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Stability and hyperstability of orthogonally ring $*$-n-derivations and orthogonally ring $*$-n-homomorphisms on C^*-algebras

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Abstract. In this paper, we investigate the generalized Hyers-Ulam-Rassias and the Isac and Rassias-type stability of the conditional of orthogonally ring $*$-n-derivation and orthogonally ring $*$-n-homomorphism on C^*-algebras. As a consequence of this, we prove the hyperstability of orthogonally ring $*$-n-derivation and orthogonally ring $*$-n-homomorphism on C^*-algebras.

Keywords: Stability and hyperstability, ring $*$-n-derivation, ring $*$-n-homomorphism, C^*-algebras

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1. Introduction

The stability problem of functional equations had been first raised by Ulam [27]. In 1941, Hyers [12] gave a first affirmative answer to the question of Ulam for Banach spaces. Hyers Theorem was generalized by Rassias [23] for linear mapping by considering an unbounded Cauchy difference. For more details about the result concerning such problems, the reader to ([8–10, 13–16, 19–22, 25]). We assume X and Y are two algebras over the real or complex filed F. An additive mapping $d : X \to X$ is said to be a ring

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n-derivation if the functional equation

$$d(x_1x_2...x_n) = d(x_1)x_2...x_n + x_1d(x_2)x_3...x_n + ... + x_1...x_{n-1}d(x_n) \tag{1}$$

for all $x_1, x_2, ..., x_n \in X$. An additive mapping $h : X \to Y$ is said to be a ring homomorphism if the functional equation $h(xy) = h(x)h(y)$ for all $x, y \in X$. In addition, h is called a ring n-homomorphism if the functional equation $h(x_1x_2...x_n) = h(x_1)h(x_2)...h(x_n)$ is valid for all $x_1, x_2, ...x_n \in X$.

Suppose that X is a real vector space (or an algebra) with $\dim X \geqslant 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;

(O_2) independence: if $x, y \in X - \{0\}$, $x \perp y$, then x, y are linearly independent;

(O_3) homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O_4) the Thalesian property: if P is a $2-$dimensional subspace (subalgebra) of X, $x \in P$ and $\lambda \in R_+$, then there exists $u_x \in P$ such that $x \perp u_x$ and $x + u_x \perp \lambda x - u_x$.

The pair $(X, \perp)$ is called an orthogonality space (algebra). By an orthogonality normed space (normed algebra) we mean an orthogonality space (algebra) having a normed structure. The orthogonal Cauchy functional equation

$$f(x + y) = f(x) + f(y), \;\; x \perp y, \tag{2}$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by Gudder and Strawther [11]. Let $(A, \perp)$ be an orthogonality normed algebra and B be an A-module. A mapping $d : A \longrightarrow B$ is an orthogonally ring derivation if d is an orthogonally additive mapping satisfying

$$d(xy) = xd(y) + d(x)y \tag{3}$$

for all $x, y \in A$ with $x \perp y$.

Let $f : \mathbb{R} \to \mathbb{R}$ and $U \in R^2$. Then we call f an orthogonally U-additive function provided that f satisfies equation (2) for all $(x, y) \in U$. In this paper, we are interested in a set U such that every orthogonally U-additive function f is an orthogonally additive function. Recently, Skof [26] consider the Hyers-Ulam stability problem [27] of a conditional Cauchy functional inequality. In particular, the result can be stated as follows: If $f : \mathbb{R} \to \mathbb{R}$ satisfies the conditional Cauchy functional inequality

$$\|f(x + y) - f(x) - f(y)\| \leqslant \epsilon \tag{4}$$

for all $x, y \in \mathbb{R}$ with $|x| + |y| \leqslant d$, then f satisfies inequality (4) for all $x, y \in \mathbb{R}$. In this paper, for a given δ we fined a set $U_\delta \in X^2$ satisfying $m(U_\delta) \leqslant \delta$ such that if f satisfies (4) for all $(x, y) \in U_\delta$, then f satisfies (4) for all $(x, y) \in X$ with $\varphi(x, y)$, replaced by $3\varphi(x, y)$ and that there exists a unique additive function $A : X \to X$ satisfying

$$\|f(x) - A(x)\| \leqslant 3\varphi(x, y) \tag{5}$$

for all $x \in X$.

