Journal of
Linear and Topological Algebra
Vol. 05*, No.* 01*,* 2016, 29- 39

Common fixed point results on vector metric spaces

G. Soleimani Rad[a*], I. Altun[b]

[a] *Young Researchers and Elite club, Central Tehran Branch, Islamic Azad University, Tehran, Iran.*
[b] *Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey.*
[b] *King Saud University, College of Science, Riyadh, Saudi Arabia.*

Received 26 January 2016; Revised 5 April 2016; Accepted 20 April 2016.

Abstract. In this paper we consider the so called a vector metric space, which is a generalization of metric space, where the metric is Riesz space valued. We prove some common fixed point theorems for three mappings in this space. Obtained results extend and generalize well-known comparable results in the literature.

Keywords: Vector metric space, Riesz space, common fixed point, weakly compatible pairs.

2010 AMS Subject Classification: 47B60, 54H25, 47H10.

1. Introduction and preliminaries

Consistent with Altun and Cevik [5, 7], the following definitions and results will be needed in the sequel.

Given a partially ordered set $(E, \leqslant)$, the notation $x < y$ means $x \leqslant y$ and $x \neq y$. An order interval $[x, y]$ is the set $\{z \in E : x \leqslant z \leqslant y\}$.

A partially ordered set $(E, \leqslant)$ is a lattice if each pair of elements has a supremum and an infimum. A real linear space E with an order relation $\leqslant$ on E which is compatible with the algebraic structure of E is called an ordered linear space. An ordered linear space E for which $(E, \leqslant)$ is a lattice is called a Riesz space or linear lattice. The cone of nonnegative elements in a Riesz space E is denoted by E_+. If (a_n) is a decreasing sequence in E such that $\inf a_n = a$, we write $a_n \downarrow a$. E is said to be Archimedean if $\frac{1}{n}a \downarrow 0$ holds

*Corresponding author.
E-mail address: gha.soleimani.sci@iauctb.ac.ir ; gh.soleimani2008@gmail.com (G. Soleimani Rad).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

for every $a \in E_+$. A sequence (b_n) is said to order convergent or o-convergent to b if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $|b_n - b| \leqslant a_n$ for all n, and written $b_n \to_o b$, where $|a| = a \vee (-a)$ for any $a \in E$. Moreover, (b_n) is said to be o-Cauchy if there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $|b_n - b_{n+p}| \leqslant a_n$ holds for all n and p. E is said to be o-Cauchy complete if every o-Cauchy sequence is o-convergent. For notations and other facts regarding Riesz spaces we refer to [3].

We begin with some important definitions.

Definition 1.1 [5, 7]Let X be a non-empty set and E be a Riesz space. The function $d : X \times X \to E$ is said to be a vector metric or E-metric if it is satisfying the following conditions:
(E_1) $d(x, y) = 0$ if and only if $x = y$;
(E_2) $d(x, y) \leqslant d(x, z) + d(y, z)$;
for all $x, y, z \in X$. Also the triple (X, d, E) is said to be vector metric space.

For arbitrary elements x, y, z and w of a vector metric space, the following statements are satisfied:

$$(Em_1)\ \ 0 \leqslant d(x, y);$$
$$(Em_2)\ \ d(x, y) = d(y, x);$$
$$(Em_3)\ \ |d(x, z) - d(y, z)| \leqslant d(x, y);$$
$$(Em_4)\ \ |d(x, z) - d(y, w)| \leqslant d(x, y) + d(z, w).$$

***Example* 1.2** [5, 7]A Riesz space E is a vector metric space with $d : E \times E \to E$ defined by $d(x, y) = |x - y|$. This vector metric is called to be absolute valued metric on E.

