.** We study spectral triples on (weighted) groups and consider functors between the categories of weighted groups and spectral triples. We study the properties of weights and the corresponding functor for spectral triples coming from discrete weighted groups.

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### 1. Introduction

Spectral triples play a major role in mathematical physics and differential geometry, specially in studying compact oriented Riemannian spin manifolds. Alain Connes showed that the Dirac operator on the square integrable sections of the spinor bundle plus the algebra of complex smooth functions on a compact oriented Riemannian spin manifold can encode both the smooth and spin structures of the manifold [8].

This notion is related to spectral geometry. In 1911, Hermann Weyl proved the following remarkable asymptotic formula describing distribution of (large) eigenvalues of the Dirichlet Laplacian in a bounded domain  $X \subseteq \mathbb{R}^d$ ,

$$N(\lambda) = (2\pi)^{-d} \omega_d \text{vol}(X) \lambda^{d/2} (1 + o(1)), \text{ as } \lambda \rightarrow +\infty,$$

where  $N(\lambda)$  is the number of those eigenvalues of the (positive) Laplacian, which are bounded above by  $\lambda$ ,  $\omega_d$  is the volume of the unit ball in  $\mathbb{R}^d$ , and  $\text{vol}(X)$  is the volume of  $X$ .

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This formula was actually conjectured independently by Arnold Sommerfeld [14] and Hendrik Lorentz [13] in 1910, who stated the Weyl's Law as a conjecture based on the book of Lord Rayleigh 'The Theory of Sound' (1887) (for more details and historical notes, see [2]).

Weyl published several papers [15–19] during 1911-1915, all devoted to the eigenvalue asymptotic for the Laplace operator (and also the elasticity operator) in a bounded domain with regular boundary [11]. Laplacian also was used in the acoustic equation [12].

Alain Connes and Henri Moscovici defined the spectral triples [9]. Connes used this idea to define a non commutative spin Riemannian manifold [6, 8]. One canonical way to build spectral triples, as suggested by Connes in [5], is to use a (discrete) group with a given length function or a weight. It is natural to ask if this correspondence is functorial. This is answered affirmatively in [3], where the authors study the condition on the weight on a group which guarantees the existence of the corresponding spectral triple. They also construct a covariant functor from a subcategory of weighted groups to the category of spectral triple (and define an appropriate notion of morphisms in the latter category). In a forthcoming paper [1], we study the spectral geometry of weighted groups and find the relation between Weyl's law (in spectral geometry) and the polynomial growth rate (in geometric group theory).

In this paper, we give a contravariant functor between the category of weighted groups and the category of spectral triples with much weaker conditions on the weight. Some classes of morphisms (weaker than weighted morphisms) are defined and their properties are investigated. The paper is organized as follows. Section 1 (the current section) was devoted to the description of the goals and motivations of the paper. In section 2 we review preliminaries on spectral triples and discrete weighted groups. In section 3, the categories of weighted groups and spectral triples are discussed and some contravariant functors between those are constructed.

## 2. spectral triples and weighted groups

Alain Connes [3, 7, 8, 10] has proposed a set of axioms for non-commutative manifolds, which are analogues to those for compact orientable spin Riemannian manifolds. this is done via the notion of a (compact) spectral triple or an (unbounded) K-cycle [3]. A (compact) spectral triple is given by a triple  $(\mathcal{A}, \mathcal{H}, D)$ , consisting of

- a unital pre- $C^*$ -algebra  $\mathcal{A}$ ;
- a representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ ;
- a (not necessarily bounded) densely defined, self-adjoint operator  $D$  on  $\mathcal{H}$ , called the Dirac operator, such that, a) the resolvent  $(D - \lambda)^{-1}$  is a compact operator, for each  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , b)  $[D, \pi(a)]_- \in \mathcal{B}(\mathcal{H})$ , for every  $a \in \mathcal{A}$ , where  $[x, y]_- := xy - yx$  denotes the commutator of  $x, y$  in  $\mathcal{B}(\mathcal{H})$ .

