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On the irreducibility of the complex specialization of the representation of the Hecke algebra of the complex reflection group G_7

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Abstract. We consider a 2-dimensional representation of the Hecke algebra $\mathcal{H}(G_7, u)$, where G_7 is the complex reflection group and u is the set of indeterminates

$$u = (x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3).$$

After specializing the indetrminates to non zero complex numbers, we then determine a necessary and sufficient condition that guarantees the irreducibility of the complex specialization of the representation of the Hecke algebra $\mathcal{H}(G_7, u)$.

Keywords: Braid group, Hecke algebra, irreducible, reflections.

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1. Introduction

Let V be a complex vector space and W a finite irreducible subgroup of $GL(V)$ generated by complex reflections. Let R be the set of reflections in W. For any element s of R, denote by H_s its pointwise fixed hyperplane. We define the set $V^{reg} = V$- $\cup_{s\in R} H_s$ and denote by $\bar{V}$ the quotient V^{reg}/W. The braid group associated to (W, V) is the fundamental group $B(W) = \pi_1(\bar{V}, \bar{x}_0)$ of $\bar{V}$ with respect to any point $\bar{x}_0 \in \bar{V}$. We choose the set of indeterminates, $u = (u_{s,j})_{s,0\leqslant j\leqslant o(s)-1}$, where s runs over the generators of W and

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$u_{s,j} = u_{t,j}$ if s and t are conjugate in W. Here $o(s)$ denotes the order of s. The cyclotomic Hecke algebra associated to W is the quotient of the group algebra $\mathbb{Z}[u, u^{-1}]BW$ by the ideal generated by the relations $\prod_{j=0}^{o(s)-1}(s - u_{s,j})$.

In [7], G. Malle and J. Michel constructed on the cyclotomic hecke algebra $\mathcal{H}(G_7, u)$ of the complex reflexion group, G_7, an irreducible representation

$$\phi : \mathcal{H}(G_7, u) \to M_2(\mathbb{C}(u^{\frac{1}{2}}, u^{-\frac{1}{2}})),$$

where u is the set of indeterminates $u = (x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3)$. In our work, we specialize the indeterminates $x_1, x_2, y_1, y_2, y_3, z_1$, z_2 and z_3 to nonzero complex numbers $\rho e^{i\alpha}$, where $\alpha \in (-\pi, \pi]$ and ρ a positive real number. We then get a representation

$$\varphi : \mathcal{H}(G_7, u) \to GL_2(\mathbb{C}).$$

In section 3, we consider the case when $x_1 = x_2$ and we show that φ is irreducible if and only if $z_1 \neq \frac{y_1 z_2}{y_2}$ and $z_1 \neq \frac{y_2 z_2}{y_1}$ (Theorem 3.4). In section 4, we assume that $x_1 \neq x_2$ and we show that φ is irreducible if and only if $x_1 y_2 z_2 \neq x_2 y_1 z_1$, $x_1 y_1 z_2 \neq x_2 y_2 z_1$, $x_1 y_2 z_1 \neq x_2 y_1 z_2$ and $x_1 y_1 z_1 \neq x_2 y_2 z_2$ (Theorem 4.5).

2. Preliminaries

Definition 2.1 [6] Let V be a complex vector space of dimension n. A complex reflection of GL(V) is a non-trivial element of GL(V) which acts trivialy on a hyperplane.

Definition 2.2 [6] Let V be a complex vector space of dimension n. A complex reflection group is the subgroup of GL(V) generated by complex reflections.

Examples of complex reflection groups include dihedral groups and symmetric groups. For $n \geqslant 3$, the dihedral group, D_n, is the group of the isometries of the plane preserving a regular polygon, with the operation being composition.

A classification of all irreducible reflection groups shows that there are 34 primitive irreducible reflection groups [8]. The starting point was with A. Cohen, who provided a data for those irreducible complex reflection groups of rank 2 [5].

Definition 2.3 [3] The complex reflection group, G_7, is an abstract group defined by the presentation

$$G_7 =< t, u, s \mid t^2 = u^3 = s^3 = 1, tus = ust = stu > .$$

Theorem 2.4 [1] The braid group of G_7 is isomorphic to the group

$$B =< s_1, s_2, s_3 \mid s_1 s_2 s_3 = s_2 s_3 s_1 = s_3 s_1 s_2 > .$$

Definitions and properties of braid groups are found in [2].