Let X be a normed orthogonal algebra space with countable dense subset E and Y Banach X-module space. For $j = 1, 2, 3, ...,$ we denote by $B_j = \{(x, y) \in X^2 : \|x - x_j\| < 1, \|y - y_j\| < 2^{-j}\}$ the rectangle with center (x_j, y_j). let $U = \bigcup_{j=0}^{\infty} B_j$ and $E \times E :=$

$\{(x_1, y_1), (x_2, y_2), (x_3, y_3), ...\}$. It is easy to see that the Lebegues measure $m(U)$ of U satisfies $m(U) \leqslant 1$. Now for $d > 0$. Let

$$U_d := U \bigcap \{(x, y) \in X^2 : \|x\| + \|y\| > d, x \perp y\}.$$

Then for a given $\delta > 0$, we can choose $d > 0$ such that $m(U) \leqslant \delta$. We first consider that stability of functional inequality (4) in the restricted domain U_d (see [1]-[7],[17]-[18],[24]).

The outline of the paper is as follows: In Sec. 2 we prove stability of orthogonally ring $*$-n-derivation and orthogonally ring $*$-n-homomorphism in C^*-algebra for the functional equation additive. In Sec. 3 we establish the hyperstability of these functional equation additive by suitable control functions.

2. Stability

Throughout this section, assume that A is a C^*-algebra with norm $\|.\|_A$ and that B is a C^*-algebra with norm $\|.\|_B$. For a given mapping $f : A \to A$, we define

$$\Delta f(z_1, z_2, ..., z_n) := f(z_1 z_2 ... z_n) - f(z_1) z_2 z_3 ... z_n - z_1 f(z_2) z_3 ... z_n - ... z_1 z_2 ... z_{n-1} f(z_n) \quad (6)$$

for all $z_i \in A$, $1 \leqslant i \leqslant n$ that are mutually orthogonal.
We prove the generalized Hyers-Ulam stability of orthogonally ring $*$-n-derivation in C^*-algebra for the functional equation additive.

Theorem 2.1 Suppose that $f : A \to A$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : A^{n+2} \to [0, \infty)$ such that

$$\|f(x + y) - f(x) - f(y) + \Delta f(z_1, z_2, ..., z_n)\|_A \leqslant \varphi(x, y, z_1, z_2, ..., z_n) \quad (7)$$

and

$$\|f(x^*) - f(x)^*\|_A \leqslant \varphi(x, x, 0, ..., 0) \quad (8)$$

for all $(x, y) \in U_d$ and $z_i \in A$, $1 \leqslant i \leqslant n$ that are mutually orthogonal. Suppose the function φ satisfying

$$\varphi(x - p - t, p + t) + \varphi(x - p - t, y + p + t) + \varphi(-p - t, y + p + t) \leqslant 3\varphi(x, y, 0, ..., 0) \quad (9)$$

for all $(x - p - t, p + t), (x - p - t, y + p + t), (-p - t, y + p + t) \in U_d$ and

$$\psi(x) = 3 \sum_{k=0}^{\infty} 2^{-k-1} \varphi(2^n x, 2^n x, 0, ..., 0) < \infty \quad (10)$$

and

$$\lim_{n\to\infty} 2^{-n} \varphi(2^n x, 2^n y, 2^n z_1, 2^n z_2, ..., 2^n z_n) = 0 \quad (11)$$

for all $(x,y) \in U_d$ and $z_i \in A$, $1 \leqslant i \leqslant n$ that are mutually orthogonal, then there exists a unique orthogonally ring $*$-n-derivation $D : A \to A$ such that

$$\|f(x) - D(x)\|_A \leqslant \psi(x). \tag{12}$$

for all $x \in A$.

Proof. For given $x, y \in A$ we choose $p \in A$ such that

$$\|p\|_A \leqslant d + \|x\|_A + \|y\|_A + 1. \tag{13}$$

We first choose $(x_{i_1}, y_{i_1}) \in E^2$ such that

$$\| -p - x_{i_1}\|_A + \|p - y_{i_1}\|_A \leqslant \frac{1}{4}, \tag{14}$$

and then we choose $(x_{i_2}, y_{i_2}) \in E^2$, $(x_{i_3}, y_{i_3}) \in E^2$ and $(x_{i_4}, y_{i_4}) \in E^2$ with $1 < i_1 < i_2 < i_3 < i_4$, step by step, satisfying

$$\|x - y_{i_1} - x_{i_2}\|_A + \|y_{i_1} - y_{i_2}\|_A < 2^{-i_1-1}, \tag{15}$$

$$\|x - y_{i_2} - x_{i_3}\|_A + \|y + y_{i_2} - y_{i_3}\|_A < 2^{-i_2-1}, \tag{16}$$