There are different possible types of morphisms between spectral triples [4]. Here we choose the one used in [3], leading to the category  $\mathfrak{Srip}$ : given two objects  $(\mathcal{A}_j, \mathcal{H}_j, D_j)$ , for  $j = 1, 2$ , a *morphism* is a pair

$$(\phi, \Phi) \in \mathbf{Mor}_{\mathfrak{Srip}}[(\mathcal{A}_1, \mathcal{H}_1, D_1), (\mathcal{A}_2, \mathcal{H}_2, D_2)]$$

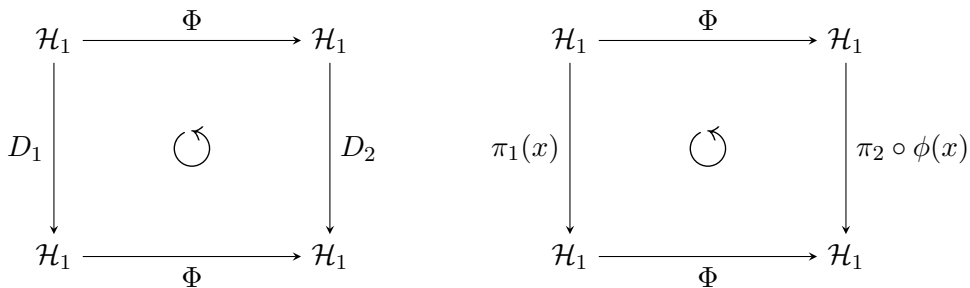
also written as,

$$(\mathcal{A}_1, \mathcal{H}_1, D_1) \xrightarrow{(\phi, \Phi)} (\mathcal{A}_2, \mathcal{H}_2, D_2),$$

where  $\phi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  is a  $*$ -morphism between the pre- $C^*$ -algebras  $\mathcal{A}_1, \mathcal{A}_2$ , and  $\Phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is a bounded linear map in  $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$  that intertwines the representations  $\pi_1, \pi_2 \circ \phi$  and Dirac operators  $D_1, D_2$ :

$$\begin{aligned} \phi_2(\phi(x)) \circ \Phi &= \Phi \circ \pi_1(x), \quad (x \in \mathcal{A}_1), \\ D_2 \circ \Phi &= \Phi \circ D_1, \end{aligned}$$

i.e., the following diagrams commute, for every  $x \in \mathcal{A}_1$ ,



Let  $G$  be a discrete group. The (complex) group algebra  $\mathbb{C}[G]$  of  $G$  is the set of complex functions on  $G$  with finite support. Here we denote this algebra by  $\mathcal{A}_G$ . This is generated by Dirac functions (measures)

$$\delta_x(y) := \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}$$

as  $x$  runs over  $G$ . The addition is defined pointwise, whereas the multiplication is given by the convolution, defined by

$$(f * g)(x) := \sum_{z \in G} f(z)g(z^{-1}x),$$

for  $f, g \in \mathcal{A}_G$ . Also the involution is defined by

$$(f^*)(x) := \overline{f(x^{-1})}.$$

These make  $\mathcal{A}_G$  into a pre- $C^*$  algebra (with an appropriate norm coming from non degenerate representations).

The assignment  $\mathcal{A}$ , sending  $G$  to  $\mathcal{A}_G$  is indeed a functor [3].

**Proposition 2.1** There exists a covariant functor  $\mathcal{A}$  from the category  $\mathfrak{Gr}$  of groups with homomorphisms, to the category  $\mathfrak{Alg}$  of associative complex unital involutive algebras with unital involutive algebra homomorphisms, which to every group  $G$  associates the group algebra  $\mathcal{A}_G$ .

On the other hand, one could see  $\mathcal{A}_G$  as a vector space and equipped it with an inner product

$$\langle f, g \rangle := \sum_{x \in G} \overline{f(x)}g(x).$$

This is a pre-Hilbert space whose completion is  $\ell^2(G)$ , here denoted by  $\mathcal{H}_G$ .

