

New best proximity point results in G -metric space

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Abstract. Best approximation results provide an approximate solution to the fixed point equation $Tx = x$, when the non-self mapping T has no fixed point. In particular, a well-known best approximation theorem, due to Fan [6], asserts that if K is a nonempty compact convex subset of a Hausdorff locally convex topological vector space E and $T : K \rightarrow E$ is a continuous mapping, then there exists an element x satisfying the condition $d(x, Tx) = \inf\{d(y, Ty) : y \in K\}$, where d is a metric on E . Recently, Hussain et al. (Abstract and Applied Analysis, Vol. 2014, Article ID 837943) introduced proximal contractive mappings and established certain best proximity point results for these mappings in G -metric spaces. The aim of this paper is to introduce certain new classes of auxiliary functions and proximal contraction mappings and establish best proximity point theorems for such kind of mappings in G -metric spaces. As consequences of these results, we deduce certain new best proximity and fixed point results in G -metric spaces. Moreover, we present certain examples to illustrate the usability of the obtained results.

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1. Introduction

Best proximity point evolves as a generalization of the concept of best approximation. A best approximation theorem guarantees the existence of an approximate solution, a best

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proximity point theorem is contemplated for solving the problem to find an approximate solution which is optimal. Given nonempty closed subsets A and B of E , when a non-self-mapping $T : A \rightarrow B$ has not a fixed point, it is quite natural to find an element x^* such that $d(x^*, Tx^*)$ is minimum. Best proximity point theorems guarantee the existence of an element x^* such that $d(x^*, Tx^*) = d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\}$; this element is called a best proximity point of T . Moreover, if the mapping under consideration is a self-mapping, best proximity point theorem reduces to a fixed point result. For some results in this direction, we refer to [1, 5, 8, 10, 19, 22] and references therein.

On the other hand, Mustafa and Sims introduced the notion of G -metric and investigated the topology of such spaces. The authors also characterized some celebrated fixed point result in the context of G -metric space. Following this initial paper, a number of authors have published so many fixed point results on the setting of G -metric space (see [3, 7, 12, 15, 17, 18, 20] and references therein). Samet et al. [21] and Jleli and Samet [11] reported that some published results can be considered as a straight consequence of the existence theorem in the setting of usual metric space. More recently, Asadi et al. [2] proved some fixed point theorems in the framework of G -metric space that cannot be obtained from the existence results in the context of associated metric space. G -metric spaces proved to be rich for fixed point theory but the best proximity problem remains open. In this paper we prove certain new best proximity point results using auxiliary functions and as consequence we deduce some recent fixed point results as corollaries.

First we recollect some necessary definition and results in this direction. The notion of G -metric spaces is defined as follows:

Definition 1.1 (See [13]) Let X be a non-empty set, $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following properties :

- (G1) $G(x, y, z) = 0$ if $x = y = z$,
- (G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
- (G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
- (G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables),
- (G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ (rectangle inequality) for all $x, y, z, a \in X$.

Then the function G is called a generalized metric, or, more specifically, a G -metric on X , and the pair (X, G) is called a G -metric space.

Note that every G -metric on X induces a metric d_G on X defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \text{ for all } x, y \in X. \quad (1)$$

For a better understanding of the subject we give the following examples of G -metrics:

Example 1.2 Let (X, d) be a metric space. The function $G : X \times X \times X \rightarrow [0, +\infty)$, defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\},$$

for all $x, y, z \in X$, is a G -metric on X .

Example 1.3 (See e.g. [13]) Let $X = [0, \infty)$. The function $G : X \times X \times X \rightarrow [0, +\infty)$, defined by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|,$$

for all $x, y, z \in X$, is a G -metric on X .

In their initial paper, Mustafa and Sims [13] also defined the basic topological concepts in G -metric spaces as follows:

Definition 1.4 (See [13]). Let (X, G) be a G -metric space, and let $\{x_n\}$ be a sequence of points of X . We say that $\{x_n\}$ is G -convergent to $x \in X$ if

$$\lim_{n,m \rightarrow +\infty} G(x, x_n, x_m) = 0,$$

that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \rightarrow x$ or $\lim_{n \rightarrow +\infty} x_n = x$.

Proposition 1.5 (See [13]). Let (X, G) be a G -metric space. The following are equivalent:

- (1) $\{x_n\}$ is G -convergent to x ,
- (2) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow +\infty$,
- (3) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow +\infty$,
- (4) $G(x_n, x_m, x) \rightarrow 0$ as $n, m \rightarrow +\infty$.

Definition 1.6 (See [13]). Let (X, G) be a G -metric space. A sequence $\{x_n\}$ is called a G -Cauchy sequence if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$, that is, $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow +\infty$.

Proposition 1.7 (See [13]). Let (X, G) be a G -metric space. Then the followings are equivalent:

- (1) the sequence $\{x_n\}$ is G -Cauchy,
- (2) for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq N$.

Definition 1.8 (See [13]) A G -metric space (X, G) is called G -complete if every G -Cauchy sequence is G -convergent in (X, G) .

Definition 1.9 Let (X, G) be a G -metric space. A mapping $F : X \times X \times X \rightarrow X$ is said to be continuous if for any three G -convergent sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converging to x , y and z respectively, $\{F(x_n, y_n, z_n)\}$ is G -convergent to $F(x, y, z)$.

Mustafa [16] extended the well-known Banach Contraction Principle Mapping in the framework of G -metric spaces as follows:

Theorem 1.10 (See [16]) Let (X, G) be a complete G -metric space and $T : X \rightarrow X$ be a mapping satisfying the following condition for all $x, y, z \in X$:

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \tag{2}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Theorem 1.11 (See [16]) Let (X, G) be a complete G -metric space and $T : X \rightarrow X$ be a mapping satisfying the following condition for all $x, y \in X$:

$$G(Tx, Ty, Ty) \leq kG(x, y, y), \tag{3}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Remark 1 We notice that the condition (2) implies the condition (3). The converse is true only if $k \in [0, \frac{1}{2})$. For details see [16].

