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Fixed points of weak ψ-quasi contractions in generalized metric spaces

K. P. R. Sastry[a], G. V. R. Babu[b], P. S. Kumar[b,*]

[a] *8-28-8/1, Tamil Street, China Waltair, Visakhapatnam-530 017, India.*
[b] *Department of Mathematics, Andhra University, Visakhapatnam-530 003, India.*

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Abstract. In this paper, we introduce the notion of weak ψ-quasi contraction in generalized metric spaces and using this notion we obtain conditions for the existence of fixed points of a self map in D-complete generalized metric spaces. We deduce some corollaries from our result and provide examples in support of our main result.

Keywords: Generalized metric space, weak ψ-quasi contraction, fixed point.

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1. Introduction

In 2015, Jleli and Samet [1] obtained a generalization of the notion of a metric space which they called a generalized metric space. They also stated and proved fixed point theorems for some contractions defined on these spaces. For more works in this direction, we refer [2–4]. Recently Sastry, Naidu, Rao and Naidu [4] have dealt with fixed point results in generalized metric spaces with less stringent conditions and obtained Banach contraction principle in generalized metric spaces as a corollary.

In this paper, we define weak ψ-quasi contraction in generalized metric spaces and prove the existence of fixed points of weak ψ-quasi contractions in D-complete generalized metric spaces and obtain results of Sastry, Naidu, Rao and Naidu [4] as corollaries.

2. Preliminaries

In the sequel, we use the following notation introduced by Jleli and Samet [1]. Let X be a non-empty set and $D : X \times X \to [0, +\infty]$ be a given mapping. For every $x \in X$, let us

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define the set $\mathcal{C}(D, X, x) = \{\{x_n\} \subset X : \lim_{n\to\infty} D(x_n, x) = 0\}$.

Definition 2.1 [1] Let X be a non-empty set and $D : X \times X \to [0, +\infty]$ be a function which satisfies the following conditions:
(2.1.1) $D(x, y) = 0$ implies $x = y$
(2.1.2) $D(x, y) = D(y, x)$ for all $x, y \in X$
(2.1.3) there exists $\lambda > 0$ such that if $x, y \in X$ and $\{x_n\} \in \mathcal{C}(D, X, x)$, then

$$D(x, y) \leqslant \lambda \limsup_{n\to\infty} D(x_n, y).$$

Then D is called a generalized metric and the pair (X, D) is called a generalized metric space with coefficient λ. In general we drop λ. It may be noted that in a generalized metric space, the distance between two points may be infinite. $D(x, y)$ is called the generalized distance between x and y.

Remark 1 *[1] Obviously, if the set $\mathcal{C}(D, X, x)$ is empty for every $x \in X$ then (X, D) is a generalized metric space if and only if* (2.1.1) *and* (2.1.2) *are satisfied.*

Definition 2.2 [1] Let (X, D) be a generalized metric space. Also, let $\{x_n\}$ be a sequence in X and $x \in X$. We say that the sequence $\{x_n\}$ is D-convergent to x, if $\{x_n\} \in \mathcal{C}(D, X, x)$; that is, $\lim_{n\to\infty} D(x_n, x) = 0$.

Proposition 2.3 [1] Let (X, D) be a generalized metric space. Also, let $\{x_n\}$ be a sequence in X and $x, y \in X$. If $\{x_n\}$ is D-convergent to x and $\{x_n\}$ is D-convergent to y, then $x = y$.

Definition 2.4 [1] Let (X, D) be a generalized metric space. Also, let $\{x_n\}$ be a sequence in X and $x \in X$. We say that the sequence $\{x_n\}$ is a D-Cauchy sequence if $\lim_{m,n\to\infty} D(x_n, x_{n+m}) = 0$.

Definition 2.5 [1] Let (X, D) be a generalized metric space. It is said to X be D-complete if every D-Cauchy sequence in X is convergent to some element in X.

Here in after, we use converges in place of D-converges when there is no confusion.