Definition 2.5 [7] Let u be the set of indeterminates $u = (x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3)$. The cyclotomic Hecke algebra $\mathcal{H}(G_7, u)$ of G_7 is the quotient of the group algebra of B over $\mathbb{Z}[u, u^{-1}]$ by the relations

$$(s_1 - x_1)(s_1 - x_2) = 0, \qquad \prod_{i=1}^{3}(s_2 - y_i) = 0, \qquad \prod_{i=1}^{3}(s_3 - z_i) = 0.$$

For more details about the Hecke algebra of G_7, see [4].

Definition 2.6 [7] Let $u = (x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3)$. The representation ϕ is defined as follows:

$$\phi : \mathcal{H}(G_7, u) \to M_2(\mathbb{C}(u^{\pm\frac{1}{2}}))$$

$$s_1 = \begin{pmatrix} x_1 \frac{y_1+y_2}{y_1y_2} - \frac{(z_1+z_2)x_2}{r} \\ 0 \qquad\qquad x_2 \end{pmatrix}, \quad s_2 = \begin{pmatrix} y_1+y_2 & \frac{1}{x_1} \\ -y_1y_2x_1 & 0 \end{pmatrix} \quad \text{and} \quad s_3 = \begin{pmatrix} 0 & \frac{-r}{y_1y_2x_1x_2} \\ r & z_1+z_2 \end{pmatrix},$$

where $r = \sqrt{x_1x_2y_1y_2z_1z_2}$.

We specialize the indeterminates $x_1, x_2, y_1, y_2, z_1, z_2$ and z_3 to nonzero complex numbers, $\rho e^{i\alpha}$, where $\alpha \in (-\pi, \pi]$ and ρ a positive real number. We then get a representation $\varphi : \mathcal{H}(G_7, u) \to GL_2(\mathbb{C})$.

Definition 2.7 Principal square root function is defined as follows:

$$z \in \mathbb{C}, \ \ z = (\rho, \alpha), \ \ \rho \geqslant 0 \ \text{ and } \ \sqrt{z} = \sqrt{\rho}e^{i\frac{\alpha}{2}} \ \text{ where } \ -\pi < \alpha \leqslant \pi.$$

Since $\alpha \in (-\pi, \pi]$, it follows that $\sqrt{z^2} = z$ for any complex number z.

3. Irreducibility of the representation φ for $x_1 = x_2$

We assume that $x_1 = x_2$ and we find a necessary and sufficient condition that guarantees the irreducibility of the representation $\varphi : \mathcal{H}(G_7, u) \to GL_2(\mathbb{C})$. Under this assumption, we have that the images of the generators of $\mathcal{H}(G_7, u)$ are

$$s_1 = \begin{pmatrix} x_2 \frac{y_1+y_2}{y_1y_2} - \frac{(z_1+z_2)x_2}{\sqrt{x_2^2y_1y_2z_1z_2}} \\ 0 \qquad\qquad x_2 \end{pmatrix}, \quad s_2 = \begin{pmatrix} y_1+y_2 & \frac{1}{x_2} \\ -y_1y_2x_2 & 0 \end{pmatrix}$$

and

$$s_3 = \begin{pmatrix} 0 & -\frac{\sqrt{x_2^2y_1y_2z_1z_2}}{x_2^2y_1y_2} \\ r & z_1+z_2 \end{pmatrix}.$$

For the matrix s_1, we denote by $s_1(i,j)$ the term of the matrix s_1 which lies in the ith row and in the jth column.

Lemma 3.1 $s_1(1,2) = 0$ if and only if $z_1 = \frac{y_1z_2}{y_2}$ or $z_1 = \frac{y_2z_2}{y_1}$.