Next we recall some notions, used in Weyl dimension theory of groups.

**Definition 2.2** Let  $G$  be a group and  $g_1, \dots, g_n \in G$ ,

- (1)  $g_1, \dots, g_n$  are independent, if for  $0 < k \in \mathbb{N}$  and  $1 \leq \alpha_1, \dots, \alpha_k \leq n$  and  $m_1, \dots, m_k \in \mathbb{Z}$  such that,  $\alpha_i \neq \alpha_{i+1}$ ,  $1 \leq i < k$ ,  $g_{\alpha_1}^{m_1} g_{\alpha_2}^{m_2} \dots g_{\alpha_k}^{m_k} = 1$  implies that  $m_i = 0$ , for all  $i$ .
- (2)  $g_1, \dots, g_n$  are semi-independent, if for  $0 < k \in \mathbb{N}$  and  $1 \leq \alpha_1, \dots, \alpha_k \leq n$  and  $m_1, \dots, m_k \in \mathbb{Z}$ ,  $g_{\alpha_1}^{m_1} g_{\alpha_2}^{m_2} \dots g_{\alpha_k}^{m_k} = 1$  implies that  $\sum_{\alpha_i=j} m_i = 0$ , for all  $j$ .

An element  $g \in G$  is (semi-)independent if and only if it is of infinite order,  $ord(g) = \infty$ .

A weight  $\omega$  on a group  $G$  is a real valued function on  $G$ ,  $\omega : G \rightarrow \mathbb{R}$  [7]. A length function on  $G$  (with identity element  $e$ ) is a weight  $\omega$  satisfying the following conditions.

- $\omega(xy) \leq \omega(x) + \omega(y)$ ,
- $\omega(x^{-1}) = \omega(x)$ ,
- $\omega(x) = 0 \iff x = e$ ,

for all  $x, y \in G$ . We have the following definitions for kinds of weights on  $G$ :

Following [3], we say that a weight  $\omega : G \rightarrow \mathbb{R}$  is a

- (1) proper weight if for any  $k \in \mathbb{N}$ , the set  $\omega^{-1}([-k, k])$  is a finite set,
- (2) Dirac weight if for any  $x \in G$ ,

$$\sup\{|\omega(y) - \omega(x^{-1}y)| : y \in G\} < \infty,$$

- (3) charged weight if  $|\omega|$  is a length function on  $G$ .

Clearly a length function is always a Dirac weight.

**Proposition 2.3** Let  $G$  be a group and  $N$  a normal subgroup of  $G$ , if  $\omega : G \rightarrow \mathbb{R}$  be a positive weight, then for the function  $\tilde{\omega} : G/N \rightarrow \mathbb{R}$  defined by

$$\tilde{\omega}(xN) := \inf\{\omega(xz) | z \in N\},$$

we have,

- (1) if  $\omega$  be proper weight, then  $\tilde{\omega}$  is a proper weight,
- (2) if  $\omega$  be Dirac weight and  $N$  be a finite subgroup, then  $\tilde{\omega}$  is a Dirac weight,
- (3) if  $\omega$  be length function, then  $\tilde{\omega}$  is a length function.

**Proof.**

- (1) Let  $\omega$  be a proper weight and  $k > 0$ , if  $\#\tilde{\omega}^{-1}[-k, k] = \infty$ , then for any coset  $[x] \in \tilde{\omega}^{-1}[-k, k]$  we can find an element  $x_0 \in \tilde{\omega}^{-1}[-k, k]$  such that,  $|\omega(x_0)| < 2k$ . Since cosets are disjoint, we have  $\#\omega^{-1}[-2k, 2k] = \infty$ , which contradicts the properness of  $\omega$ .
- (2) Let  $N = \{z_1, \dots, z_m\}$  and  $x, y \in G$ , since  $\omega$  is a Dirac weight, for any index  $i, j, k$

