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A note on the new basis in the mod 2 Steenrod algebra

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Abstract. The Mod 2 Steenrod algebra is a Hopf algebra that consists of the primary cohomology operations, denoted by Sq^n, between the cohomology groups with $\mathbb{Z}_2$ coefficients of any topological space. Regarding to its vector space structure over $\mathbb{Z}_2$, it has many base systems and some of the base systems can also be restricted to its sub algebras. On the contrary, in addition to the work of Wood, in this paper we define a new base system for the Hopf subalgebras $\mathcal{A}(n)$ of the mod 2 Steenrod algebra which can be extended to the entire algebra. The new base system is obtained by defining a new linear ordering on the pairs $(s+t, s)$ of exponents of the atomic squares $Sq^{2^s(2^t-1)}$ for the integers $s \geqslant 0$ and $t \geqslant 1$.

Keywords: Steenrod Algebra, cohomology operations, Hopf algebra.

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1. Introduction

The mod 2 Steenrod algebra $\mathcal{A}$ consists of the Steenrod squares

$$Sq^i : H^n(X;\mathbb{Z}_2) \longrightarrow H^{n+i}(X;\mathbb{Z}_2) \qquad \text{for } i \geqslant 0$$

which were introduced by Norman Steenrod [8] in 1947. These squares are the cohomology operations satisfying the naturality property. Furthermore they commute with the suspension maps hence they are stable (for the details, please refer to [7] and the references therein).

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The Steenrod algebra was applied to the Hopf invariant 1 problem and the vector fields on spheres and the Steenrod operations are also used for computing the homotopy groups of n-spheres. Therefore it is one of the important tools in algebraic topology. It has a deep structure which has still not been completely solved.

The Steenrod algebra has a Hopf algebraic structure [4] and is the union of the finite Hopf subalgebras, $\mathcal{A}(n)$'s. Hence every element of the Steenrod algebra belongs to $\mathcal{A}(n)$ for some n and the $\mathcal{A}(n)$s are finite dimensional vector spaces. This means that every element in positive grading is nilpotent. Some researchers in this area investigate the nilpotence height of some operations but this is still an open problem, and we do not know the nilpotence height of all elements of the Steenrod algebra. The more we know about the structure of the sub Hopf algebras, we'll be closer to consider the problems in the entire Steenrod algebra. Monks [5] has defined a special base system by constructing a family of certain Milnor elements [4] for the mod 2 Steenrod algebra. Arnon [1] has also defined base systems for the Hopf subalgebra that can be extended to whole mod 2 Steenrod algebra and Karaca [2] has generalized the Arnon base systems to odd prime cases. Further, Palmieri and Zhang [6] have constructed a family of bases for the mod p Steenrod algebra from the elements called iterated commutators.

Wood [10] has defined the atomic squares which are of the form $Sq^{2^s(2^t-1)}$ and introduced the Y and Z base systems for the Hopf subalgebra by giving a linear ordering on the pairs (s,t) and $(s+t,t)$ respectively. Wood has also stated the Problem 4.19 in [9] that "Which orderings of atomic square give rise to bases of the $\mathcal{A}(n)$?" This paper is one of the examples of his problem.

The paper is organized as follows: Firstly we provide a brief information about the mod 2 Steenrod algebra and the Y and Z base systems introduced by Wood [10]. Then we introduce the new base system, N basis, which can be used alternatively in place of Y and Z basis.

2. Preliminaries

For each pair of nonnegative integers i, n there is a group homomorphism

$$Sq^i : H^n(X;\mathbb{Z}_2) \longrightarrow H^{n+i}(X;\mathbb{Z}_2)$$

between the cohomology groups of a topological space X, called the ith Steenrod squares. They are stable cohomology operations, that is, they commute with suspension maps. All the Steenrod squares form an algebraic structure which is known as the mod 2 Steenrod algebra $\mathcal{A}$ subject to the Adem relations

$$Sq^iSq^j = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k}Sq^k$$

for $i < 2j$, where $\lfloor \frac{i}{2} \rfloor$ denotes the greatest integer which is less than or equal to $\frac{i}{2}$ and the binomial coefficient is taken in mod 2. The grading of the Steenrod square Sq^i is i and for the composition of the Steenrod squares $Sq^{i_1}Sq^{i_2}...Sq^{i_k}$, the grading is $i_1+i_2+...+i_k$.

The Steenrod algebra $\mathcal{A}$ has a natural co-product map $\psi : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$, where $Sq^i \mapsto \psi(Sq^i) = \sum_{j+k=i} Sq^j \otimes Sq^k$ over the Steenrod squares [4]. Since ψ is linear, then it can be extended to the whole algebra therefore it makes $\mathcal{A}$ into a co-commutative Hopf algebra.