Proof. We show that if $s_1(1,2) = 0$ then $z_1 = \frac{y_1z_2}{y_2}$ or $z_1 = \frac{y_2z_2}{y_1}$. Assume that $s_1(1,2) = 0$. This implies that $\frac{y_1+y_2}{y_1y_2} = \frac{(z_1+z_2)x_2}{x_2\sqrt{y_1y_2z_1z_2}}$. This implies that $(y_1+y_2)\sqrt{y_1y_2z_1z_2} = (z_1+z_2)y_1y_2$. Using $y_1y_2 = (\sqrt{y_1y_2})^2$, we get $(y_1+y_2)\sqrt{z_1z_2} = (z_1+z_2)\sqrt{y_1y_2}$. Squaring both sides, we obtain $(y_1+y_2)^2z_1z_2 = (z_1+z_2)^2y_1y_2$. This implies that $z_1 = \frac{y_1z_2}{y_2}$ or $z_1 = \frac{y_2z_2}{y_1}$. On the other hand, direct computations show that if $z_1 = \frac{y_1z_2}{y_2}$ or $z_1 = \frac{y_2z_2}{y_1}$ then $s_1(1,2) = 0$. ■

We now determine a sufficient condition for irreduciblity.

Proposition 3.2 The representation φ is irreducible if $z_1 \neq \frac{y_1 z_2}{y_2}$ and $z_1 \neq \frac{y_2 z_2}{y_1}$.

Proof. Using the hypothesis and Lemma 3.1, we get $s_1(1,2) \neq 0$. Let S be a non trivial proper invariant subspace of $\mathbb{C}^2$. The eigenspace of s_1 is generated by e_1. This implies that S is of the form $< v >$, where $v = ae_1$ for some non-zero complex number a. S is invariant implies that $s_2 v = (a(y_1 + y_2), -ax_2y_1y_2) \in S$, which is a contradiction. Therefore S is irreducible. ■

We determine a necessary condition for irreducibility.

Proposition 3.3 The representation φ is reducible if $z_1 = \frac{y_1 z_2}{y_2}$ or $z_1 = \frac{y_2 z_2}{y_1}$.

Proof. In each case, we show that the 1-dimensional subspace M generated by the vector $u = (-\frac{1}{x_2 y_2}, 1)$ is invariant.
Case1. $z_1 = \frac{y_1 z_2}{y_2}$. Substituting in Definition 2.6, we get

$$s_1 = \begin{pmatrix} x_2 & 0 \\ 0 & x_2 \end{pmatrix}, \quad s_2 = \begin{pmatrix} y_1 + y_2 & \frac{1}{x_2} \\ -x_2 y_1 y_2 & 0 \end{pmatrix} \quad \text{and} \quad s_3 = \begin{pmatrix} 0 & -\frac{z_2}{x_2 y_2} \\ x_2 y_1 z_2 & z_2 + \frac{y_1 z_2}{y_2} \end{pmatrix}.$$

It is easy to see that $s_2 u = y_1 u$ and $s_3 u = z_2 u$. This implies that M is invariant.
Case2. $z_1 = \frac{y_2 z_2}{y_1}$. Substituting in Definition 2.6, we get

$$s_1 = \begin{pmatrix} x_2 & 0 \\ 0 & x_2 \end{pmatrix}, \quad s_2 = \begin{pmatrix} y_1 + y_2 & \frac{1}{x_2} \\ -x_2 y_1 y_2 & 0 \end{pmatrix} \quad \text{and} \quad s_3 = \begin{pmatrix} 0 & -\frac{z_2}{x_2 y_1} \\ x_2 y_2 z_2 & z_2 + \frac{y_2 z_2}{y_1} \end{pmatrix}.$$

It is also easy to see that $s_2 u = y_1 u$ and $s_3 u = z_2 \frac{y_2}{y_1} u$. This implies that M is invariant.
■

Here we have proved the following theorem:

Theorem 3.4 The representation φ is irreducible if and only if $z_1 \neq \frac{y_1 z_2}{y_2}$ and $z_1 \neq \frac{y_2 z_2}{y_1}$.

4. Irreducibility of the representation φ for $x_1 \neq x_2$

We assume that $x_1 \neq x_2$ and we find a necessary and sufficient condition that guarantees the irreducibility of the representation φ: $\mathcal{H}(G_7, u) \to GL_2(\mathbb{C})$. For simplicity, we denote by w the term

$$(x_1 - x_2)^2 y_1^2 y_2^2 z_1 z_2 + [(y_1 + y_2)r - x_1 y_1 y_2 (z_1 + z_2)][(y_1 + y_2)r - x_2 y_1 y_2 (z_1 + z_2)] \quad (1)$$

Lemma 4.1 The complex number w, defined in (1), is different from zero if and only if $x_1 y_2 z_2 \neq x_2 y_1 z_1$, $x_1 y_1 z_2 \neq x_2 y_2 z_1$, $x_1 y_2 z_1 \neq x_2 y_1 z_2$ and $x_1 y_1 z_1 \neq x_2 y_2 z_2$.