we have a bound named  $M_{i,j,k}$  such that

$$|\omega(xz_i z_j z_k^{-1} z_k y) - \omega(z_k y)| \leq M_{i,j,k},$$

for any  $y \in G$ . Put  $M = \max\{M_{i,j,k} | i, j, k\}$ , then

$$\begin{aligned} |\tilde{\omega}([x][y]) - \tilde{\omega}([y])| &\leq \sup\{|\omega(xz_i z_j z_k^{-1} z_k y) - \omega(z_k y)|\} \\ &\leq \max\{M_{i,j,k} | i, j, k\} \\ &= M, \end{aligned}$$

which is shown that  $\tilde{\omega}$  is a Dirac weight.

(3) For  $x, y \in G$  and  $z, w \in N$ ,

$$\begin{aligned} \tilde{\omega}([x][y]) &\leq \omega(xzyw) \\ &\leq \omega(xz) + \omega(yw) \end{aligned}$$

thus,

$$\begin{aligned} \tilde{\omega}([x][y]) &\leq \inf\{\omega(xz) | z \in N\} + \inf\{\omega(yw) | w \in N\} \\ &= \tilde{\omega}([x]) + \tilde{\omega}([y]). \end{aligned}$$

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As a concrete (and typical) example, if  $G = \mathcal{F}^X$  is the free group on a set of generators  $X$ , then  $G$  has a length function. When moreover,  $X$  is countable set,  $G$  also has a proper length function. Since every group  $G$  is a quotient of a free group, it follows that every (discrete) group  $G$  has a length function, and every countably generated (discrete) group  $G$  has a proper length function. Conversely, if a group  $G$  has a proper weight, then  $G$  must be countable.

### 3. Category of weighted groups and spectral triples

For constructing a category for weighted groups one needs an appropriate notion of morphism between weighted groups which respects the weight function. In [3] certain morphisms between weighted groups are defined as follows.

**Definition 3.1** [3] Let  $(G, \omega)$  and  $(H, \lambda)$  be to weighted groups. A map  $\phi : G \rightarrow H$  is called

(1) weighted if

$$\phi^\#(\lambda) = \lambda \circ \phi = \omega,$$

and co-weighted if there is a weighted homomorphism  $\rho : H \rightarrow G$  such that

$$\phi \circ \rho = \iota_H,$$

(2) isometric if

$$|\lambda \circ \phi| = |\omega|,$$

and sub-isometric if

$$|\lambda \circ \phi| \leq |\omega|.$$

Here we introduce some other relevant classes of morphisms which are important for geometric purposes.

**Definition 3.2** Let  $(G, \omega)$  and  $(H, \lambda)$  be two weighted groups. A map  $\phi : G \rightarrow H$  is called

- bounded if there exist a constant  $d \geq 0$  such that for any  $x \in G$ ,

$$\omega(x) - d \leq \lambda(\phi(x)) \leq \omega(x) + d,$$

- semi-bounded if there exist constants  $d \geq 0$  and  $a, b, c > 0$ , such that for any  $x \in G$ ,

$$a.\omega(x) - d \leq b.\lambda(\phi(x)) \leq c.\omega(x) + d,$$

- quasi-bounded map if there exist constants  $d \geq 0$  and  $a, b, c, r, s, t > 0$ , such that for any  $x \in G$ ,

$$a.|\omega(x)|^r - d \leq b.|\lambda(\phi(x))|^s \leq c.|\omega(x)|^t + d.$$

Let us show that most of the properties of weights are stable under morphisms with appropriate boundedness conditions.

**Proposition 3.3** Let  $(G, \omega)$  and  $(H, \lambda)$  be two groups and  $\phi : G \rightarrow H$  be a bounded (semi-bounded or quasi-bounded) morphism.