Lemma 1.12 [16] By the rectangle inequality (G5) together with the symmetry (G4), we have

$$G(x, y, y) = G(y, y, x) \leq G(y, x, x) + G(x, y, x) = 2G(y, x, x). \quad (4)$$

2. Main Results

Recall that every G -metric on X induces a metric d_G on X defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \text{ for all } x, y \in X. \quad (5)$$

Let (X, G) be a G -metric space. Suppose that A and B are nonempty subsets of a G -metric space (X, G) . We define the following sets:

$$\begin{aligned} A_0 &= \{x \in A : d_G(x, y) = d_G(A, B) \text{ for some } y \in B\} \\ B_0 &= \{y \in B : d_G(x, y) = d_G(A, B) \text{ for some } x \in A\} \end{aligned} \quad (6)$$

where $d_G(A, B) = \inf\{d_G(x, y) : x \in A, y \in B\}$.

Definition 2.1 Let (X, G) be a G -metric space and A and B be two nonempty subsets of X . Then B is said to be approximately compact with respect to A if every sequence $\{y_n\}$ in B , satisfying the condition $d_G(x, y_n) \rightarrow d_G(x, B)$ for some x in A , has a convergent subsequence.

We assume that

$$\Psi = \{\psi : [0, \infty) \rightarrow [0, \infty) \text{ such that } \psi \text{ is non-decreasing and continuous}\}$$

where $\psi(t) = 0$ if and only if $t = 0$, and

$$\Phi = \{\phi : [0, \infty) \rightarrow [0, \infty) \text{ such that } \phi \text{ is lower semi-continuous}\}$$

where $\phi(t) > 0, t > 0$, and $\phi(0) \geq 0$.

Definition 2.2 We say that $f : [0, \infty)^2 \rightarrow R$ is a function of C -class if f is continuous and

$$(1) f(s, t) \leq s$$

$$(2) f(s, t) = s \implies s = 0 \text{ or } t = 0$$

for all $s, t \in \mathbb{R}$. Note that for some f we have that $f(0, 0) = 0$.

Example 2.3 Let $s, t \in [0, \infty)$, then

$$(1) f(s, t) = s - t, f(s, t) = s \implies t = 0;$$

$$(2) f(s, t) = ks, 0 < k < 1, f(s, t) = s \implies s = 0;$$

$$(3) f(s, t) = \frac{s}{(1+t)^r}; r \in (0, \infty), f(s, t) = s \implies s = 0 \text{ or } t = 0;$$

$$(4) f(s, t) = \log_a(t + a^s)/(1+t), a > 1, f(s, t) = s \implies s = 0 \text{ or } t = 0;$$

$$(5) f(s, t) = \log_a(1 + a^s)/2, a > 1, f(s, t) = s \implies s = 0;$$

$$(6) f(s, t) = (s+l)^{(1/(1+t)^r)} - l, l > 1, r \in (0, \infty), f(s, t) = s \implies t = 0;$$

$$(7) f(s, t) = s \log_{t+a} a, a > 1, f(s, t) = s \implies s = 0 \text{ or } t = 0;$$

$$(8) f(s, t) = s - \left(\frac{1+s}{2+s}\right)\left(\frac{t}{1+t}\right), f(s, t) = s \implies t = 0;$$

$$(9) f(s, t) = s\beta(s), \beta : [0, \infty) \rightarrow [0, 1), f(s, t) = s \implies s = 0;$$

- (10) $f(s, t) = s - \frac{t}{k+t}, f(s, t) = s \Rightarrow t = 0;$
- (11) $f(s, t) = s - \varphi(s), f(s, t) = s \Rightarrow s = 0,$ here $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0;$
- (12) $f(s, t) = sh(s, t), f(s, t) = s \Rightarrow s = 0,$ here $h : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $h(t, s) < 1$ for all $t, s > 0;$
- (13) $f(s, t) = s - (\frac{2+t}{1+t})t, f(s, t) = s \Rightarrow t = 0.$

Definition 2.4 Let A and B be two nonempty subsets of a G -metric space (X, G) . Let $T : A \rightarrow B$ be a non-self mapping. We say T is a G - ψ - ϕ - f -proximal contractive mapping if for $x, y, u, u^*, v \in A$

$$\left. \begin{aligned} d_G(u, Tx) &= d_G(A, B) \\ d_G(u^*, Tu) &= d_G(A, B) \\ d_G(v, Ty) &= d_G(A, B) \end{aligned} \right\} \implies \psi(G(u, u^*, v)) \leq f(\psi(G(x, u, y)), \phi(G(x, u, y))) \quad (7)$$

holds where $\psi \in \Psi$ and $\phi \in \Phi$.

Following is our first main result.

Theorem 2.5 Let A, B be two nonempty subsets of a G -metric space (X, G) such that (A, G) is a complete G -metric space, A_0 is nonempty and B is approximatively compact with respect to A . Assume that $T : A \rightarrow B$ is a G - ψ - ϕ - f -proximal contractive mapping such that $T(A_0) \subset B_0$. Then T has a unique best proximity point i.e., there exists unique $z \in A$ such that $d_G(z, Tz) = d_G(A, B)$.

Proof. Since the subset A_0 is not empty, we take x_0 in A_0 . Taking $Tx_0 \in T(A_0) \subseteq B_0$ into account, we can find $x_1 \in A_0$ such that $d_G(x_1, Tx_0) = d_G(A, B)$. Further, since $Tx_1 \in T(A_0) \subseteq B_0$, it follows that there is an element x_2 in A_0 such that $d_G(x_2, Tx_1) = d_G(A, B)$. Recursively, we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$d_G(x_{n+1}, Tx_n) = d_G(A, B) \text{ for all } n \in \mathbb{N} \cup \{0\} \quad (8)$$

This show that $\left\{ \begin{aligned} d_G(u, Tx) &= d_G(A, B), \\ d_G(u^*, Tu) &= d_G(A, B), \\ d_G(v, Ty) &= d_G(A, B) \end{aligned} \right.$ where $x = x_{n-1}, u = x_n, u^* = x_{n+1}$ and $y = x_n, v = x_{n+1}$. Therefore from (7) we have,

$$\begin{aligned} \psi(G(x_n, x_{n+1}, x_{n+1})) &\leq f(\psi(G(x_{n-1}, x_n, x_n)), \phi(G(x_{n-1}, x_n, x_n))) \\ &\leq \psi(G(x_{n-1}, x_n, x_n)) \end{aligned} \quad (9)$$

which implies $G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n)$. So the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is decreasing sequence in \mathbb{R}^+ and thus it is convergent to $t \in \mathbb{R}^+$. We claim that $t = 0$. Suppose, to the contrary, that $t > 0$. Taking limit as $n \rightarrow \infty$ in (9) we get,

$$\psi(t) \leq f(\psi(t), \phi(t))$$

which implies $\psi(t) = 0$ or $\phi(t) = 0$. That is, $t = 0$ which is a contrary. Hence, $t = 0$. i.e.,

$$\lim_{n \rightarrow \infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

We shall show that $\{x_n\}_{n=0}^\infty$ is a G -Cauchy sequence.