Definition 1.3 [5, 7]
(i) A sequence (x_n) in a vector metric space (X, d, E) vectorial converges or E-converges to some $x \in E$ (we write $x_n \to^{d,E} x$), if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $d(x_n, x) \leqslant a_n$ for all n;
(ii) A sequence (x_n) is called E-Cauchy sequence if there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leqslant a_n$ holds for all n and p;
(iii) A vector metric space X is called E-complete if each E-Cauchy sequence in X E-converges to a limit in X.

Lemma 1.4 [5, 7]We have following properties for the E-convergence sequence $\{x_n\}$ in vector metric space X:
(a) The limit of the sequence $\{x_n\}$ is unique;
(b) Every subsequence of (x_n) E-converges to x;
(c) If $x_n \to^{d,E} x$ and $y_n \to^{d,E} y$, then $d(x_n, y_n) \to_o d(x, y)$.

Lemma 1.5 [5] If E is a Riesz space and $a \leqslant ka$ where $a \in E_+$ and $k \in [0, 1)$, then $a = 0$.

Remark 1 [5, 7]
(i) *The difference between vector metric and Zabrejko's metric defined in [18] is that the Riesz space has also a lattice structure;*
(ii) *One of the differences between vector metric and Huang-Zhang's metric given in [10] is that there exists a cone due to the natural existence of ordering on Riesz space. The other difference is that vector metric omits the requirement for the vector space to be a Banach space;*

(iii) Set $E = \mathbf{R}$, the concepts of vectorial convergence and convergence in metric coincide. If $X = E$ and d is absolute valued vector metric, then vectorial convergence and convergence in order are same. In the case set $E = \mathbf{R}$, the concepts of E-Cauchy sequence and Cauchy sequence are the same.

For see more details on fixed point theorems in cone metric spaces, one can review some of paper in this field such as [4, 10, 11, 14, 15, 17] and references contained therein. Recently, also, many authors have studied on common fixed point theorems for weakly compatible pairs (see [1, 2, 8, 9, 12, 13, 16] and references contained therein).

Definition 1.6 [13]Let $f, g : X \to X$ be mappings of a set X. If $fw = gw = z$ for some $w \in X$, then w is called a coincidence point of f and g, and z is called a point of coincidence of f and g.

Definition 1.7 [13]Let $f, g : X \to X$ be mappings of a set X. Then f and g are said to be weakly compatible if they commute at every coincidence point.

Lemma 1.8 [1] Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $z = fw = gw$, then z is the unique common fixed point of f and g.

2. Main results

The following theorems and corollaries are the vector metric version for some fixed point results of Jungck [12], Arshad et al. [6], Abbas et al. [2] and Rahimi et al. [16].

Theorem 2.1 Let X be a vector metric space with E is Archimedean. Suppose the mappings $f, g, T : X \to X$ satisfy the following conditions:
(i) for all $x, y \in X$

$$d(fx, gy) \leqslant k u_{x,y}(f, g, T) \tag{1}$$

where $k \in (0, 1)$ is a constant and

$$u_{x,y}(f, g, T) \in \Big\{ d(Tx, Ty), d(fx, Tx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Tx)] \Big\}; \tag{2}$$

(ii) $f(X) \cup g(X) \subset T(X)$;
(iii) one of $f(X)$, $g(X)$, or $T(X)$ is a E-complete subspace of X.

Then $\{f, T\}$ and $\{g, T\}$ have a unique point of coincidence in X. Moreover, if $\{f, T\}$ and $\{g, T\}$ are weakly compatible, then f, g, and T have a unique common fixed point in X.