- (1) If  $\lambda$  be a Dirac weight, so is  $\omega$ . Conversely, if  $\phi$  is surjective and  $\omega$  is a Dirac weight, then so is  $\lambda$ .
- (2) If  $\phi$  has finite kernel and  $\lambda$  is a proper weight, so is  $\omega$ . Conversely if  $\phi$  is surjective and  $\omega$  is a proper weight, then so is  $\lambda$ .

**Proof.** Let  $\phi$  is bounded, then there exist  $d \geq 0$  such that, for any  $x \in G$ ,

$$\omega(x) - d \leq \lambda \circ \phi(x) \leq \omega(x) + d.$$

If  $\lambda$  be a Dirac weight, then for any  $z \in H$  there exist a  $M_z \geq 0$  such that, for any  $u \in H$ ,

$$|\lambda(zu) - \lambda(u)| \leq M_z.$$

For  $x, y \in G$ ,

$$\begin{aligned} \pm(\omega(xy) - \omega(y)) &\leq \pm(\lambda(\phi(xy)) - \lambda(\phi(y))) + 2d \\ &= \pm(\lambda(\phi(x)\phi(y)) - \lambda(\phi(y))) + 2d \\ &\leq |\lambda(\phi(x)\phi(y)) - \lambda(\phi(y))| + 2d \\ &\leq M_{\phi(x)} + 2d, \end{aligned}$$

hence,  $\omega$  is also Dirac.

For the converse, let  $\phi$  be an epimorphism and  $\omega$  be a Dirac weight, then for any  $x \in G$ , there exist  $M_x \geq 0$  such that, for any  $y \in G$ ,

$$|\omega(xy) - \omega(y)| \leq M_x.$$

For  $z, u \in H$ , choose  $x, y \in G$  with  $\phi(x) = z$  and  $\phi(y) = u$ . Then

$$\begin{aligned} \pm(\lambda(zu) - \lambda(u)) &\leq \pm(\omega(xy) - \omega(y)) + 2d \\ &\leq |\omega(xy) - \omega(y)| + 2d \\ &\leq M_x + 2d, \end{aligned}$$

hence,  $\lambda$  is also Dirac.

Next, if  $\omega$  is proper, then for any  $k \geq 0$  and  $z \in H$  such that  $|\lambda(z)| \leq k$ , choose  $x \in G$  with  $\phi(x) = z$ . Then

$$\lambda(z) + d \leq \omega(x) \leq \lambda(z) + d,$$

thus  $x \in \omega^{-1}([-k-d, k+d])$ . This gives an injective map from  $\lambda^{-1}([-k, k])$  to  $\omega^{-1}([-k-d, k+d])$ , hence  $\#\lambda^{-1}([-k, k]) \leq \#\omega^{-1}([-k-d, k+d])$ , thus  $\lambda$  is proper.

For the converse, let  $\phi$  have a finite kernel and  $\lambda$  be proper, then as above,

$$\#\{x \in G | x \in \omega^{-1}([-k, k])\} \leq \#\ker(\phi) \times \#\lambda^{-1}([-k-d, k+d]),$$

thus  $\omega$  is a proper weight.

Similar proofs also work for the cases of semi-bounded or quasi-bounded morphisms.

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On the other hand, properties such as properness or being Dirac does not pass from one weight to another, when we apply morphisms with weak type of boundedness. We illustrate this by an example. Let  $G = H = \mathbb{Z}$ , and  $\phi = Id_G$ , if we choose  $\lambda(n) = |n|$  and

$$\omega(n) = \begin{cases} n & , \text{if } n \geq 0. \\ -n & , \text{if } n = 2k + 1, K < 0. \\ n & , \text{if } n = 2k, K < 0. \end{cases} \tag{1}$$

It is easy to see that  $|\omega| = |\lambda| = \lambda$ , which show that the morphism is sub-bounded. But  $\omega(2k + 1) - \omega(2k) = 4k + 1$ , where  $k < 0$ , so for  $x = 1$ ,  $M_x = \infty$ , thus  $\omega$  is not a Dirac weight.