Suppose, to the contrary, that there exists $\varepsilon > 0$, and a sequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) \geq \varepsilon \quad (10)$$

with $n(k) \geq m(k) > k$. Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (10). Hence,

$$G(x_{m(k)}, x_{m(k)+1}, x_{n(k)-1}) < \varepsilon \quad (11)$$

By Proposition 1.5 (iii) and (G5) we have

$$\begin{aligned} \varepsilon &\leq G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) = G(x_{n(k)}, x_{m(k)}, x_{m(k)+1}) \\ &\leq G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)+1}, x_{m(k)}) \\ &\leq G(x_{m(k)}, x_{m(k)+1}, x_{n(k)-1}) + 2s_{n(k)-1} \\ &\leq \varepsilon + 2s_{n(k)-1}. \end{aligned} \quad (12)$$

Letting $k \rightarrow \infty$ in (26) we derive that

$$\lim_{k \rightarrow \infty} G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) = \varepsilon. \quad (13)$$

Also, by Proposition 1.5 (iii) and (G5) we obtain the following inequalities

$$\begin{aligned} G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) &\leq G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, x_{m(k)+1}, x_{n(k)}) \\ &= G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{n(k)}, x_{m(k)-1}, x_{m(k)+1}) \\ &\leq G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) \\ &\quad + G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}) \\ &\leq 2s_{m(k)-1} + 2s_{n(k)-1} + G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}) \end{aligned} \quad (14)$$

and

$$\begin{aligned} G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}) &\leq G(x_{n(k)-1}, x_n, x_n) + G(x_{n(k)}, x_{m(k)-1}, x_{m(k)+1}) \\ &= G(x_{n(k)-1}, x_n, x_n) + G(x_{m(k)-1}, x_{m(k)+1}, x_{n(k)}) \\ &\leq G(x_{n(k)-1}, x_n, x_n) + G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) \\ &\quad + G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}) \\ &= s_{n(k)-1} + s_{m(k)-1} + G(x_{m(k)}, x_{m(k)+1}, x_{n(k)}). \end{aligned} \quad (15)$$

Letting $k \rightarrow \infty$ in (14) and (15) and applying (27) we find that

$$\lim_{k \rightarrow \infty} G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}) = \varepsilon. \tag{16}$$

Again by Proposition 1.5 (iii) and (G5) we have,

$$\begin{aligned} G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)+1}) &= G(x_{m(k)+1}, x_{m(k)-1}, x_{n(k)-1}) \\ &= G(x_{m(k)+1}, x_{m(k)-1}, x_{n(k)-1}) \\ &\leq G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{m(k)-1}, x_{n(k)-1}) \\ &= G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}) \\ &\leq 2s_{m(k)} + G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}) \end{aligned} \tag{17}$$

and

$$\begin{aligned} G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}) &= G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}) \\ &\leq G(x_{m(k)-1}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{m(k)}, x_{n(k)-1}) \\ &\leq G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) \\ &\quad G(x_{m(k)+1}, x_{m(k)}, x_{n(k)-1}) \\ &= s_{m(k)-1} + s_{m(k)} + G(x_{m(k)+1}, x_{m(k)}, x_{n(k)-1}) \\ &= s_{m(k)-1} + s_{m(k)} + G(x_{m(k)}, x_{m(k)+1}, x_{n(k)-1}) \\ &< s_{m(k)-1} + s_{m(k)} + \varepsilon. \end{aligned} \tag{18}$$

Taking limit as $k \rightarrow \infty$ in (17) and (18) and applying (16) we have,

$$\lim_{k \rightarrow \infty} G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}) = \varepsilon. \tag{19}$$

By (7) with $x = x_{m(k)-1}$, $u = x_{m(k)-1}$, $u^* = x_{m(k)}$, $y = x_{n(k)-1}$, $v = x_{n(k)}$ we have,

$$\psi(G(x_{m(k)}, x_{m(k)+1}, x_{n(k)})) \leq f(\psi(G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1})), \phi(G(x_{m(k)-1}, x_{m(k)}, x_{n(k)-1}))).$$

Taking limit as $k \rightarrow \infty$ in the above inequality we have,

$$\psi(\varepsilon) \leq f(\psi(\varepsilon), \phi(\varepsilon))$$

so $\psi(\varepsilon) = 0$ or $\phi(\varepsilon) = 0$ which implies $\varepsilon = 0$ which is a contradiction. Thus,

$$\lim_{m,n \rightarrow \infty} G(x_m, x_{m+1}, x_n) = 0.$$

That is $\{x_n\}_0^\infty$ is a Cauchy sequence. Since, (A, G) is a complete G -metric space, so there exists $z \in A$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$. On the other hand, for all $n \in \mathbb{N}$, we can write

$$\begin{aligned} d_G(z, B) &\leq d_G(z, Tx_n) \\ &\leq d_G(z, x_{n+1}) + d_G(x_{n+1}, Tx_n) \\ &= d_G(z, x_{n+1}) + d_G(A, B). \end{aligned}$$

Taking the limit as $n \rightarrow +\infty$ in the above inequality, we get

$$\lim_{n \rightarrow +\infty} d_G(z, Tx_n) = d_G(z, B) = d_G(A, B). \quad (20)$$

Since, B is approximatively compact with respect to A , so the sequence, $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ that converges to some $y^* \in B$. Hence,

$$d_G(z, y^*) = \lim_{n \rightarrow \infty} d_G(x_{n_k+1}, Tx_{n_k}) = d_G(A, B)$$

and so $z \in A_0$. Now, since, $Tz \in T(A_0) \subseteq B_0$, there exists $w \in A_0$ such that $d_G(w, Tz) = d_G(A, B)$.