$$\|y - y_{i_3} - x_{i_4}\|_A + \|y_{i_3} - y_{i_4}\|_A < 2^{-i_3-1}, \tag{17}$$

Let

$$t_1 = y_{i_1} - p, \qquad t_2 = y_{i_2} - y_{i_1},$$
$$t_3 = y_{i_3} - y_{i_2} - y, \quad t_4 = y_{i_4} - y_{i_3}$$

and $t = t_1 + t_2 + t_3 + t_4$. Then from (14)-(17) we have

$$\|t_1\|_A < \frac{1}{4},\ \ \|t_2\|_A < 2^{-i_1-1},\ \ \|t_3\|_A < 2^{-i_2-1},\ \ \|t_4\|_A < 2^{-i_3-1},\ \ \|t\|_A < \frac{1}{2}. \tag{18}$$

Thus, from (13), (14) and (18) we get

$$\| -p - t\|_A + \|p + t\|_A \geqslant 2(\|p\|_A - \|t\|_A) \geqslant 2(\|p\|_A - \frac{1}{2}) > 2d \geqslant d \tag{19}$$

and

$$\| -p - t - x_{i_1}\|_A \leqslant \|p - x_{i_1}\|_A + \|t\|_A < \frac{1}{4} + \frac{1}{2} < 1 \tag{20}$$

and

$$\|p + t - y_{i_1}\|_A = \|t_2 + t_3 + t_4\|_A < 2^{-i_1-1} + 2^{-i_2-1} + 2^{-i_3-1} < 2^{-i_1}. \tag{21}$$

Inequalities (19)-(21) imply that

$$(-p - t, p + t) \in U_d. \tag{22}$$

Also from the inequalities

$$\|x-p-t\|_A+\|p+t\|_A \geqslant 2(\|p\|_A-\|x\|_A-\|t\|_A)$$
$$> 2(\|p\|-\|x\|-\frac{1}{2}) > d,$$

and

$$\|x-p-t-x_{i_2}\|_A \leqslant \|x-y_{i_1}-x_{i_2}\|_A+\|t_2\|_A+\|t_3\|_A+ \ |t_4\|_A$$
$$< \frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32} < \frac{1}{2}$$

and

$$\|p+t-y_{i_2}\|_A = \|t_3+t_4\|_A < 2^{-i_2-1}+2^{-i_3-1} < 2^{-i_2},$$

we have

$$(x-p-t, p+t) \in U_d. \tag{23}$$

Similarly, using the followings

$$\|x-p-t-x_{i_3}\|_A \leqslant \|x-y_{i_2}-x_{i_3}\|_A+\|t_3\|_A+\|t_4\|_A < 1,$$
$$\|y+p+t-y_{i_3}\|_A = \|t_4\|_A < 2^{-i_3},$$
$$\|-p-t-x_{i_4}\|_A \leqslant \|y-y_{i_3}-x_{i_4}\|_A+\|t_4\|_A < 1,$$
$$\|y+p+t-y_{i_4}\|_A = 0,$$

we have

$$(x-p-t, y+p+t), (-p-t, y+p+t) \in U_d. \tag{24}$$

Now, form (23), (24), (9) and putting $z_i = 0$ for all $1 \leqslant i \leqslant n$ in (7) we have

$$\begin{aligned}\|f(x+y)-f(x)-f(y)\| \leqslant & \|-f(x)+f(x-p-t)+f(p+t)\|_A \\ & + \|f(x+y)-f(x-p-t)-f(y+p+t)\|_A \\ & + \|-f(y)+f(-p-t)+f(y+p+t)\|_A \\ \leqslant & 3\varphi(x, y, 0, ..., 0)\end{aligned} \tag{25}$$

for all $x, y \in A$. Setting $x = y$ in (25) we get

$$\|\frac{1}{2}f(2x)-f(x)\|_A \leqslant \frac{3}{2}\varphi(x, x, ..., 0) \tag{26}$$

for all $x \in A$. By induction, we can show that

$$\|2^{-n}f(2^n x)-f(x)\|_A \leqslant 3\sum_{k=0}^{n-1} 2^{-k-1}(2^k x, 2^k x, 0, ..., 0) \tag{27}$$

for all x in A. Replacing x by $2^m x$ in (27), we get

$$\|f\frac{(2^{n+m}x)}{2^{m+n}} - \frac{f(2^m x)}{2^m}\|_A \leqslant 3\sum_{k=0}^{n+m-1} 2^{-k-1}\varphi(2^k x, 2^k x, 0, ..., 0) \tag{28}$$