If  $\omega, \lambda$  are two weight on a group  $G$ , then we say that they are boundedly (semi-boundedly or quasi-boundedly) equivalent and write  $\omega \underset{b}{\sim} \lambda$  ( $\omega \underset{sb}{\sim} \lambda$  or  $\omega \underset{qb}{\sim} \lambda$ ), if the identity map is bounded (semi-bounded or quasi-bounded) on  $G$ .

Next let us remind how a spectral triple is defined on a weighted group [3, 7].

**Proposition 3.4** [3] To every pair  $(G, \omega)$  where  $G$  is a discrete countable group and  $\omega$  is a weight function on  $G$ , we associate a triple  $(\mathcal{A}_G, \mathcal{H}_G, D_\omega)$  as follows:

- $\mathcal{A}_G$  is the group algebra of  $G$ ,
- $\mathcal{H}_G$  is the Hilbert space of  $G$ ,
- the representation of the algebra  $\mathcal{A}_G$  on  $\mathcal{H}_G$  is the left-regular representation  $\pi_G : \mathcal{A}_G \rightarrow \mathcal{B}(\mathcal{H}_G)$ ,

- the Dirac operator  $D_\omega$  is the pointwise multiplication operator by the weight function  $\omega_G$ , i.e.,

$$(D_\omega \xi)(x) := \omega(x)\xi(x), \quad (x \in G),$$

defined on the domain  $\{\xi \in \mathcal{H}_G \mid \sum_{x \in G} |\omega(x)\xi(x)|^2 < \infty\}$ .

The triple  $(\mathcal{A}_G, \mathcal{H}_G, D_\omega)$  is a spectral triple if and only if the weight  $\omega$  is proper and Dirac.

In [3], the authors studied a category of spectral triples and showed that there is a covariant functor from the category of (discrete) groups with length function to this category. In this section we define a new contravariant functors between these categories.

Here we deal with four categories (and their variants). In the first category  $\mathfrak{Gr}$ , the objects are discrete groups with Dirac proper weights and the morphisms are weighted group homomorphisms with some additional properties. The second and third categories  $\mathfrak{Alg}$  and  $\mathfrak{Hil}$  consists of unital  $*$ -algebras with unital  $*$ -homomorphisms, and Hilbert spaces with linear isometries, in each case with some additional properties on morphisms. The fourth category  $\mathfrak{Tripl}$  is the category of spectral triples as described in [3].

The next result is proved in [3, Theorem 3.18].

**Theorem 3.5** There exists a covariant functor  $\mathfrak{T}$ , from the category  $\mathfrak{Gr}$  of proper Dirac-weighted countable groups with weighted monomorphisms to the category of spectral triples, associating to every  $(G, \omega)$  the canonical triple  $(\mathcal{A}_G, \mathcal{H}_G, D_\omega)$ .

Note that for a monomorphism  $\phi : (G, \omega) \rightarrow (H, \lambda)$ , one can define  $\mathcal{A}_\phi : \mathcal{A}_G \rightarrow \mathcal{A}_H$  and  $\mathcal{H}_\phi : \mathcal{H}_G \rightarrow \mathcal{H}_H$  by extending  $\phi$  on the span of the set  $\{\delta_x \mid x \in G\}$ , and then to the algebra or Hilbert space it generates.

Here we first define a contravariant functor between categories  $\mathfrak{Gr}$  and  $\mathfrak{Alg}$ .

**Proposition 3.6** There exist a contravariant functor  $\mathcal{F}$  from the category  $\mathfrak{Gr}$  to the category  $\mathfrak{Alg}$  which associates to every group  $G$  its group algebra  $\mathcal{A}_G$  and to every finite kernel epimorphism  $\phi : G \rightarrow H$  the morphism  $\mathcal{F}(\phi) : \mathcal{A}_H \rightarrow \mathcal{A}_G$  defined by

$$\mathcal{F}(\phi)(f) = \frac{1}{N} f \circ \phi, \quad (f \in \mathcal{A}_H),$$

where  $N$  is the cardinal of  $\ker(\phi)$ .