Proof. Suppose x_0 is an arbitrary point of X. Since $f(X) \subset T(X)$, there exists $x_1 \in X$ such that $f(x_0) = T(x_1) = y_1$. Since $g(X) \subset T(X)$, there exists $x_2 \in X$ such that $g(x_1) = T(x_2) = y_2$. If we continue in this manner, then

$$\begin{aligned} &\exists\, x_{2n+1} \in X \qquad s.t \qquad y_{2n+1} = fx_{2n} = Tx_{2n+1} \\ &\exists\, x_{2n+2} \in X \qquad s.t \qquad y_{2n+2} = gx_{2n+1} = Tx_{2n+2}, \end{aligned}$$

for $n = 0, 1, \cdots$.
We first show that

$$d(y_{2n+1}, y_{2n+2}) \leqslant kd(y_{2n}, y_{2n+1}) \tag{3}$$

for all n. From 1, we have

$$d(y_{2n+1}, y_{2n+2}) = d(fx_{2n}, gx_{2n+1}) \leqslant ku_{x_{2n},x_{2n+1}}(f, g, S, T)$$

for $n = 1, 2, \cdots$, where

$$\begin{aligned} u_{x_{2n},x_{2n+1}}(f, g, T) \in & \Big\{d(Tx_{2n}, Tx_{2n+1}), d(fx_{2n}, Tx_{2n}), d(gx_{2n+1}, Tx_{2n+1}) \\ & , \frac{d(fx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Tx_{2n})}{2}\Big\} \\ = & \Big\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}) \\ & , \frac{d(y_{2n+1}, y_{2n+1}) + d(y_{2n+2}, y_{2n})}{2}\Big\} \\ = & \Big\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \frac{d(y_{2n+2}, y_{2n})}{2}\Big\}. \end{aligned}$$

If $u_{x_{2n},x_{2n+1}}(f, g, T) = d(y_{2n}, y_{2n+1})$, then clearly (3) holds. If $u_{x_{2n},x_{2n+1}}(f, g, T) = d(y_{2n+1}, y_{2n+2})$, then according to Lemma 1.5, $d(y_{2n+1}, y_{2n+2}) = 0$ and clearly (3) holds. Finally, suppose that $u_{x_{2n},x_{2n+1}}(f, g, T) = \frac{d(y_{2n}, y_{2n+2})}{2}$. Then, we have

$$d(y_{2n+1}, y_{2n+2}) \leqslant \frac{k}{2} d(y_{2n}, y_{2n+1}) + \frac{1}{2} d(y_{2n+1}, y_{2n+2})$$

which implies that (3) holds. Similarly, we have

$$d(y_{2n+2}, y_{2n+3}) \leqslant kd(y_{2n+1}, y_{2n+2}). \tag{4}$$

Therefore, from (3) and (4), we get

$$d(y_n, y_{n+1}) \leqslant k^n d(y_0, y_1). \tag{5}$$

By using (5), for all n and p, we have

$$\begin{aligned} d(y_n, y_{n+p}) & \leqslant d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p}) \\ & \leqslant (k^n + k^{n+1} + \cdots + k^{n+p-1}) d(y_0, y_1) \\ & \leqslant \frac{k^n}{1-k} d(y_0, y_1). \end{aligned}$$

Since E is Archimedean then $\{y_n\}$ is an E-Cauchy sequence. Suppose that $T(X)$ is complete. Then there exists a v in $T(X)$, such that

$$Tx_{2n} = y_{2n} \to^{d,E} v \quad \text{and} \quad Tx_{2n+1} = y_{2n+1} \to^{d,E} v.$$

Hence there exists a sequence $\{a_n\}$ in E such that $a_n \downarrow 0$, and $d(Tx_{2n}, v) \leqslant a_n$ and $d(Tx_{2n+1}, v) \leqslant a_{n+1}$. Since T is a self-map on X, there exist $w \in X$ such that $Tw = v$. Now, we prove that $gw = v$. For this, consider

$$d(v, gw) \leqslant d(v, fx_{2n}) + d(fx_{2n}, gw) \leqslant a_{n+1} + ku_{x_{2n},w}(f, g, T),$$

where

$$\begin{aligned} u_{x_{2n},w}(f,g,T) \in \Big\{ & d(Tx_{2n}, Tw), d(fx_{2n}, Tx_{2n}), d(gw, Tw) \\ & , \frac{d(fx_{2n}, Tw) + d(gw, Tx_{2n})}{2} \Big\} \\ = \Big\{ & d(y_{2n}, v), d(y_{2n+1}, y_{2n}), d(gw, v) \\ & , \frac{d(y_{2n+1}, v) + d(gw, y_{2n})}{2} \Big\}. \end{aligned}$$

for all n. There are four possibilities.