From (7) with $x = x_n$, $u = x_{n+1}$, $u^* = x_{n+2}$, $y = z$ and $v = w$ we have,

$$\psi(G(x_{n+1}, x_{n+2}, w)) \leq f(\psi(G(x_n, x_{n+1}, z)), \phi(G(x_n, x_{n+1}, z))).$$

Taking limit as $n \rightarrow \infty$ we get,

$$\psi(G(z, z, w)) \leq f(\psi(0), \phi(0)) \leq \psi(0) = 0.$$

Then $G(z, z, w) = 0$. i.e., $w = z$. Thus $d_G(z, Tz) = d_G(A, B)$. Therefore T has a best proximity point. To prove uniqueness, suppose that $p \neq q$, such that $d_G(p, Tp) = d_G(A, B)$ and $d_G(q, Tq) = d_G(A, B)$. Now by (7) with $x = u = u^* = p$ and $y = v = q$ we get,

$$\psi(G(p, p, q)) \leq f(\psi(G(p, p, q)), \phi(G(p, p, q)))$$

which implies $\psi(G(p, p, q)) = 0$ or $\phi(G(p, p, q)) = 0$. i.e., $p = q$. ■

Example 2.6 Let $X = [0, \infty)$ and $G(x, y, z) = \frac{1}{4}(|x - y| + |y - z| + |x - z|)$ be a G -metric on X . Then $d_G(x, y) = |x - y|$. Let $A = \{3, 4, 5, 6, 7\}$ and $B = \{9, 10, 11, 12, 13\}$. Define $T : A \rightarrow B$ by

$$T(x) = \begin{cases} 9, & \text{if } x = 7 \\ x + 6, & \text{otherwise} \end{cases}$$

Also define $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ by $\psi(t) = t, \phi(t) = a^{\frac{t}{2}}$ for $a > 0$ and $f : [0, \infty)^2 \rightarrow \mathbb{R}$, $f(s, t) = \log_a \frac{t+a^s}{1+t}$.

Clearly, $d_G(A, B) = 2, A_0 = \{7\}, B_0 = \{9\}$ and $TA_0 \subset B_0$. Let $d_G(u, Tx) = d_G(A, B) = 2$ and

$d_G(v, Ty) = d_G(A, B) = 2$. Then $(u, x), (v, y) \in \{(7, 7), (7, 3)\}$.

Also, if $d_G(u^*, Tu) = d_G(A, B) = 2$, then $u^* = 7$. Therefore, if

$d_G(u, Tx) = d_G(A, B)$

$d_G(u^*, Tu) = d_G(A, B)$
 $d_G(v, Ty) = d_G(A, B)$
 then $u = u^* = v = 7$, so $\psi(G(u, u^*, v)) = 0$.

Hence,

$$\psi(G(u, u^*, v)) = 0 \leq \frac{1}{2}G(x, u, y) = \log_a \frac{a^{\frac{1}{2}G(x, u, y)} + a^{G(x, u, y)}}{1 + a^{\frac{1}{2}G(x, u, y)}} = \log_a \frac{\phi(G(x, u, y)) + a^{\psi(G(x, u, y))}}{1 + \phi(G(x, u, y))}.$$

That is,

$d_G(u, Tx) = d_G(A, B)$
 $d_G(u^*, Tu) = d_G(A, B)$
 $d_G(v, Ty) = d_G(A, B)$

↓

$$\psi(G(u, u^*, v)) \leq \log_a \frac{\phi(G(x, u, y)) + a^{\psi(G(x, u, y))}}{1 + \phi(G(x, u, y))}.$$

Thus T is a G - ψ - ϕ - f -proximal contractive mapping. All conditions of Theorem 2.5 hold true and T has a unique best proximity point. Here, $z = 7$ is unique best proximity point of T .

If in Theorem 2.5 we take $\psi(t) = t$, $f(s, t) = rs$ where $0 \leq r < 1$, then we deduce the following corollary.

Corollary 2.7 Let A, B be two nonempty subsets of a G -metric space (X, G) such that (A, G) is a complete G -metric space, A_0 is nonempty and B is approximatively compact with respect to A . Assume that $T : A \rightarrow B$ is a non-self mapping such that $T(A_0) \subseteq B_0$ and for $x, y, u, u^*, v \in A$,

$$\left. \begin{aligned} d_G(u, Tx) &= d_G(A, B) \\ d_G(u^*, Tu) &= d_G(A, B) \\ d_G(v, Ty) &= d_G(A, B) \end{aligned} \right\} \implies G(u, u^*, v) \leq rG(x, u, y)$$

holds where $0 \leq r < 1$. Then T has a unique best proximity point. i.e., there exists unique $z \in A$ such $d_G(z, Tz) = d_G(A, B)$.

If in Theorem 2.5 we take $f(s, t) = s - t$, we obtain following corollary.

Corollary 2.8 (Theorem 16 of [9]) Let A, B be two nonempty subsets of a G -metric space (X, G) such that (A, G) is a complete G -metric space, A_0 is nonempty and B is approximatively compact with respect to A . Let $T : A \rightarrow B$ be a non-self mapping such that $T(A_0) \subset B_0$. If for $x, y, u, u^*, v \in A$

$$\left. \begin{aligned} d_G(u, Tx) &= d_G(A, B) \\ d_G(u^*, Tu) &= d_G(A, B) \\ d_G(v, Ty) &= d_G(A, B) \end{aligned} \right\} \implies \psi(G(u, u^*, v)) \leq \psi(G(x, u, y)) - \phi(G(x, u, y))$$

holds where $\psi \in \Psi$ and $\phi \in \Phi$. Then T has a unique best proximity point i.e., there exists unique $z \in A$ such that $d_G(z, Tz) = d_G(A, B)$.

If in Theorem 2.5 we take $f(s, t) = \frac{s}{1+t}$, we obtain following corollary.

Corollary 2.9 Let A, B be two nonempty subsets of a G -metric space (X, G) such that (A, G) is a complete G -metric space, A_0 is nonempty and B is approximatively compact

with respect to A . Let $T : A \rightarrow B$ be a non-self mapping such that $T(A_0) \subset B_0$. If for $x, y, u, u^*, v \in A$

$$\left. \begin{array}{l} d_G(u, Tx) = d_G(A, B) \\ d_G(u^*, Tu) = d_G(A, B) \\ d_G(v, Ty) = d_G(A, B) \end{array} \right\} \implies \psi(G(u, u^*, v)) \leq \frac{\psi(G(x, u, y))}{1 + \phi(G(x, u, y))}$$

holds where $\psi \in \Psi$ and $\phi \in \Phi$. Then T has a unique best proximity point i.e., there exists unique $z \in A$ such that $d_G(z, Tz) = d_G(A, B)$.