**Proof.** The condition on the kernel of  $\phi$  guarantees that  $\mathcal{F}(\phi)$  takes functions of finite support on  $H$  to functions of finite support on  $G$ . We only need to check that  $\mathcal{F}(\phi)$  is an algebra homomorphism. For  $f, g \in \mathcal{A}_H$  and  $x \in G$ , we have

$$\begin{aligned} [\mathcal{F}(\phi)(f) * \mathcal{F}(\phi)(g)](x) &= \frac{1}{N^2} [(f \circ \phi) * (g \circ \phi)](x) \\ &= \frac{1}{N^2} [N \cdot \sum_{y \in \text{Im}(\phi)} f(y)g(y^{-1}\phi(x))] \\ &= \frac{1}{N} \sum_{y \in H} f(y)g(y^{-1}\phi(x)) \\ &= \mathcal{F}(\phi)(f * g)(x). \end{aligned}$$





This functor is a base for our contravariant functor to the category of spectral triples, introduced later. The following counterexample shows that we need the condition that the kernel is finite. We use the abbreviation  $\phi^\# := \mathcal{F}(\phi)$ .

**Example 3.7** Let  $G = \mathbb{Z} \times \mathbb{Z}$  and  $H = \mathbb{Z}$  and define  $\phi : G \rightarrow H; \phi(m, n) = m$ . This is a homomorphisms, but we have

$$\phi^\#(\delta_0) = \delta_0 \circ \phi = \sum_{n=-\infty}^{+\infty} \delta_n,$$

which is not a finite support function.

Next we define another functor between categories  $\mathfrak{Gr}$  and  $\mathfrak{Hil}$ .

**Proposition 3.8** There is a contravariant functor  $\mathcal{F}$  from the category  $\mathfrak{Gr}$  of groups with finite kernel epimorphism to the category  $\mathfrak{Hil}$  of Hilbert spaces with linear isometries, which associates to each group  $G$  the Hilbert space  $\ell^2(G)$  and to a finite kernel epimorphism  $\phi : G \rightarrow H$  the map

$$\mathcal{F}(\phi) : g \mapsto \frac{1}{\sqrt{N}}g \circ \phi \quad (g \in \ell^2(H)),$$

where  $N = \# \ker(\phi)$ . This functor is exact.

**Proof.** For the first statement, observe that if  $\xi, \eta \in \ell^2(H)$ ,

$$\begin{aligned} \langle \mathcal{F}(\phi)(\xi), \mathcal{F}(\phi)(\eta) \rangle &= \frac{1}{N} \sum_{x \in G} \xi \circ \phi(x) \overline{\eta \circ \phi(x)} \\ &= \frac{1}{N} \sum_{\phi(x)=y \in Im(\phi)} N \cdot \xi(y) \overline{\eta(y)} \\ &= \langle \xi, \eta \rangle. \end{aligned}$$

For exactness, one just needs to recall that every exact sequence in the first category is of the form

$$0 \rightarrow G \rightarrow G \rightarrow 0 \rightarrow 0.$$



Finally we can define a contravariant functor between categories  $\mathfrak{Gr}$  and  $\mathfrak{Trip}$ .

**Proposition 3.9** There exist a contravariant functor  $\mathcal{H}$  from the subcategory of  $\mathfrak{Gr}$ , consisting of proper Dirac weighted groups with weighted finite kernel homomorphisms to the category  $\mathfrak{Trip}$  of spectral triples, which associates  $(\mathbb{C}[G], \ell^2(G), D_\omega)$  to  $(G, \omega)$ , where  $D_\omega$  is the corresponding Dirac operator.