Definition 2.6 Let $f : X \to X$ be a self map of X and $x \in X$. Write $f^1(x) = f(x)$ and $f^{n+1}(x) = f(f^n(x))$ for $n = 1, 2, \cdots$. For convenience, we write $x = f^0(x)$, $x_1 = f^1(x)$ and $x_{n+1} = f(x_n)$ for $n = 1, 2, \cdots$. Then $\{x_n\}$ is called the sequence of iterates of f at x.

We use the following two results in Section 3.

Theorem 2.7 [4] Let (X, D) be a generalized metric space. Suppose $\{x_n\} \subset X, x \in X$ and $x_n \to x$. Then $D(x, x) = 0$.

Theorem 2.8 [4] Let (X, D) be a generalized metric space and $x \in X$. Suppose $\mathcal{C}(D, X, x) \neq \phi$. Then $D(x, x) = 0$.

Now we give examples of generalized metric spaces.

***Example* 2.9** Let $X = [0,1]$ and $D : X \times X \to [0,\infty]$ be given by

$$D(x,y) = \begin{cases} y & \text{if } x = 0 \text{ and for any } y \\ x & \text{if } y = 0 \text{ and for any } x \\ 1 & \text{if } x \text{ and } y \text{ are rational and non-zero} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Then (X,D) is a generalized metric space with $\lambda = 2$.

***Example* 2.10** Let $X = [0,1]$ and $D : X \times X \to [0,\infty]$ be given by

$$D(x,y) = \begin{cases} |x-y|+1 & \text{if } x \neq y \\ 0 & \text{otherwise.} \end{cases}$$

Then (X,D) is a generalized metric space with $\lambda = 1$.

***Example* 2.11** Let $X = \{1, \frac{1}{2}, \frac{1}{3}, ...\} \cup \{0\}$ and $D : X \times X \to [0,\infty]$ be given by $D(1,x) = D(x,1) = \infty$ if $x = 0$ or $\frac{1}{n}$ for $n = 1,2,3,\ldots$, and otherwise $D(x,y) = |x-y|$. Then (X,D) is a generalized metric space with $\lambda = 1$.

3. Main results

We start with the following notation which we use in the subsequent development.
Suppose $\lambda > 1$. We write
$\Psi_\lambda = \{\psi : [0,\infty] \to [0,\infty] | \psi \text{ is non-decreasing}, \psi(t) = 0 \iff t = 0,\ \psi(t) < \frac{t}{\lambda} \text{ for } t > 0\}$
and
$\Psi_1 = \{\psi : [0,\infty] \to [0,\infty] | \psi$ is non-decreasing, right continuous, $\psi(t) = 0 \iff t = 0$ and $\psi(t) < t$ for $t > 0\}$.

Lemma 3.1 If $\psi \in \Psi_\lambda$, then $\lim\limits_{n\to\infty} \psi^n(t) = 0$.

Proof. Let $\psi \in \Psi_\lambda$.
Case (i): Suppose $\lambda = 1$. Then $\psi(t) < t$ for $t > 0$. Since $\{\psi^n(t)\}$ is decreasing it converges to some $r \geqslant 0$. Suppose $\epsilon > 0$. Then $r \leqslant \psi^n(t) < r + \epsilon$ for sufficiently large n, which implies that $\psi(r) \leqslant \psi^{n+1}(t) \leqslant \psi(r+\epsilon)$. Hence $\psi(r) < r \leqslant \psi^{n+1}(t) \leqslant \psi(r+\epsilon) \to \psi(r)$ as $\epsilon \to 0$ (since ψ is right continuous). Therefore, $\psi(r) = r$ so that $r = 0$. Hence $\psi^n(t) \to 0$ as $n \to \infty$.
Case (ii): Suppose $\lambda > 1$. Then $\psi(t) < \frac{t}{\lambda}$ for $t > 0$. Now, $\psi(t) < \frac{t}{\lambda}$, which implies that $\psi^2(t) = \psi(\psi(t) \leqslant \psi(\frac{t}{\lambda}) < \frac{t}{\lambda^2}$. By induction, we have $\psi^n(t) < \frac{t}{\lambda^n} \to 0$ as $n \to \infty$. Therefore, $\psi^n(t) \to 0$ as $n \to \infty$. Thus, in both the cases $\psi^n(t) \to 0$ as $n \to \infty$. ■

Now, we introduce weak ψ-quasi contraction in generalized metric spaces.