If in Theorem 2.5 we take $f(s, t) = s \log_{t+a} a, a > 1$, we obtain;

Corollary 2.10 Let A, B be two nonempty subsets of a G -metric space (X, G) such that (A, G) is a complete G -metric space, A_0 is nonempty and B is approximatively compact with respect to A . Let $T : A \rightarrow B$ be a non-self mapping such that $T(A_0) \subset B_0$. If for $x, y, u, u^*, v \in A$

$$\left. \begin{array}{l} d_G(u, Tx) = d_G(A, B) \\ d_G(u^*, Tu) = d_G(A, B) \\ d_G(v, Ty) = d_G(A, B) \end{array} \right\} \implies \psi(G(u, u^*, v)) \leq \psi(G(x, u, y)) \log_{a+\phi(G(x, u, y))} a$$

holds where $a > 1, \psi \in \Psi$ and $\phi \in \Phi$. Then T has a unique best proximity point i.e., there exists unique $z \in A$ such that $d_G(z, Tz) = d_G(A, B)$.

If in Theorem 2.5 we take $f(s, t) = \log(t + a^s)/(1 + t), a > 1$, we obtain following corollary.

Corollary 2.11 Let A, B be two nonempty subsets of a G -metric space (X, G) such that (A, G) is a complete G -metric space, A_0 is nonempty and B is approximatively compact with respect to A . Let $T : A \rightarrow B$ be a non-self mapping such that $T(A_0) \subset B_0$. If for $x, y, u, u^*, v \in A$

$$\left. \begin{array}{l} d_G(u, Tx) = d_G(A, B) \\ d_G(u^*, Tu) = d_G(A, B) \\ d_G(v, Ty) = d_G(A, B) \end{array} \right\} \implies \psi(G(u, u^*, v)) \leq \log_a \frac{\phi(G(x, u, y)) + a^{\psi(G(x, u, y))}}{1 + \phi(G(x, u, y))}$$

holds where $a > 1, \psi \in \Psi$ and $\phi \in \Phi$. Then T has a unique best proximity point i.e., there exists unique $z \in A$ such that $d_G(z, Tz) = d_G(A, B)$.

Definition 2.12 Let A and B be two nonempty subsets of a G -metric space (X, G) . Let $T : A \cup B \rightarrow A \cup B$ be a non-self mapping such that $T(A) \subset B, T(B) \subset A$. We say T is generalized G - ψ - ϕ - f -proximal cyclic weak contractive mapping if for $x, u, u^* \in A, v, y \in B$

$$\left. \begin{aligned} d_G(u, Tu^*) &= d_G(A, B) \\ d_G(u^*, Tx) &= d_G(A, B) \\ d_G(v, Ty) &= d_G(A, B) \end{aligned} \right\} \implies \psi(G(u^*, u, v)) \leq f(\psi(M(x, v, y)), \phi(M(x, v, y))) \quad (21)$$

holds where $\psi \in \Psi$, $\phi \in \Phi$ and $M(x, v, y) = \max\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$.

Theorem 2.13 Let A, B be two nonempty subsets of a G -metric space (X, G) such that (A, G) , (B, G) are complete G -metric spaces, A_0 is nonempty and B is approximatively compact with respect to A . Assume that $T : A \cup B \rightarrow A \cup B$ is a G - ψ - ϕ - f -proximal cyclic weak contractive mapping such that $T(A) \subset B$, $T(B) \subset A$ and $T(A_0) \subset B_0$. Then T has a best proximity point.

Proof. If x_0 in A_0 , then $x_1 = Tx_0 \in T(A_0) \subseteq B$, so $d_G(x_0, Tx_0) = d_G(x_0, x_1) = d_G(A, B)$. Further, since $x_2 = Tx_1 \in T(B_0) \subseteq A$, it follows that $d_G(x_1, Tx_1) = d_G(x_1, x_2) = d_G(A, B)$. Recursively, we obtain a sequence $\{x_n\}$ in $A \cup B$ satisfying

$$d_G(x_n, x_{n+1}) = d_G(A, B) \text{ for all } n \in \mathbb{N} \cup \{0\} \quad (22)$$

This implies that $\begin{cases} d_G(u^*, Tx) = d_G(A, B), \\ d_G(u, Tu^*) = d_G(A, B), \\ d_G(v, Ty) = d_G(A, B) \end{cases}$ where $x = x_{n-1}$, $u = x_{n+1}$, $u^* = x_{n+1}$ and $y = x_n$, $v = x_n$. Therefore from (21) we have,

$$\begin{aligned} \psi(G(x_{n+1}, x_{n+1}, x_n)) &\leq f(\psi(M(x_{n-1}, x_n, x_n)), \phi(M(x_{n-1}, x_n, x_n))) \\ &\leq \psi(M(x_{n-1}, x_n, x_n)) \end{aligned}$$

where

$$\begin{aligned} M(x_{n-1}, x_n, x_n) &= \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n)\} \\ &= \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} \\ &= \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}. \end{aligned}$$

If

$$M(x_{n-1}, x_n, x_n) = G(x_n, x_{n+1}, x_{n+1})$$

then we have that

$$\psi(G(x_n, x_{n+1}, x_{n+1})) = \psi(G(x_{n+1}, x_{n+1}, x_n)) \leq f(\psi(G(x_n, x_{n+1}, x_{n+1})), \phi(G(x_n, x_{n+1}, x_{n+1})))$$

therefore $\psi(G(x_n, x_{n+1}, x_{n+1})) = 0$ or $\phi(G(x_n, x_{n+1}, x_{n+1})) = 0$. Thus we obtain that $G(x_n, x_{n+1}, x_{n+1}) = 0$ therefore $x_n = x_{n+1}$ and this implies that each x_n is fixed point, that is a contradiction.