Case 1.

$$d(v, gw) \leqslant kd(y_{2n}, v) + a_{n+1} \leqslant a_n + a_{n+1} \leqslant 2a_n.$$

Case 2.

$$d(v, gw) \leqslant kd(y_{2n+1}, y_{2n}) + a_{n+1} \leqslant a_{n+1} + 2a_n \leqslant 3a_n.$$

Case 3.

$$d(v, gw) \leqslant kd(v, gw) + a_{n+1} \leqslant kd(v, gw) + a_n.$$

Thus $d(v, gw) \leqslant \frac{1}{1-k} a_n$.

Case 4.

$$\begin{aligned} d(v, gw) &\leqslant \frac{d(y_{2n+1}, v) + d(gw, y_{2n})}{2} + a_{n+1} \\ &\leqslant \frac{1}{2} d(v, gw) + 2a_n. \end{aligned}$$

Thus $d(v, gw) \leqslant 4a_n$.

Since the infimum of sequences on the right side of last inequality are zero, then $d(v, gw) = 0$, that is, $gw = v$. Therefore, $gw = Tw = v$, that is, v is a point of coincidence of mappings g and T, and w is a coincidence point of mappings g and T.
Now, we show that $fw = v$. Consider

$$d(fw, v) \leqslant d(fw, gx_{2n+1}) + d(gx_{2n+1}, v) \leqslant a_n + ku_{w,x_{2n+1}}(f, g, T),$$

where

$$u_{w,x_{2n+1}}(f,g,T) \in \Big\{d(Tw,Tx_{2n+1}), d(fw,Tw), d(gx_{2n+1},Tx_{2n+1}) \\ , \frac{d(fw,Tx_{2n+1})+d(gx_{2n+1},Tw)}{2}\Big\}$$

for all n. There are four possibilities.

Case 1.

$$d(fw,v) \leqslant a_n + d(v,y_{2n+1}) \leqslant a_n + a_{n+1} \leqslant 2a_n.$$

Case 2.

$$d(fw,v) \leqslant a_n + kd(fw,v).$$

Thus $d(fw,v) \leqslant \frac{1}{1-k}a_n$.

Case 3.

$$d(fw,v) \leqslant a_n + kd(y_{2n+2},y_{2n+1}) \leqslant 3a_n.$$

Case 4.

$$\begin{aligned} d(fw,v) &\leqslant a_n + \frac{d(fw,y_{2n+1})+d(gx_{2n+2},v)}{2} \\ &\leqslant 2a_n + \frac{1}{2}d(fw,v). \end{aligned}$$

Thus $d(fw,v) \leqslant 4a_n$.

Since the infimum of sequences on the right side of last inequality are zero, then $d(fw,v)=0$, that is, $fw=v$. Therefore, $fw=Tw=v$, that is, v is a point of coincidence of mappings f and T, and w is a coincidence point of mappings f and T. Now we shall show that v is unique point of coincidence of pairs $\{f,T\}$ and $\{g,T\}$. Let v' be also a point of coincidence of these three mappings, then $fw'=gw'=Tw'=v'$ for $w' \in X$. Now, we have

$$d(v,v') = d(fw,gw') \leqslant ku_{w,w'}(f,g,T),$$

where

$$\begin{aligned} u_{w,w'}(f,g,T) &\in \Big\{d(Tw,Tw'), d(fw,Tw), d(gw',Tw'), \frac{d(fw,Tw')+d(gw',Tw)}{2}\Big\} \\ &= \Big\{0, d(v,v')\Big\}. \end{aligned}$$