Definition 3.2 Let (X,D) be a generalized metric space with coefficient λ, $\psi \in \Psi_\lambda$ and $f : X \to X$ be a mapping. We write

$$M(x,y) = \max\{D(x,y), D(x,fx), D(y,fy), D(x,fy), D(y,fx)\}.$$

We say that f is a weak ψ-quasi contraction if

$$D(fx,fy) \leqslant \psi(M(x,y)) \text{ for every } x,y \in X. \tag{1}$$

Now we state and prove our main result.

Theorem 3.3 Let (X, D) be a D-complete generalized metric space and $f : X \to X$ be a weak ψ-quasi contraction. Suppose that there exists $x_0 \in X$ such that

$$\sup_n D(x_0, f^n(x_0)) < \alpha < \infty, \tag{2}$$

$$D(f^n(x_0), f^{n+1}(x_0)) \leqslant \psi^n(\alpha) \tag{3}$$

for $n = 0, 1, 2, \cdots$. Then $\{f^n(x_0)\}$ is Cauchy and hence converges to $w \in X$. If $\limsup_n D(f^n(x_0, fw)) < \infty$, then w is a fixed point of f. Moreover, if w' is another fixed point of f such that $D(w, w') < \infty$ and $D(w', w') < \infty$, then $w = w'$.

Proof. Frist we show that

$$D(f^n(x_0), f^n(x_0)) \leqslant \psi^n(\alpha) \text{ for every } \; n. \tag{4}$$

The result is true for $n = 0$, by (2). Now, assume the truth for n. i.e.,

$$D(f^n(x_0), f^n(x_0)) \leqslant \psi^n(\alpha). \tag{5}$$

We show that $D(f^{n+1}(x_0), f^{n+1}(x_0)) \leqslant \psi^{n+1}(\alpha)$. We have

$$D(f^{n+1}(x_0), f^{n+1}(x_0)) = D(f(f^n(x_0)), f(f^n(x_0))) \leqslant \psi(M(f^n(x_0), f^n(x_0))$$

where, by (3) and (5),

$$\begin{aligned} M(f^n(x_0), f^n(x_0) = \max\{&D(f^n(x_0), f^n(x_0)), D(f^n(x_0), f^{n+1}(x_0)), D(f^n(x_0), f^{n+1}(x_0)),\\ &D(f^n(x_0), f^{n+1}(x_0)), D(f^n(x_0), f^{n+1}(x_0))\}\\ \leqslant\ & \max\{\psi^n(\alpha), \psi^n(\alpha), \psi^n(\alpha), \psi^n(\alpha), \psi^n(\alpha)\}\\ =\ & \psi^n(\alpha). \end{aligned}$$

Therefore, $D(f^{n+1}(x_0), f^{n+1}(x_0)) \leqslant \psi(\psi^n(\alpha)) = \psi^{n+1}(\alpha)$. Hence, (4) holds for every n. Now, we show that

$$D(f^n(x_0), f^{n+m}(x_0)) \leqslant \psi^n(\alpha) \;\text{ for }\; m, n = 0, 1, 2, \cdots . \tag{6}$$

If $n = 0$ then $D(x_0, f^m(x_0)) < \alpha = \psi^0(\alpha)$ by (2). Assume that (6) is true for n. i.e.,

$$D(f^n(x_0), f^{n+m}(x_0)) \leqslant \psi^n(\alpha) \;\text{ for }\; m = 0, 1, 2, \cdots . \tag{7}$$

We prove

$$D(f^{n+1}(x_0), f^{n+1+m}(x_0)) \leqslant \psi^{n+1}(\alpha) \text{ for } \; m = 0, 1, 2, \cdots . \tag{8}$$