for all $n, m \in \mathbb{N}$ and $x \in A$. Hence, $\{2^{-n}f(2^n x)\}$ is a cauchy sequence in complete space A. Now, let D defined by

$$D(x) := \lim_{n\to\infty} 2^{-n} f(2^n x). \tag{29}$$

Taking the limit in (27) as $n \to \infty$, we obtain the inequality

$$\|D(x) - f(x)\|_A \leqslant \psi(x)$$

for all $x \in A$. It follow from (7), (11) and (29) that

$$\begin{aligned}\|D(x+y) - D(x) - D(y)\|_A &= \lim_{n\to\infty} 2^{-n}\|f(2^n(x+y)) - f(2^n x) - f(2^n y)\|_A \\ &\leqslant \lim_{n\to\infty} 2^{-n}\varphi(2^n x, 2^n y, 0, ..., 0) = 0.\end{aligned}$$

Also,

$$\begin{aligned}\|\Delta D(z_1, z_2, ..., z_n)\|_A &= \lim_{n\to\infty} 2^{-n^2}\|\Delta f(z_1, z_2, ..., z_n)\|_A \\ &\leqslant \lim_{n\to\infty} 2^{-n^2}\varphi(0, 0, 2^n z_1, ..., 2^n z_n) \\ &\leqslant \lim_{n\to\infty} 2^{-n}\varphi(0, 0, 2^n z_1, ..., 2^n z_n) = 0.\end{aligned}$$

It follows from (8), (11) and (29) that

$$\begin{aligned}\|D(x^*) - D(x)^*\|_A &= \lim_{n\to\infty} 2^{-n}\|f(2^n x^*) - f(2^n x)^*\|_A \\ &\leqslant \lim_{n\to\infty} 2^{-n}\varphi(2^n x, 2^n x, 0, ..., 0) = 0.\end{aligned}$$

Now, let $D' : A \to A$ by another orthogonally ring $*$-n-derivation satisfying $\|D'(x) - f(x)\| \leqslant \psi(x)$ for all x in A. Then, we get

$$\begin{aligned}\|D(x) - D'(x)\|_A &= \lim_{n\to\infty} 2^{-n}\|D(2^n x) - D'(2^n x)\|_A \\ &\leqslant \lim_{n\to\infty} 2^{-n}(3\sum_{k=0}^{\infty} 2^{-k}\varphi(2^{k+m}x, 2^{k+m}x, 0, ..., 0)) \\ &\leqslant \lim_{n\to\infty} 2^{-k}\varphi(2^k x, 2^k x, 0, ..., 0) = 0.\end{aligned}$$

Therefor $D'(x) = D(x)$ for all $x \in A$. ■

Corollary 2.2 Let $\phi : [0, \infty) \to [0, \infty)$ be a function satisfying the following condition:
i) $\lim_{r\to\infty} \frac{\phi(r)}{r} = 0$,

ii) $\phi(rs) < \phi(r)\phi(s)$ for all $r, s \in [0, \infty)$,
iii) $\phi(r) < r$ for all $r > 1$.
If function $f : A \to A$ with $f(0) = 0$ and satisfying the inequalities

$$\|f(x+y) - f(x) - f(y) + \Delta f(z_1, z_2, ..., z_n)\|_A \leqslant \theta(\phi(\|x\|_A) + \phi(\|y\|_A) + \phi(\|z_1\|_A) + ... + \phi(\|z_n\|_A)), \tag{30}$$

and

$$\|f(x^*) - f(x)^*\|_A \leqslant 2\theta(\phi(\|x\|_A) \tag{31}$$

for all $\theta \geqslant 0$, for all $(x, y) \in U_d$ and $z_i \in A$ for $1 \leqslant i \leqslant n$ that are mutually orthogonal. Then there exists a unique orthogonally ring $*$-n-derivation function $D : A \to A$ such that

$$\|f(x) - D(x)\|_A \leqslant 3\frac{2\theta}{2 - \phi(2)}\psi(\|x\|_A) \tag{32}$$

for all x in A.