**Proof.** We need to show that for weighted groups  $(G, \omega)$  and  $(H, \lambda)$  and a weighted finite kernel homomorphism  $\phi : G \rightarrow H$ , the corresponding diagram commutes. This

follows if one observes that for a basic element  $\delta_y$  in  $\ell^2(H)$ , we have

$$\begin{aligned}
 D_\omega \circ \mathcal{H}(\phi)(\delta_y) &= D_\omega\left(\frac{1}{\sqrt{N}} \sum_{\phi(x)=y} \delta_x\right) \\
 &= \frac{1}{\sqrt{N}} \sum_{\phi(x)=y} D_\omega(\delta_x) \\
 &= \frac{1}{\sqrt{N}} \sum_{\phi(x)=y} \omega(x)\delta_x \\
 &= \lambda(y) \frac{1}{\sqrt{N}} \sum_{\phi(x)=y} \delta_x \\
 &= \mathcal{H}(\phi)(\lambda(y)\delta_y) \\
 &= \mathcal{H}(\phi) \circ D_\lambda(\delta_y).
 \end{aligned}$$

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## References

- [1] M. Amini, K. Shamsolkotabi, Spectral geometry of weighted groups, preprint.
- [2] W. Arendt, R. Nittka, W. Peter, F. Steiner, Weyl's law: Spectral properties of the Laplacian in Mathematics and Physics, in: *Mathematical analysis of evolution, information, and complexity*, W. Arendt and W.P. Schleich (Eds.), Wiley-VCH, Weinheim, (2009), 1-71.
- [3] P. Bertozzini, R. Conti, W. Lewkeeratiyutkul, A category of spectral triples and discrete groups with length function, *Osaka J. Math.* 43 (2) (2006), 327-350.
- [4] P. Bertozzini, R. Conti, W. Lewkeeratiyutkul, Categorical non-commutative geometry, *Journal of Physics: Conference Series*. 346 (2012), 1-9.
- [5] A. Connes, *Noncommutative geometry*, Academic Press, New York, 1994.
- [6] A. Connes, Noncommutative geometry and reality, *J. Math. Phys.* 36 (11) (1995), 6194-6231.
- [7] A. Connes, Compact metric spaces, Fredholm modules and hyperfiniteness, *Ergodic Theory Dynam. Systems.* 9 (1989), 207-220.
- [8] A. Connes, On the spectral characterization of manifolds, (2008), arXiv:0810.2088.
- [9] A. Connes, H. Moscovici, The local index formula in noncommutative geometry, *Geometry and Funct. Analysis.* 5 (1995), 174-243.
- [10] J. M. Gracia-Bondia, H. Figueroa, J.C. Varilly, *Elements of noncommutative geometry*, Birkhäuser, Boston, 2000.
- [11] V. Ivrii, 100 years of Weyl's law, *Bulletin of Mathematical Sciences.* 6 (2016), 379-452.
- [12] M. Kac, Can one hear the shape of a drum?, *Amer. Math. Monthly.* 73 (1966), 1-23.
- [13] H. A. Lorentz, Alte und neue fragen der Physik, *Physikal. Zeitschr.* 11 (1910), 1234-1257.
- [14] A. Sommerfeld, Die Greensche funktion der schwingungsgleichung für ein beliebiges gebiet, *Physikal. Zeitschr.* 11 (1910), 1057-1066.
- [15] H. Weyl, Über die asymptotische verteilung der eigenwerte, *Nachr. Konigl. Ges. Wiss. Göttingen.* (1911), 110-117.
- [16] H. Weyl, Das asymptotische verteilungsgesetz linearen partiellen differentialgleichungen, *Math. Ann.* 71 (1912), 441-479.
- [17] H. Weyl, Über die abhängigkeit der eigenschwingungen einer membran von deren begrenzung, *J. Reine Angew. Math.* 141 (1912), 1-11.
- [18] H. Weyl, Über die randwertaufgabe der strahlungstheorie und asymptotische spektralgeometrie, *J. Reine Angew. Math.* 143 (1913), 177-202.
- [19] H. Weyl, Das asymptotische verteilungsgesetz der eigenschwingungen eines beliebig gestalteten elastischen körpers, *Rend. Circ. Mat. Palermo.* 39 (1915), 1-49.