The result is true if $m = 0$, by (4). Now assume (8) is true for m. i.e.,

$$D(f^{n+1}(x_0), f^{n+1+m}(x_0)) \leqslant \psi^{n+1}(\alpha). \tag{9}$$

We must prove that

$$D(f^{n+1}(x_0), f^{n+1+m+1}(x_0)) \leqslant \psi^{n+1}(\alpha). \tag{10}$$

Now, we have

$$D(f^{n+1}(x_0), f^{n+1+m+1}(x_0)) = D(f(f^n(x_0)), f(f^{n+1+m}(x_0))) \leqslant \psi(M(f^n(x_0), f^{n+1+m}(x_0)),$$

where, by (3), (7) and (9),

$$\begin{aligned} M(f^n(x_0), f^{n+1+m}(x_0)) &= \max\{D(f^n(x_0), f^{n+1+m}(x_0)), D(f^n(x_0), f^{n+1}(x_0)), \\ &\quad D(f^{n+1+m}(x_0), f^{n+1+m+1}(x_0)), D(f^n(x_0), f^{n+1+m+1}(x_0)), \\ &\quad D(f^{n+1+m}(x_0), f^{n+1}(x_0))\} \\ &\leqslant \max\{\psi^n(\alpha), \psi^n(\alpha), \psi^{n+1+m}(\alpha), \psi^n(\alpha), \psi^{n+1}(\alpha)\} \\ &= \psi^n(\alpha). \end{aligned}$$

Therefore, $D(f^{n+1}(x_0), f^{n+1+m+1}(x_0)) \leqslant \psi(\psi^n(\alpha)) = \psi^{n+1}(\alpha)$. Hence, (10) holds for $m+1$, which in turn (6) holds for every $m, n = 0, 1, 2, \cdots$. Now, let $m \to \infty$ and $n \to \infty$ in (6). Then, by Lemma 3.1, we have

$$\lim_{n,m\to\infty} D(f^n(x_0), f^{n+m}(x_0)) \leqslant \lim_{n\to\infty} \psi^n(\alpha) = 0.$$

Therefore $\{f^n(x_0)\}$ is Cauchy and hence converges to a limit w (say) in X. Thus $f^n(x_0) \to w$ as $n \to \infty$. Now,

$$D(f^{n+1}(x_0), fw) = D(f(f^n(x_0), fw) \leqslant \psi(M(f^n(x_0), w)), \tag{11}$$

where

$$\begin{aligned} M(f^n(x_0), w) &= \max\{D(f^n(x_0), w), D(f^n(x_0), f^{n+1}(x_0)), D(w, fw), \\ &\quad D(f^n(x_0), fw), D(w, f^{n+1}(x_0))\} \\ &\leqslant \max\{\epsilon, \epsilon, \lambda\overline{\lim}D(f^n(x_0), fw), \overline{\lim}D(f^n(x_0), fw)\} \end{aligned}$$

for large n. We write $\mu = \overline{\lim}D(f^n(x_0), fw)$. Let $n \to \infty$ in (11). Then we get $\mu \leqslant \psi(\max\{\epsilon, \lambda\mu, \mu\})$ for large n. Therefore, $\mu \leqslant \psi(\epsilon) < \epsilon$ for small $\epsilon > 0$. Hence, $\mu = 0$. i.e., $\overline{\lim}D(f^n(x_0), fw) = 0$ (since $\epsilon > 0$ is arbitrary and $\limsup_n D(f^n(x_0), fw) < \infty$ by hypothesis). Thus, $f^n(x_0) \to fw$ as $n \to \infty$. Hence, $fw = w$ such that w is a fixed point of f. Suppose w' is also a fixed point of f such that $D(w, w') < \infty$ and $D(w', w') < \infty$. Now, $D(w, w') = D(fw, fw') \leqslant \psi(M(w, w'))$, where

$$M(w, w') = \max\{D(w, w'), D(w, w), D(w', w'), D(w, w'), D(w, w')\} = D(w, w').$$

Therefore, $D(w, w') \leqslant \psi(D(w, w')) < D(w, w')$. Hence, $D(w, w') = 0$ such that uniqueness of the fixed point follows. ■

The following is an example in support of Theorem 3.3.