Milnor [4] also showed that the Steenrod algebra has a filtration by finite dimensional sub algebras and is a finite dimensional vector space in each grading (we say in algebraic topology that Steenrod algebra is finite type) so the dual algebra also has a Hopf algebraic structure. Knowing what the dual algebra is and the base system of it provides to determine the base system of the Steenrod algebra.

For $n \geqslant 0$, let $\mathcal{A}(n)$ be the Hopf subalgebras which are defined by the profile functions h: $\{1, 2, ...\} \longrightarrow \{0, 1, ..., \infty\}$ [3], where $t \longmapsto h(t) = \max\{n + 2 - t, 0\}$. Then each $\mathcal{A}(n)$ is generated as an algebra by $\{P_t^s \ : \ s + t \leqslant n + 1\}$ where P_t^s is a Milnor basis $Sq(0, ..., 0, 2^s, 0, ,)$ such that 2^s occupies position t [3, 5]. Note that the grading of P_t^s is $2^s(2^t - 1)$.

Let's associate the $Sq^{2^s(2^t-1)}$ to P_t^s. We call the number $2^s(2^t - 1)$ as *atomic numbers* and the squares $Sq^{2^s(2^t-1)}$ as *atomic squares* [9].

3. A New Base System

Wood considers two base systems for $\mathcal{A}(n)$ which can be extended to $\mathcal{A}$ and constructed by the Steenrod squares. Let

$$W_n^k = Sq^{2^k(2^{n+1}-1)} Sq^{2^k(2^n-1)} ... Sq^{2^k}$$

and define

$$Y_n = W_0^n W_1^{n-1} ... W_n^0.$$

For instance,

$$Y_1 = Sq^2 Sq^3 Sq^1$$

$$Y_2 = Sq^4 Sq^6 Sq^2 Sq^7 Sq^3 Sq^1$$

$$Y_3 = Sq^8 Sq^{12} Sq^4 Sq^{14} Sq^6 Sq^2 Sq^{15} Sq^7 Sq^3 Sq^1.$$

Wood proves that Y_n is the top element for $\mathcal{A}(n)$ where the top element means the monomial in the subalgebra with a maximum length, length is the number of Steenrod operations in the monomial and the set of $2^{(n+1)(n+2)/2}$ monomials obtained by selecting all subsets of atomic factors in Y_n, in the given order, is an additive basis for $\mathcal{A}(n)$ [10].

Another base system constructed by Wood for $\mathcal{A}(n)$ is as follows: Let

$$X_n = Sq^{1.2^n} Sq^{3.2^{n-1}} Sq^{7.2^{n-2}} ... Sq^{2^{n+1}-1}$$

and define

$$Z_n := X_n X_{n-1} ... X_1 X_0.$$

For instance,

$$Z_1 = Sq^2 Sq^3 Sq^1$$

$$Z_2 = Sq^4 Sq^6 Sq^7 Sq^2 Sq^3 Sq^1$$

$$Z_3 = Sq^8 Sq^{12} Sq^{14} Sq^{15} Sq^4 Sq^6 Sq^7 Sq^2 Sq^3 Sq^1.$$

In this case element Z_n is again the top element for $\mathcal{A}(n)$ in terms of grading and the set of $2^{(n+1)(n+2)/2}$ monomials obtained by selecting all subsets of atomic factors in Z_n, in the given order, is, again, an additive basis for $\mathcal{A}(n)$ [10].

The motivation of the constructions for the base systems Y and Z for $\mathcal{A}(n)$ is based on considering the atomic squares $Sq^{2^s(2^t-1)}$ associated with P^s_t and imposing and ordering for the pairs (s,t). Wood call the orderings as Y-order and Z-order for the base system Y and Z respectively. The Y-order is a left lexicographic order on the pairs (s,t) and Z-order is a left lexicographic order on the pairs $(s+t,s)$.

There is a practical tool for finding the top elements Y_n and Z_n in $\mathcal{A}(n)$ under the given order. For this, we will use the sequence of terms in $1,3,7,\cdots,2^n-1$ form. In this case, we take the first $n+1$ term from this sequence and write it down in the first row. Then in the following row, we place the double of the terms in the previous row but the number of terms in each row will be one-less than the number of previous terms. We will continue this process until the number of terms in the last row is 1. These numbers will be the exponents of our top element Y_n or Z_n.

For instance consider $\mathcal{A}(3)$. In this case, we will write down the terms $1,3,7,15$ in the first row In the second row, we will write down the doubles of 1, 3 and 7 , which are 2, 6 and 14. In the following line, we will write down the doubles of 2 and 6, which are 4 and 12. In the last line, we will only write down the double of 4, 8. Now, in which sequence should we read these numbers? If we start reading from the bottom line to the top line, from right to left, this gives us the exponents for the top element Y_n in $\mathcal{A}(n)$. For example, we want to read the top element Y_3 in $\mathcal{A}(3)$. We start off from the bottom line, 8. Then we go up to the upper line and read from right to left, 12 and 4. We go up again to the upper line and read from right to left, 14, 6, 2, and it goes on like this and this will give us the exponents for the top element Y_3. Notice that, if we read from the bottom to the top diagonally, this will give us the top element Z_n.