Hence we have that

$$M(x_{n-1}, x_n, x_n) = G(x_{n-1}, x_n, x_n)$$

so

$$\psi(G(x_n, x_{n+1}, x_{n+1})) \leq f(\psi(G(x_{n-1}, x_n, x_n)), \phi(G(x_{n-1}, x_n, x_n))) \leq \psi(G(x_{n-1}, x_n, x_n))$$

which implies $G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n)$. So the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is decreasing sequence in \mathbb{R}^+ and thus it is convergent to $t \in \mathbb{R}^+$. We claim that $t = 0$. Suppose, to the contrary, that $t > 0$. Taking limit as $n \rightarrow \infty$ in above we get,

$$\psi(t) \leq f(\psi(t), \phi(t))$$

which implies $\psi(t) = 0$ or $\phi(t) = 0$. That is, $t = 0$ which is a contrary. Hence, $t = 0$. i.e.,

$$\lim_{n \rightarrow \infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \quad (23)$$

We shall show that $\{x_n\}_{n=0}^{\infty}$ is a G -Cauchy sequence. Suppose, to the contrary, that there exists $\varepsilon > 0$, and a sequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geq \varepsilon \quad (24)$$

with $n(k) \geq m(k) > k$. Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) > k$ and satisfying (24). Hence,

$$G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) < \varepsilon \quad (25)$$

By Proposition 1.5 (iii) and (G5) we have

$$\begin{aligned} \varepsilon &\leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ &\leq G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) \\ &< \varepsilon + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) \end{aligned} \quad (26)$$

Letting $k \rightarrow \infty$ in (26) we derive that

$$\lim_{k \rightarrow \infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \quad (27)$$

Observe that for every $k \in N$; there exist $s(k)$ satisfying $0 \leq s(k) \leq m$ such that

$$n(k) - m(k) + s(k) \equiv 1 \pmod{m}. \quad (28)$$

Therefore for large enough values of k we have $r(k) = m(k) - s(k) > 0$ and $x_{r(k)}$ and $x_{n(k)}$ lie in the set A and B respectively.

Next using (21) with $x = x_{r(k)}$, $u = x_{n(k)+1}$, $u^* = x_{r(k)}$, $y = x_{n(k)}$ and $v = x_{n(k)}$

$$\psi(G(x_{r(k)}, x_{n(k)+1}, x_{n(k)})) \leq f(\psi(M(x_{r(k)}, x_{n(k)}, x_{n(k)})), \phi(M(x_{r(k)}, x_{n(k)}, x_{n(k)}))) \leq \psi(M(x_{r(k)}, x_{n(k)}, x_{n(k)})) \tag{29}$$

where

$$\begin{aligned} M(x_{r(k)}, x_{n(k)}, x_{n(k)}) &= \max\{G(x_{r(k)}, x_{n(k)}, x_{n(k)}), G(x_{r(k)}, Tx_{r(k)}, Tx_{r(k)}), G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)})\} \\ &= \max\{G(x_{r(k)}, x_{n(k)}, x_{n(k)}), G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}), G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1})\}. \end{aligned}$$

Employing rectangle inequality repeatedly, we obtain

$$\begin{aligned} G(x_{r(k)}, x_{n(k)}, x_{n(k)}) &\leq G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}) + G(x_{r(k)+1}, x_{n(k)}, x_{n(k)}) \\ &\leq G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}) + G(x_{r(k)+1}, x_{r(k)+2}, x_{r(k)+2}) + G(x_{r(k)+2}, x_{n(k)}, x_{n(k)}) \\ &\leq \left[\sum_{i=r}^{m-1} G(x_{i(k)}, x_{i(k)+1}, x_{i(k)+1}) \right] + G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \end{aligned}$$

or

$$G(x_{r(k)}, x_{n(k)}, x_{n(k)}) - G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq \sum_{i=r}^{m-1} G(x_{i(k)}, x_{i(k)+1}, x_{i(k)+1})$$

On letting $k \rightarrow \infty$ and using (23), (29) we deduce that

$$\lim_{k \rightarrow \infty} G(x_{r(k)}, x_{n(k)}, x_{n(k)}) = \lim_{k \rightarrow \infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{30}$$

Using rectangle inequality again, we have

$$G(x_{r(k)}, x_{n(k)+1}, x_{n(k)}) \leq G(x_{r(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)+1}, x_{n(k)}).$$

On letting $k \rightarrow \infty$ and using (30), we deduce that

$$\lim_{k \rightarrow \infty} G(x_{r(k)}, x_{n(k)+1}, x_{n(k)}) = \varepsilon. \tag{31}$$

Now passing to limit as $k \rightarrow \infty$ in (29) and using (23), (30), (31), we get

$$\psi(\varepsilon) \leq f(\psi(\max\{\varepsilon, 0, 0\}), \phi(\max\{\varepsilon, 0, 0\})) = f(\psi(\varepsilon), \phi(\varepsilon))$$

and hence $\psi(\varepsilon) = 0$ or $\phi(\varepsilon) = 0$, therefore $\varepsilon = 0$ which contradicts the assumption that $\{x_n\}$ is not G-Cauchy. Thus $\{x_n\}$ is a Cauchy sequence.

Since A and B is complete there exist $z \in A \subset A \cup B$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$. On the other hand for each $n \in N$, we have

$$\begin{aligned} d_G(z, B) &\leq d_G(z, Tx_n) = d_G(z, x_{n+1}) \leq d_G(z, x_n) + d_G(x_n, x_{n+1}) \\ &\leq d_G(z, x_n) + d_G(A, B) \leq d_G(z, x_n) + d_G(z, B). \end{aligned}$$

Taking limit as $n \rightarrow \infty$ in above inequality, we get

$$d_G(z, B) \leq \lim_{n \rightarrow \infty} d_G(z, Tx_n) = d_G(z, B) = d_G(A, B).$$

Since, B is approximatively compact with respect to A , so the sequence, $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ that converges to some $y^* \in B \subset A \cup B$.

Hence

$$d_G(z, y^*) = \lim_{n \rightarrow \infty} d_G(x_{n_k}, Tx_{n_k}) = d_G(A, B)$$

and so $z \in A_0$. Now, since, $Tz \in T(A_0) \subseteq B_0$, there exists $w \in A_0$ such that $d_G(w, Tz) = d_G(A, B)$.