Hence, $d(v,v')=0$, that is, $v=v'$. If $\{f,T\}$ and $\{g,T\}$ are weakly compatible, then v is a unique common fixed point of f, g, and T by Lemma 1.8. The proofs for the cases in which $f(X)$ or $g(X)$ is complete are similar. ■

Corollary 2.2 Let X be a vector metric space with E is Archimedean. Suppose the mappings $f,T : X \to X$ satisfy the following conditions:

(i) for all $x, y \in X$

$$d(fx, fy) \leqslant ku_{x,y}(f, T)$$

where $k \in (0, 1)$ is a constant and

$$u_{x,y}(f, T) \in \{d(Tx, Ty), d(fx, Tx), d(fy, Ty), \frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\};$$

(ii) $f(X) \subset T(X)$;
(iii) one of $f(X)$ or $T(X)$ is a E-complete subspace of X.
Then $\{f, T\}$ have a unique point of coincidence in X. Moreover, if $\{f, T\}$ are weakly compatible, then f and T have a unique common fixed point in X.

***Example* 2.3** Let $E = \mathbf{R}^2$ with coordinatwise ordering defined by $(x_1, y_1) \leqslant (x_2, y_2)$ if and only if $x_1 \leqslant x_2$ and (y_1, y_2), $X = \mathbf{R}$ and $d(x, y) = (|x - y|, c|x - y|)$ with $c > 0$. Define the mappings $fx = x^2 + 2$ and $Tx = 3x^2$. Now, for all $x, y \in X$, we have

$$d(fx, fy) = \frac{1}{3}d(Tx, Ty) \leqslant ku_{x,y}(f, T)$$

with $u_{x,y}(f, T) = d(Tx, Ty)$ for $k =\in [1/3, 1)$. Moreover,

$$f(X) = [2, \infty) \subset [0, \infty) = T(X)$$

and $f(X)$ is E-complete subspace of X. Therefore all conditions of Corollary 2.2 are satisfied. Consequently, f and T have a unique point of coincidence in X. $v = 3 \in X$ is unique point of coincidence of f and T and $x_1 = 1$ and $x_2 = -1$ are coincidence points of f and T. Note that f and T have not a common fixed point, because they are not weakly compatible.

Remark 2 *Let $E = \mathbf{R}^2$ with coordinatwise ordering defined by $(x_1, y_1) \leqslant (x_2, y_2)$ if and only if $x_1 < x_2$ or $x_1 = x_2$ and (y_1, y_2). since $\mathbf{R}^2$ is not Archimedean with lexicographical ordering, then we can not use this ordering for above example.*

The following corollary extends well known Fishers result [9] to vector metric spaces with E is Archimedean.

Corollary 2.4 Let X be a vector metric space with E is Archimedean. Suppose the mappings $f, g, T : X \to X$ satisfy

$$d(fx, gy) \leqslant kd(Tx, Ty)$$

for all $x, y \in X$, where $k < 1$. If $f(X) \cup g(X) \subset T(X)$, and one of $f(X)$, $g(X)$, or $T(X)$ is a E-complete subspace of X, Then $\{f, T\}$ and $\{g, T\}$ have a unique point of coincidence in X. Moreover, if $\{f, T\}$ and $\{g, T\}$ are weakly compatible, then f, g, and T have a unique common fixed point in X.

Theorem 2.5 Let X be a vector metric space with E is Archimedean. Suppose the mappings $f, g, T : X \to X$ satisfy the following conditions:

(i) for all $x, y \in X$

$$d(fx, gy) \leqslant k_1 d(Tx, Ty) + k_2 d(fx, Tx) + k_3 d(gy, Ty) \\ + k_4 d(fx, Ty) + k_5 d(gy, Tx) \tag{6}$$

where k_i for $i = 1, 2, \cdots, 5$ are nonnegative constants with

$$k_1 + k_2 + k_3 + 2\max\{k_4, k_5\} < 1;$$

(ii) $f(X) \cup g(X) \subset T(X)$;
(iii) one of $f(X)$, $g(X)$, or $T(X)$ is a E-complete subspace of X.