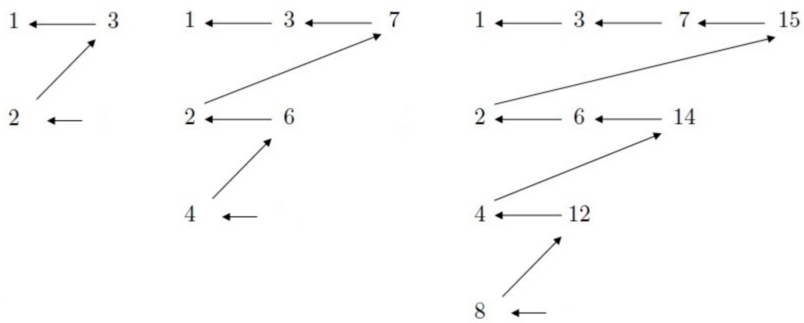

Figure 1. Y_1, Y_2 and Y_3

Unfortunately not all orderings on this number set yield a top element because of the Adem relations, the coefficient can be a multiple of 2 so a monomial can vanish. Wood offers the Problem 4.19 in his paper [9] that in which ordering on this number set give rise to bases for $\mathcal{A}(n)$ that can be also extended to whole algebra? For this, we have made an ordering by applying the 'Cantor Diagonalization Method' to the pair $(s+t,s)$ of atomic squares $Sq^{2^s(2^t-1)}$.

For instance the pair $(4,0)$ is larger than the pair $(2,1)$ so the atomic squares Sq^{15} which is associated with $(4,0)$ will take precedence of Sq^2 which is associated with $(2,1)$ while we form a new basis element.

Again, there is a practical tool for finding the new top element. We order the number set as we considered above. The new top element is as follows: We start reading from

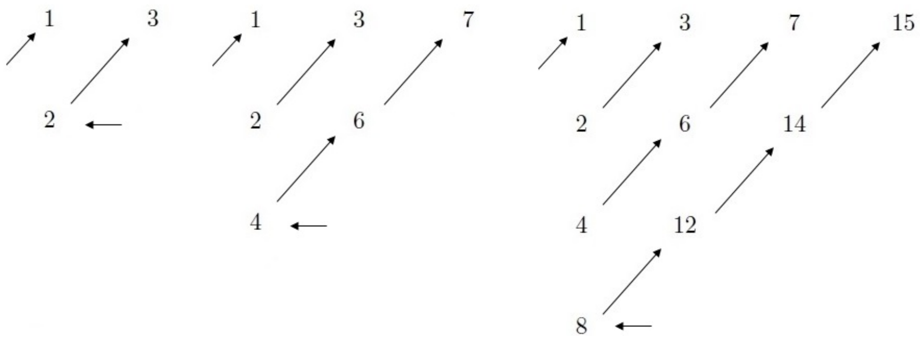

Figure 2. Z_1, Z_2 and Z_3

s+t \ s	0		1		2		3		4		5		6		7		8	...
1	(1,0)	→	(1,1)		(1,2)	→	(1,3)		(1,4)	→	(1,5)		(1,6)	→	(1,7)		(1,8)	
		↙		↗		↙		↗		↙		↗		↙		↗		
2	(2,0)		(2,1)		(2,2)		(2,3)		(2,4)		(2,5)		(2,6)		(2,7)			
	↓	↗		↙		↗		↙		↗		↙		↗				
3	(3,0)		(3,1)		(3,2)		(3,3)		(3,4)		(3,5)		(3,6)					
		↙		↗		↙		↗		↙		↗						
4	(4,0)		(4,1)		(4,2)		(4,3)		(4,4)		(4,5)							
	↓	↗		↙		↗		↙		↗								
5	(5,0)		(5,1)		(5,2)		(5,3)		(5,4)									
		↙		↗		↙		↗										
6	(6,0)		(6,1)		(6,2)		(6,3)											
	↓	↗		↙		↗												
7	(7,0)		(7,1)		(7,2)													
		↙		↗														
8	(8,0)		(8,1)															
	↓	↗																
9	(9,0)																	
⋮																		

Table 1. Cantor Diagonalization Method' to the pair $(s+t, s)$ of atomic squares $Sq^{2^s(2^t-1)}$

the bottom row. We go up and move from right to the left. Then we go up again and go to the right 2 units and keep going on like this until the element ends diagonally. Then go up and continue from the least significant digit. Go down and move to the left 2 units until the element ends diagonally again. Then move to the left 1 unit do the same procedures of going up and down. This ordering will also define a top element for $\mathcal{A}(n)$. For instance if we would like to find the top element N_3, we go from 8 to 12 and 12 to 4. Then we go up and go to the right 2 units, 14. there are no more moves to go up and move to the right 2 units so we go up and continue from 15. Then go down and move to the left 2 units, 6. There is no move remained for going down and moving to the left 2 units so move left on the same line, 2. Then go up again, move to the right two units, 7. Since it's not impossible to go up/down, move to the left again, 3, and continue to move to the left, 1.