Proof. It follows (7) by setting

$$\varphi(x, y, z_1, z_2, ..., z_n) = \theta(\phi(\|x\|_A) + \phi(\|y\|_A) + \phi(\|z_1\|_A)... + \phi(\|z_n)\|_A))$$

for all $(x, y) \in U_d$, and $z_i \in A$, that are mutually orthogonal. ■

For a given mapping $f : A \to B$, we define

$$\Delta f(z_1, z_2, ..., z_n) := f(z_1 z_2 ... z_n) - f(z_1)f(z_2)...f(z_n)$$

for all $z_i \in A$, $1 \leqslant i \leqslant n$ that are mutually orthogonal. We prove the generalized Hyers-Ulam stability of orthogonally ring $*$-n-homomorphism in C^*-algebra for the functional equations additive.

Theorem 2.3 Suppose that $f : A \to B$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : A^{n+2} \to [0, \infty)$ such that

$$\|f(x+y) - f(x) - f(y) + \Delta f(z_1, z_2, ..., z_n)\|_B \leqslant \varphi(x, y, z_1, z_2, ..., z_n) \tag{33}$$

and

$$\|f(x^*) - f(x)^*\|_B \leqslant \varphi(x, x, 0, ..., 0) \tag{34}$$

for all $(x, y) \in U_d$ and $z_i \in A$, $1 \leqslant i \leqslant n$ that are mutually orthogonal. Suppose a function φ satisfying

$$\varphi(x-p-t, p+t) + \varphi(x-p-t, y+p+t) + \varphi(-p-t, y+p+t) \leqslant 3\varphi(x, y, 0, ..., 0) \tag{35}$$

for all $(x-p-t, p+t), (x-p-t, y+p+t), (-p-t, y+p+t) \in U_d$. If a function φ

satisfying

$$\psi(x) = 3\sum_{k=0}^{\infty} 2^{-k-1}\varphi(2^n x, 2^n x, 0, ..., 0) < \infty, \tag{36}$$

and

$$\lim_{n\to\infty} 2^{-n}\varphi(2^n x, 2^n y, 2^n z_1, 2^n z_2, ..., 2^n z_n) = 0 \tag{37}$$

for all $(x, y) \in U_d$ and $z_i \in A$, $1 \leqslant i \leqslant n$ that are mutually orthogonal, then there exists a unique orthogonally ring $*$-n-homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leqslant \psi(x). \tag{38}$$

for all $x \in A$.

Proof. By the reasoning as that in the proof Theorem 2.1 there exists a unique orthogonally ring $*$-n-homomorphism mapping $H : A \to B$ satisfying (38). The mapping $H : A \to B$ is given by $H(x) := \lim_{n\to\infty} 2^{-n} f(2^n x)$ for all $x \in A$. It follows from (33),

$$\begin{aligned}\|\Delta H(z_1, z_2, ..., z_n)\|_B &= \lim_{n\to\infty} 2^{-n^2}\|\Delta f(z_1, z_2, ..., z_n)\|_B \\ &\leqslant \lim_{n\to\infty} 2^{-n^2}\varphi(0, 0, 2^n z_1, ..., 2^n z_n) \\ &\leqslant \lim_{n\to\infty} 2^{-n}\varphi(0, 0, 2^n z_1, ..., 2^n z_n) = 0\end{aligned}$$

for all $z_i \in A$, $1 \leqslant i \leqslant n$. ■

Corollary 2.4 Let A and B be two C^*-algebras with norm and let $\phi : [0, \infty) \to [0, \infty)$ be a function satisfying the following condition:
i) $\lim_{r\to\infty} \frac{\phi(r)}{r} = 0$,
ii) $\phi(rs) < \phi(r)\phi(s)$ for all $r, s \in [0, \infty)$,
iii) $\phi(r) < r$ for all $r > 1$.
If function $f : A \to B$ with $f(0) = 0$ and satisfying the inequalities

$$\begin{aligned}\|f(x + y) - f(x) - f(y) + \Delta f(z_1, z_2, ..., z_n)\|_B \leqslant \theta(\phi(\|x\|_A) + \phi(\|y\|_A) \\ + \phi(\|z_1\|_A) + ... + \phi(\|z_n\|_A),\end{aligned} \tag{39}$$

and

$$\|f(x^*) - f(x)^*\|_B \leqslant 2\theta(\psi(\|x\|_A)) \tag{40}$$

for all $\theta \geqslant 0$ and for all $(x, y) \in U_d$ and $z_i \in A$, $1 \leqslant i \leqslant n$ that are mutually orthogonal, then there exists a unique orthogonally ring $*$-n-homomorphism function $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leqslant 3\frac{2\theta}{2 - \phi(2)}\psi(\|x\|_A) \tag{41}$$

for all x in A.