From (21) with $x = x_{n-1}$, $u = w$, $u^* = z$, $y = x_n$ and $v = x_n$ we have,

$$\begin{aligned} \psi(G(z, w, x_n)) &\leq f(\psi(M(x_{n-1}, x_n, x_n)), \phi(M(x_{n-1}, x_n, x_n))) = \\ &f(\psi(\max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n)\}), \\ &\phi(\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\})) \end{aligned}$$

Taking limit as $n \rightarrow \infty$ we get,

$$\psi(G(z, w, z)) \leq f(\psi(0), \phi(0)) \leq \psi(0) = 0.$$

Then $G(z, z, w) = 0$. i.e., $w = z$. Thus $d_G(z, Tz) = d_G(A, B)$. Therefore T has a best proximity point. ■

Example 2.14 Let $X = [0, \infty)$ and $G(x, y, z) = \frac{1}{4}(|x-y| + |y-z| + |x-z|)$ be a G -metric on X . Then $d_G(x, y) = |x - y|$.

Let $A = \{0, 4, 8\}$ and $B = \{2, 6, 10\}$. Define $T : A \cup B \rightarrow A \cup B$ by

$$T(x) = 0, \text{ if } x = 10$$

$$x + 2, \text{ otherwise.}$$

Also define $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ by $\psi(t) = t, \phi(t) = a^{\frac{t}{2}}$ and, $f : [0, \infty)^2 \rightarrow R$, $f(s, t) = \log_a \frac{t+a^s}{1+t}$.

Clearly, $d_G(A, B) = 2$, and $TA \subset B, TB \subset A$, then T is a generalized G - ψ - ϕ - f -proximal cyclic weak contraction for $u = u^* = 4, x = 0 \in A$, and $v = 2, y = 10 \in B$, we have

$$\psi(G(u^*, u, v)) = G(u^*, u, v) = 1,$$

and

$$\begin{aligned} M(x, v, y) &= \max\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\} \\ &= \max\{G(0, 2, 10), G(0, 2, 2), G(10, 0, 0)\} = 5 \end{aligned}$$

so

$$\begin{aligned} \psi(G(u^*, u, v)) &= 1 \leq \frac{1}{2}M(x, v, y) = \log_a \frac{a^{\frac{1}{2}M(x, v, y)} + a^{M(x, v, y)}}{1 + a^{\frac{1}{2}M(x, v, y)}} \\ &= \log_a \frac{\phi(M(x, v, y)) + a^{\psi(M(x, v, y))}}{1 + \phi(M(x, v, y))}. \end{aligned}$$

Thus

$$\begin{aligned} d_G(u^*, Tx) &= d_G(A, B) \\ d_G(u, Tu^*) &= d_G(A, B) \\ d_G(v, Ty) &= d_G(A, B) \\ &\Downarrow \\ \psi(G(u, u^*, v)) &\leq \log_a \frac{\phi(M(x, u, y)) + a^{\psi(M(x, u, y))}}{1 + \phi(M(x, u, y))}. \end{aligned}$$

Hence T is a generalized G - ψ - ϕ - f -proximal cyclic weak cyclic contractive mapping. All conditions of Theorem 2.13 hold true and T has a best proximity point. Here, $z = 0$ is best proximity point of T .

If in Theorem 2.13, we take $f(s, t) = s - t$, we obtain following corollary.

Corollary 2.15 Let A, B be two nonempty subsets of a G -metric space (X, G) such that $(A, G), (B, G)$ are complete G -metric space, A is nonempty and B is approximatively compact with respect to A . Assume that $T : A \cup B \rightarrow A \cup B$ is a such that $T(A) \subset B, T(B) \subset A$ and $T(A_0) \subset B_0$. If for $x, u, u^* \in A, v, y \in B$

$$\begin{aligned} d_G(u^*, Tx) &= d_G(A, B) \\ d_G(u, Tu^*) &= d_G(A, B) \\ d_G(v, Ty) &= d_G(A, B) \end{aligned}$$

\Downarrow

$$\psi(G(u^*, u, v)) \leq \psi(M(x, v, y)) - \phi(M(x, v, y))$$

holds where $\psi \in \Psi$ and $\phi \in \Phi, M(x, v, y) = \max\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$

Then T has a unique best proximity point.

If in Theorem 2.13, we take $f(s, t) = \frac{s}{(1+t)^r}, r \in (0, \infty)$, we obtain following corollary.

Corollary 2.16 Let A, B be two nonempty subsets of a G -metric space (X, G) such that $(A, G), (B, G)$ are complete G -metric spaces, A is nonempty and B is approximatively compact with respect to A . Assume that $T : A \cup B \rightarrow A \cup B$ is a such that $T(A) \subset B, T(B) \subset A$ and $T(A_0) \subset B_0$. If for $x, u, u^* \in A, v, y \in B$

$$d_G(u^*, Tx) = d_G(A, B)$$

$$d_G(u, Tu^*) = d_G(A, B)$$

$$d_G(v, Ty) = d_G(A, B)$$

↓

$$\psi(G(u^*, u, v)) \leq \frac{\psi(M(x, v, y))}{(1 + \phi(M(x, v, y)))^r}, r \in (0, \infty)$$

holds where $\psi \in \Psi$ and $\phi \in \Phi$, $M(x, v, y) = \max\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$. Then T has a best proximity point.

If in Theorem 2.13, we take $f(s, t) = s \log_{tr+a} a$, we obtain following corollary.

Corollary 2.17 Let A, B be two nonempty subsets of a G -metric space (X, G) such that (A, G) , (B, G) are a complete G -metric spaces, A is nonempty and B is approximatively compact with respect to A . Assume that $T : A \cup B \rightarrow A \cup B$ is a such that $T(A) \subset B$, $T(B) \subset A$ and $T(A_0) \subset B_0$. If for $x, u, u^* \in A, v, y \in B$

$$d_G(u^*, Tx) = d_G(A, B)$$

$$d_G(u, Tu^*) = d_G(A, B)$$

$$d_G(v, Ty) = d_G(A, B)$$

↓

$$\psi(G(u^*, u, v)) \leq \psi(M(x, v, y)) \log_{\phi(M(x, v, y))^{r+a}} a, a > 1, r > 0$$

holds where $\psi \in \Psi$ and $\phi \in \Phi$, $M(x, v, y) = \max\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$. Then T has a best proximity point.

Remark 2 Several more best proximity results can be obtained from Theorems 2.5 and 2.13 using the other functions f mentioned in Example 2.3, and/or some other concrete choices of $\psi \in \Psi$ and $\phi \in \Phi$.

3. Application to fixed point theory

In this section, as an application of our best proximity results we here derive certain new fixed point results.