Then $\{f, T\}$ and $\{g, T\}$ have a unique point of coincidence in X. Moreover, if $\{f, T\}$ and $\{g, T\}$ are weakly compatible, then f, g, and T have a unique common fixed point in X.

Proof. We define sequences $\{x_n\}$ and $\{y_n\}$ as in the proof of Theorem 2.1. From (10), we have

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\ &\leqslant k_1 d(y_{2n}, y_{2n+1}) + k_2 d(y_{2n+1}, y_{2n}) + k_3 d(y_{2n+2}, y_{2n+1}) \\ &\quad + k_4 d(y_{2n+1}, y_{2n+1}) + k_5 d(y_{2n+2}, y_{2n}). \end{aligned}$$

Consequently,

$$d(y_{2n+1}, y_{2n+2}) \leqslant \alpha d(y_{2n}, y_{2n+1}) \tag{7}$$

where $\alpha = \frac{k_1+k_2+k_5}{1-k_3-k_5} < 1$. Similarly,

$$\begin{aligned} d(y_{2n+3}, y_{2n+2}) &= d(fx_{2n+2}, gx_{2n+1}) \\ &\leqslant k_1 d(y_{2n+2}, y_{2n+1}) + k_2 d(y_{2n+3}, y_{2n+2}) + k_3 d(y_{2n+2}, y_{2n+1}) \\ &\quad + k_4 d(y_{2n+3}, y_{2n+1}) + k_5 d(y_{2n+2}, y_{2n+2}). \end{aligned}$$

Consequently,

$$d(y_{2n+3}, y_{2n+2}) \leqslant \alpha d(y_{2n+2}, y_{2n+1}) \tag{8}$$

where $\alpha = \frac{k_1+k_3+k_4}{1-k_2-k_4} < 1$. From (7) and (8), we have

$$d(y_n, y_{n+1}) \leqslant \alpha^n d(y_0, y_1).$$

By the same arguments as in Theorem 2.1 we conclude that $\{y_n\}$ is a E-Cauchy sequence. Suppose that $T(X)$ is complete. Then there exists a v in $T(X)$, such that

$$Tx_{2n} = y_{2n} \to^{d,E} v \quad \text{and} \quad Tx_{2n+1} = y_{2n+1} \to^{d,E} v.$$

Hence there exists a sequence $\{a_n\}$ in E such that $a_n \downarrow 0$, and $d(Tx_{2n}, v) \leqslant a_n$ and $d(Tx_{2n+1}, v) \leqslant a_{n+1}$. Since T is a self-map on X, there exist $w \in X$ such that $Tw = v$.

Now, we prove that $fw = v$. For this, consider

$$\begin{aligned}
d(fw, v) &\leqslant d(fw, gx_{2n+1}) + d(gx_{2n+1}, v) \\
&\leqslant k_1 d(Tw, Tx_{2n+1}) + k_2 d(fw, Tw) + k_3 d(gx_{2n+1}, Tx_{2n+1}) \\
&\quad + k_4 d(fw, Tx_{2n+1}) + k_5 d(gx_{2n+1}, Tw) + d(gx_{2n+1}, v) \\
&\leqslant (k_1 + k_3 + k_4) d(v, Tx_{2n+1}) + (k_2 + k_4) d(fw, v) \\
&\quad + (k_3 + k_5 + 1) d(gx_{2n+1}, v).
\end{aligned}$$

Consequently,

$$d(fw, v) \leqslant \frac{k_1 + 2k_3 + k_4 + k_5 + 1}{1 - k_2 - k_4} a_n,$$

for all n. Thus $d(fw, v) = 0$, i.e. $fw = v$. Therefore, $fw = Tw = v$, that is, v is a point of coincidence of mappings f and T, and w is a coincidence point of mappings f and T. Now, we show that $gw = v$.