Proof. It follows (33) by putting

$$\varphi(x, y, z_1, z_2, ..., z_n) = \theta(\phi(\|x\|_A) + \phi(\|y\|_A) + \phi(\|z_1\|_A) + ... + \phi(\|z_n)\|_A)$$

for all $(x, y) \in U_d$ and $z_i \in A$, that are mutually orthogonal. ■

3. Hyperstability

In this section, assume that A is a C^*-algebra with norm $\|.\|_A$ and that B is a C^*-algebra with norm $\|.\|_B$. Now we are going to establish the hyperstability of the orthogonally ring $*$-n-derivation and orthogonally ring $*$-n-homomorphism in normed C^*-algebras for the functional equation additive.

Theorem 3.1 Let A and B be two normed C^*-algebras and $\varphi : A^{n+2} \to [0, \infty)$ be a function such that

$$\varphi(x, y, 0, ..., 0) = 0 \tag{42}$$

$$\lim_{n\to\infty} 2^{-n}\varphi(2^n x, 2^n y, 2^n z_1, 2^n z_2, ..., 2^n z_n) = 0 \tag{43}$$

for all $(x, y) \in U_d$ and $z_i \in A$, $1 \leqslant i \leqslant n$ that are mutually orthogonal. Suppose $f : A \to A$ is a mapping that

$$\|f(x + y) - f(x) - f(y) + \Delta f(z_1, z_2, ..., z_n)\|_B \leqslant \varphi(x, y, z_1, z_2, ..., z_n) \tag{44}$$

for all$(x, y) \in U_d$ and $z_i \in A$, $1 \leqslant i \leqslant n$ that are mutually orthogonal. Then f is a orthogonally ring $*$-n-derivation or orthogonally ring $*$-n-homomorphism.

Proof. Because $\varphi(x, y, 0, ..., 0) = 0$ for all $(x, y) \in U_d$. Like the proof Theorem 2.3, we have $f(2x) = 2f(x)$ and induction we infer that $f(2^n) = 2^n f(x)$. There for $D(x) = f(x)$ for all $x \in A$. Thus f is a orthogonally ring $*$-n-derivation or orthogonally ring $*$-n-homomorphism between C^*-algebra with norm. The other case is similar. ■

Corollary 3.2 Let θ, p by real number such that $\theta > 0$, $P < \frac{1}{3}$ and X and Y be two normed C^*-algebra. Let $f : A \to A$ be a mapping with $f(0) = 0$ such that

$$\begin{aligned} \|f(x + y) - f(x) - f(y) + \Delta f(z_1, z_2, ..., z_n)\|_B \leqslant \theta(&\|z_i\|_A^p + \|x\|_A^p\|y\|_A^p\|z_i\|_A^p \\ &+ \|x\|_A^p\|z_i\|_A^p + \|y\|_A^p\|z_i\|_A^p) \end{aligned} \tag{45}$$

for all $(x, y) \in U_d$ and $z_i \in A$, $1 \leqslant i \leqslant n$ that are mutually orthogonal. Then f is an orthogonally ring $*$-n-derivation.

Proof. It follows by Theorem 3.1 by putting

$$\varphi(x, y, z_1, z_2, ..., z_n) = \theta(\|z_i\|_A^p + \|x\|_A^p\|y\|_A^p\|z_i\|_A^p + \|x\|_A^p\|z_i\|_A^p + \|y\|_A^p\|z_i\|_A^p).$$

■

Corollary 3.3 Let θ, p by real number such that $\theta > 0$, $P < \frac{1}{3}$ and X and Y be two normed C^*-algebra. Let $f : A \to B$ be a mapping with $f(0) = 0$ such that

$$\|\Delta f(x+y) - f(x) - f(y) + \Delta f(z_1, z_2, ..., z_n)\|_B \leqslant \theta(\|z_i\|_A^p + \|x\|_A^p \|y\|_A^p \|z_i\|_A^p + \|x\|_A^p \|z_i\|_A^p + \|y\|_A^p \|z_i\|_A^p \tag{46}$$

for all $(x, y) \in U_d$ and $z_i \in A$, $1 \leqslant i \leqslant n$ that are mutually orthogonal. Then f is an orthogonally ring $*$-n-homomorphism.

Proof. It follows by Theorem 3.1 by putting

$$\varphi(x, y, z_1, z_2, ..., z_n) = \theta(\|z_i\|_A^p + \|x\|_A^p \|y\|_A^p \|z_i\|_A^p + \|x\|_A^p \|z_i\|_A^p + \|y\|_A^p \|z_i\|_A^p).$$

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