Note that if,
$$\left. \begin{array}{l} d_G(u, Tx) = d_G(A, B) \\ d_G(u^*, Tu) = d_G(A, B) \\ d_G(v, Ty) = d_G(A, B) \end{array} \right\} \text{ and } A = B = X, \text{ then } u = Tx, u^* = Tu \text{ and}$$

$v = Ty$. That is, $u^* = T^2x$. Therefore, if in Theorem 2.5 we take $A = B = X$, we deduce the following result.

Theorem 3.1 Let (X, G) be a complete G -metric space and $T : X \rightarrow X$ be a mapping satisfying the following condition, for all $x, y \in X$ where $\psi \in \Psi$ and $\phi \in \Phi$,

$$\psi(G(Tx, T^2x, Ty)) \leq f(\psi(G(x, Tx, y)), \phi(G(x, Tx, y))).$$

Then T has a unique fixed point.

If in Theorem 3.1, we take $f(s, t) = s - t$, then we obtain following fixed point result.

Theorem 3.2 (Theorem 2.3 of [2]) Let (X, G) be a complete G -metric space and $T : X \rightarrow X$ be a mapping satisfying the following condition, for all $x, y \in X$ where $\psi \in \Psi$ and $\phi \in \Phi$,

$$\psi(G(Tx, T^2x, Ty)) \leq \psi(G(x, Tx, y)) - \phi(G(x, Tx, y)).$$

Then T has a unique fixed point.

If in Theorem 3.1 we take $f(s, t) = \frac{s}{1+t}$, we obtain following corollary.

Corollary 3.3 Let (X, G) be a complete G -metric space and $T : X \rightarrow X$ be a mapping satisfying the following condition, for all $x, y \in X$ where $\psi \in \Psi$ and $\phi \in \Phi$,

$$\psi(G(Tx, T^2x, Ty)) \leq \frac{\psi(G(x, Tx, y))}{1 + \phi(G(x, Tx, y))}.$$

Then T has a unique fixed point.

Remark 3 Several more fixed point results can be obtained from Theorems 2.13 and 3.1 using other functions f mentioned in Example 2.3, and/or some other concrete choices of $\psi \in \Psi$ and $\phi \in \Phi$.

References

- [1] A. Amini-Harandi, N. Hussain, F. Akbar, Best proximity point results for generalized contractions in metric spaces, *Fixed Point Theory and Applications*. (2013) 2013:164.
- [2] M. Asadi, E. Karapinar, P. Salimi, A new approach to G -metric and related fixed point theorems, *Journal of Inequalities and Applications*. (2013), 2013:454.
- [3] H. Aydi, E. Karapinar, P. Salimi, Some fixed point results in GP-metric spaces, *J. Appl. Math.* (2012), Article ID 891713 .
- [4] B. S. Choudhury, P. Maity, Best proximity point results in generalized metric spaces, *Vietnam J. Mathematics*, 10.1007/s10013-015-0141-3.
- [5] C. Di Bari, T. Suzuki, C. Vetro, Best proximity points for cyclic Meir-Keeler contractions, *Nonlinear Analysis*. 69 (11) (2008), 3790-3794.
- [6] K. Fan, Extensions of two fixed point theorems of F.E. Browder, *Mathematische Zeitschrift*. 112 (3) (1969), 234-240.
- [7] N. Hussain, E. Karapinar, P. Salimi, P. Vetro, Fixed point results for G^m -Meir-Keeler contractive and G - (α, ψ) -Meir-Keeler contractive mappings, *Fixed Point Theory and Applications*. (2013), 2013:34.
- [8] N. Hussain, M. A. Kutbi, P. Salimi, Best proximity point results for modified α - ψ -proximal rational contractions, *Abstract and Applied Analysis*. (2013), Article ID 927457, 14 pages.
- [9] N. Hussain, A. Latif, P. Salimi, Best proximity point results in G -metric spaces, *Abstract and Applied Analysis*. (2014), Article ID 837943, 8 pages.
- [10] N. Hussain, A. Latif, P. Salimi, Best proximity point results for modified Suzuki α - ψ -proximal contractions, *Fixed Point Theory and Applications*. (2014), 2014:10.
- [11] M. Jleli, B. Samet, Remarks on G -metric spaces and fixed point theorems, *Fixed Point Theory Appl.* (2012), 2012:210.
- [12] F. Moradlou, P. Salimi, P. Vetro, Some new extensions of Edelstein-Suzuki-type fixed point theorem to G -metric and G -cone metric spaces, *Acta Mathematica Scientia*. 33 (4) (2013), 1049-1058.
- [13] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, *J. Nonlinear Convex Anal.* 7 (2006), 289-297.
- [14] Z. Mustafa, B. Sims, Fixed point theorems for contractive mappings in complete G -metric spaces, *Fixed Point Theory Appl.* (2009), Article ID 917175, 10 pages.
- [15] Z. Mustafa, V. Parvaneh, M. Abbas, J.R. Roshan, Some coincidence point results for generalized (ψ, ϕ) -weakly contractive mappings in ordered G -metric spaces, *Fixed Point Theory and Applications*. (2013), 2013:326.
- [16] Z. Mustafa, A new structure for generalized metric spaces with applications to fixed point theory, Ph.D. Thesis, The University of Newcastle, Australia, 2005.
- [17] S. Radenović, P. Salimi, S. Pantelić, J. Vujaković, A note on some tripled coincidence point results in G -metric spaces. *Int. J. Math. Sci. Eng. Appl.* 6 (2012), 23-38.
- [18] A. Razani, V. Parvaneh, On generalized weakly G -contractive mappings in partially ordered G -metric spaces, *Abstr. Appl. Anal.* (2012), Article ID 701910, 18 pages.
- [19] S. Sadiq Basha, Extensions of Banach's contraction principle. *Numer. Funct. Anal. Optim.* 31 (2010), 569-576.
- [20] P. Salimi, P. Vetro, A result of Suzuki type in partial G -metric spaces, *Acta Mathematica Scientia*. 34B (2) (2014), 1-11.
- [21] B. Samet, C. Vetro, F. Vetro, Remarks on G -metric spaces, *Int. J. Anal.* (2013), Article ID 917158, 6 pages.
- [22] T. Suzuki, M. Kikkawa, C. Vetro, The existence of best proximity points in metric spaces with the property UC, *Nonlinear Anal. Theory, Methods & Applications*. 71 (7-8) (2009), 2918-2926.