$$\begin{aligned}
d(v, gw) &\leqslant d(v, fx_{2n}) + d(fx_{2n}, gw) \\
&\leqslant d(v, fx_{2n}) + k_1 d(Tx_{2n}, Tw) + k_2 d(fx_{2n}, Tx_{2n}) + k_3 d(gw, Tw) \\
&\quad + k_4 d(fx_{2n}, Tw) + k_5 d(gw, Tx_{2n}) \\
&\leqslant (k_1 + k_2 + k_5) d(v, Tx_{2n}) + (k_3 + k_5) d(gw, v) + (k_2 + k_4 + 1) d(fx_{2n}, v).
\end{aligned}$$

Consequently,

$$d(gw, v) \leqslant \frac{k_1 + 2k_2 + k_4 + k_5 + 1}{1 - k_3 - k_5} a_n,$$

for all n. Thus $d(gw, v) = 0$, i.e. $gw = v$. Therefore, $gw = Tw = v$, that is, v is a point of coincidence of mappings g and T, and w is a coincidence point of mappings g and T. Now we shall show that v is unique point of coincidence of pairs $\{f, T\}$ and $\{g, T\}$. Let v' be also a point of coincidence of these three mappings, then $fw' = gw' = Tw' = v'$ for $w' \in X$. Now, we have

$$\begin{aligned}
d(v, v') &= d(fw, gw') \\
&\leqslant k_1 d(Tw, Tw') + k_2 d(fw, Tw) + k_3 d(gw', T') \\
&\quad + k_4 d(fw, Tw') + k_5 d(gw', Tw)
\end{aligned}$$

Hence, $d(v, v') = 0$, that is, $v = v'$. If $\{f, T\}$ and $\{g, T\}$ are weakly compatible, then v is a unique common fixed point of f, g and T by Lemma 1.8. The proofs for the cases in which $g(X)$, or $T(X)$ is complete are similar. ■

Corollary 2.6 Let X be a vector metric space with E is Archimedean. Suppose the mappings $f, T : X \to X$ satisfy the following conditions:
(i) for all $x, y \in X$

$$d(fx, fy) \leqslant k_1 d(Tx, Ty) + k_2 d(fx, Tx) + k_3 d(fy, Ty) + k_4 d(fx, Ty) + k_5 d(fy, Tx) \quad (9)$$

where k_i for $i = 1, 2, \cdots, 5$ are nonnegative constants with

$$k_1 + k_2 + k_3 + 2\max\{k_4, k_5\} < 1;$$

(ii) $f(X) \subset T(X)$;
(iii) one of $f(X)$ or $T(X)$ is a E-complete subspace of X.

Then $\{f, T\}$ have a unique point of coincidence in X. Moreover, if $\{f, T\}$ are weakly compatible, then f and T have a unique common fixed point in X.

Corollary 2.7 Let X be a E-complete vector metric space with E is Archimedean. Suppose the mapping $f : X \to X$ satisfies the following condition:

$$d(fx, fy) \leqslant k_1 d(x, y) + k_2 d(fx, x) + k_3 d(fy, y) + k_4 d(fx, y) + k_5 d(fy, x) \tag{10}$$

for all $x, y \in X$, where k_i for $i = 1, 2, \cdots, 5$ are nonnegative constants with

$$k_1 + k_2 + k_3 + k_4 + k_5 < 1.$$

Then f has a unique fixed point in X.