***Example* 3.4** Let $X = [0,1]$ with the generalized metric D defined as in example 2.9. We define $fx = 0$ for all $x \in [0,1]$, and define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \frac{t}{3}$ for $t \geqslant 0$. Then f satisfies all the conditions of Theorem 3.3 with $x_0 = 1$ and 0 is the unique fixed point of f. But, if we define $fx = 1$ for all $x \in [0,1]$ then

$$D(fx, fy) = D(1,1) = 1 \nleq \psi(M(1,1)) = \psi(1)$$

for any $\psi \in \Psi_2$ so that the inequality (1) fails to hold for this constant function.

The following example shows that there may be two fixed points w and w' with $D(w, w') = \infty$.

***Example* 3.5** Let $X = \{1, \frac{1}{2}, \frac{1}{3}, ...\} \cup \{0\}$ and $D : X \times X \to [0, \infty]$ be given by $D(1,x) = D(x,1) = \infty$ if $x = 0$ or $\frac{1}{n}$ for $n = 1, 2, \cdots$ and otherwise, $D(x,y) = |x - y|$. Define

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ \frac{x}{2} & \text{otherwise.} \end{cases}$$

and $\psi(t) = kt$ for $t \geqslant 0$ and $k < 1$. Then 0 and 1 are fixed points, $D(0,1) = \infty$ and all the hypotheses of Theorem 3.3 are satisfied.

The following theorems are corollaries of our main result.

Theorem 3.6 [4] Let (X, D) be a D-complete generalized metric space and $f : X \to X$ be a k-quasi contraction with $k\lambda < 1$. Suppose that there exists $x_0 \in X$ such that

$$\sup_n D(x_0, f^n(x_0)) < \alpha < \infty,$$
$$D(f^n(x_0), f^{n+1}(x_0)) \leqslant k^n \alpha$$

for every n. Then $\{f^n(x_0)\}$ is Cauchy sequence and hence converges $w \in X$. If $\limsup_n D(f^n(x_0, fw)) < \infty$, then w is a fixed point of f. Moreover, if w' is another fixed point of f such that $D(w, w') < \infty$ and $D(w', w') < \infty$, then $w = w'$.

Theorem 3.7 [4] Let (X, D) be a D-complete generalized metric space and $f : X \to X$ be a R-type contraction. Suppose $q \geqslant 2, k\lambda < 1$ and there exists $x_0 \in X$ such that

$$\sup_n D(x_0, f^n(x_0)) < \alpha < \infty,$$
$$D(f^n(x_0), f^{n+1}(x_0)) \leqslant k^n \alpha$$

for each $n \in \mathbb{N}$. Then $\{f^n(x_0)\}$ is Cauchy sequence and hence converges $w \in X$. If $\limsup_n D(f^n(x_0, fw)) < \infty$, then w is a fixed point of f. Moreover, if w' is another fixed point of f such that $D(w, w') < \infty$ and $D(w', w') < \infty$ then $w = w'$.

The following example shows the significance of the coefficient λ and the set Ψ_λ of functions, in proving the existence of fixed points of weak ψ-quasi contractions. In other words, fixed point may not exist if the condition $\psi(t) < \frac{t}{\lambda}$ is violated.

***Example* 3.8** [2] Let $X = [0,1] \cup \{2\}$ and $D : X \times X \to [0, \infty]$ be given by

$$D(x,y) = \begin{cases} 10 & \text{if either } (x,y) = (0,2) \text{ or } (x,y) = (2,0) \\ |x-y| & \text{otherwise.} \end{cases}$$

Then (X, D) is a generalized metric space with $\lambda = 5$. Define

$$f(x) = \begin{cases} 2 & \text{if } x = 0 \\ \frac{x}{2} & \text{otherwise} \end{cases}$$

and $\psi(t) = \frac{t}{2}$ for $t \geqslant 0$. In this example, $\psi(t) < \frac{t}{\lambda}$ ($\lambda = 5$ in this case) is violated and f does not have a fixed point.

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