For instance,

$$N_1 = Sq^2 Sq^3 Sq^1$$

$$N_2 = Sq^4 Sq^6 Sq^2 Sq^7 Sq^3 Sq^1$$

$$N_3 = Sq^8 Sq^{12} Sq^4 Sq^{14} Sq^{15} Sq^6 Sq^2 Sq^7 Sq^3 Sq^1.$$

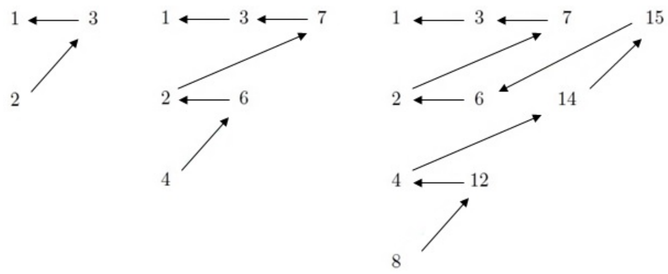

Figure 3. N_1, N_2 and N_3

Theorem 3.1 If we order the atomic numbers as shown in the figures then N_n is the top element of $\mathcal{A}(n)$ and the set of $2^{(n+1)(n+2)/2}$ monomials obtained by selecting all subsets of atomic factors in N_n, in the given order, is an additive basis for $\mathcal{A}(n)$.

To prove the theorem, we use some results obtained by Wood in [10]. Taking $i = 2^a - 1$, $j = 2^b - 1$ in the Adem relations, the following theorem was proved by Lucas theorem.

Theorem 3.2 [10] For $a \leqslant b$

$$Sq^{2^a-1}Sq^{2^b-1} = \sum_{0<c<a} Sq^{2^a+2^b-2^c-1}Sq^{2^c-1}.$$

Theorem 3.3 [10] For any monomial in the Steenrod squares having at least one factor with odd superfix can be written as a sum of monomials in each of which the last factor has odd superfix.

Theorem 3.4 [10] If n is odd, then Sq^n can be written as a sum of monomials in each of which the last factor has superfix of the form $2^k - 1$.

Suppose $I = (n_1, n_2, ..., n_k)$ is the finite sequence of nonnegative integers and

$$Sq^I = Sq^{n_1}Sq^{n_2}...Sq^{n_k}$$

be the monomials corresponding to I. Also, let Sq^{2I} is the monomial obtained by Sq^I by duplication. Wood [10] states that Adem relation defined on Sq^{2I} has the same effect as defined on the corresponding relation on Sq^I modulo terms which have at least one factor with the odd superfix.

Proof of Theorem 3.1 We must show that the theorem is true for the grading $n+1$ while it is true for $n > 0$. Take a monomial Sq^I in grading $n+1$ where

$$I = (n_1, n_2, ..., n_k).$$

If n_i is an even number for all $1 \leqslant i \leqslant k$, then the grading of the monomial $Sq^{I/2}$ is less than or equal to $n+1$. By the inductive hypothesis, $Sq^{I/2}$ is an N-basis element so the duplication of it is again N-basis element. So assume that there exists a component in the sequence I which is an odd number. By Theorem 3.3 and Theorem 3.4, we can iterate this odd component to the last component n_k in I as $2^j - 1$ for $j \geqslant 1$. Then the monomial Sq^I can be written as

$$Sq^J Sq^{2^k-1}$$

where the grading of Sq^J is less than or equal to n. If all the components of J is even then we are done because in this case Sq^I becomes an N-basis monomial. If there exists a component in the sequence J which is an odd number, then by Theorem 3.3 and Theorem 3.4, we can iterate this odd component to the last component of J which is of the form $2^m - 1$ for $m \geqslant 1$. Now the monomial Sq^I can be written as

$$Sq^L Sq^{2^m-1} Sq^{2^k-1}.$$

If $m > k$ then we are done again. Otherwise Theorem 3.2 is applied to the iterated squares $Sq^{2^m-1}Sq^{2^k-1}$ and this yields that Sq^I can be written as a sum of monomials of the form

$$Sq^K Sq^{2^\ell-1}$$

where $\ell < k$. This entire process is repeated unless $k = 1$ and this completes the proof.

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