***Example* 2.8** Let $E = \mathbf{R}^2$ with coordinatwise ordering. Also, as in [7, 10], let

$$X = \{(x, 0) \in \mathbf{R}^2 | 0 \leqslant x \leqslant 1\} \cup \{(0, x) \in \mathbf{R}^2 | 0 \leqslant x \leqslant 1\}.$$

The mapping $d : X \times X \to E$ is defined by

$$d((x, 0), (y, 0)) = (\frac{4}{3}|x - y|, |x - y|),$$
$$d((0, x), (0, y)) = (|x - y|, \frac{2}{3}|x - y|),$$
$$d((x, 0), (0, y)) = (\frac{4}{3}x + y, x + \frac{2}{3}y).$$

Then X is E-complete vector metric space. Consider $f : X \to X$ with $f(x, 0) = (0, x)$ and $f(0, x) = (x/2, 0)$, then f satisfies the relation (10) with $k_1 = 3/4$ and $k_2 = k_3 = k_4 = k_5 = 0$. According to Corollary 2.7, f has a unique fixed point in X. But f is not a contractive mapping in real value d metric on X. Therefore we can not apply the Banach fixed point theorem on metric space.

Remark 3 *Note that if $X = E$ and d is absolute valued vector metric, then we can obtain common fixed point results of Riesz space E.*

Acknowledgements

The authors are grateful to the editor-in-chief and referees for their accurate reading and their helpful suggestions.

References

[1] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. 341 (2008), 416-420.

[2] M. Abbas, B.E. Rhoades and T. Nazir, *Common fixed points for four maps in cone metric spaces*, Appl. Math. Comput. 216 (2010), 80-86.
[3] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis*, Springer-Verlag, Berlin, 1999.
[4] I. Altun, *Common fixed point theorems for weakly increasing mappings on ordered uniform spaces*, Miskolc Math. Notes. 12 (1) (2011), 3-10.
[5] I. Altun and C. Cevik, *Some common fixed point theroems in vector metric spaces*, Filomat. 25 (1) (2011), 105-113.
[6] M. Arshad, A. Azam and P. Vetro, *Some common fixed point results in cone metric spaces*, Fixed Point Theory Appl. (2009), Article ID 493965.
[7] C. Cevik and I. Altun, *Vector metric space and some properties*, Topol. Met. Nonlin. Anal. 34 (2) (2009), 375-382.
[8] A.S. Cvetković, M.P. Stanić, S. Dimitrijević and S. Simić, *Common fixed point theorems for four mappings on cone metric type space*, Fixed Point Theory Appl. (2011), Article ID 589725.
[9] B. Fisher, *Four mappings with a common fixed point*, J. Univ. Kuwait Sci. 8 (1981), 131-139.
[10] L.G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. 332 (2007), 1467-1475.
[11] S. Janković, Z. Kadelburg and S. Radenović, *On cone metric spaces; a survey*, Nonlinear Anal. 74 (2011), 2591-2601.
[12] G. Jungck, *Common fixed points for commuting and compatible maps on compacta*, Proc. Am. Math. Soc. 103 (1988), 977-983.
[13] G. Jungck and B.E. Rhoades, *Fixed point theorems for occasionally weakly compatible mappings*, Fixed Point Theory. (7) (2) (2006), 287-296.
[14] H. Rahimi, B.E. Rhoades, S. Radenović and G. Soleimani Rad, *Fixed and periodic point theorems for T-contractions on cone metric spaces*, Filomat. 27 (5) (2013), 881-888.
[15] H. Rahimi and G. Soleimani Rad, *Common fixed point theorems and c-distance in ordered cone metric spaces*, Ukrainian Mathematical Journal. 65 (12) (2014), 1845-1861.
[16] H. Rahimi, P. Vetro and G. Soleimani Rad, *Some common fixed point results for weakly compatible mappings in cone metric type space*, Miskolc Math. Notes. 14 (1) (2013), 233-243.
[17] S. Rezapour and R. Hamlbarani, *Some note on the paper cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. 345 (2008), 719-724.
[18] P.P. Zabreiko, *K-metric and K-normed linear spaces: survey*, Collect. Math. 48 (1997), 825-859.