Proof. Simple calculations show that $w = \alpha\beta$, where

$$\alpha = x_2 y_1 y_2 z_1 + x_1 y_1 y_2 z_2 - (y_1 + y_2)r$$
$$\beta = x_1 y_1 y_2 z_1 + x_2 y_1 y_2 z_2 - (y_1 + y_2)r.$$

Assume that $w = 0$. This implies that $\alpha = 0$ or $\beta = 0$. If $\alpha = 0$, then

$$x_2y_1y_2z_1 + x_1y_1y_2z_2 = (y_1 + y_2)r.$$

Squaring both sides, we get

$$y_1y_2(-x_2y_2z_1 + x_1y_1z_2)(-x_2y_1z_1 + x_1y_2z_2) = 0.$$

This implies that $x_1y_1z_2 = x_2y_2z_1$ or $x_1y_2z_2 = x_2y_1z_1$. If $\beta = 0$, then

$$x_1y_1y_2z_1 + x_2y_1y_2z_2 = (y_1 + y_2)r.$$

Squaring both sides, we get

$$y_1y_2(x_1y_2z_1 - x_2y_1z_2)(x_1y_1z_1 - x_2y_2z_2) = 0.$$

This implies that $x_1y_2z_1 = x_2y_1z_2$ or $x_1y_1z_1 = x_2y_2z_2$. On the other hand, we assume that any of the following conditions holds true.

$$x_1y_2z_2 = x_2y_1z_1, \ \ x_1y_1z_2 = x_2y_2z_1, \ \ x_1y_2z_1 = x_2y_1z_2 \ \text{ or } \ x_1y_1z_1 = x_2y_2z_2$$

Under direct computations, we easily verify that $w = 0$. ■

We now give a sufficient condition for the irreducibility of the representation φ.

Proposition 4.2 The representation ϕ is irreducible if $x_1y_2z_2 \neq x_2y_1z_1$, $x_1y_1z_2 \neq x_2y_2z_1$, $x_1y_2z_1 \neq x_2y_1z_2$ and $x_1y_1z_1 \neq x_2y_2z_2$.

Proof. If the term $s_1(1,2) = \frac{y_1+y_2}{y_1y_2} - \frac{x_2(z_1+z_2)}{r}$ equals zero, then neither e_1 nor e_2 is a common eigenvector for s_2 and s_3. This implies that the representation is irreducible. We note that under this case, we have that the complex number w is not zero and hence, by Lemma 4.1, we also have that $x_1y_2z_2 \neq x_2y_1z_1$, $x_1y_1z_2 \neq x_2y_2z_1$, $x_1y_2z_1 \neq x_2y_1z_2$ and $x_1y_1z_1 \neq x_2y_2z_2$. If $s_1(1,2) = \frac{y_1+y_2}{y_1y_2} - \frac{x_2(z_1+z_2)}{r}$ is not zero, we diagonalize the matrix S_1 by the invertible matrix

$$T = \begin{pmatrix} 1 & \frac{\frac{y_1+y_2}{y_1y_2} - \frac{x_2(z_1+z_2)}{r}}{x_2-x_1} \\ 0 & 1 \end{pmatrix}.$$

We get

$$T^{-1}s_1T = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}.$$

We then conjugate s_2 by the matrix T. We get

$$T^{-1}s_2T = \begin{pmatrix} M & w \\ -x_1y_1y_2 & P \end{pmatrix},$$

where

$$M = -\frac{x_2(-x_1y_1y_2z_1 - x_1y_1y_2z_2 + y_1r + y_2r)}{(x_1 - x_2)r},$$

$$P = -\frac{x_1(-x_2y_1y_2z_1 - x_2y_1y_2z_2 - y_1r - y_2r)}{(x_1 - x_2)r}.$$

By conjugating s_3 by T, we get

$$T^{-1}s_3T = \begin{pmatrix} A & B \\ r & C \end{pmatrix},$$

where

$$A = \frac{(y_1 + y_2)r - x_2y_1y_2(z_1 + z_2)}{(x_1 - x_2)y_1y_2}$$

and

$$\begin{aligned} B = \frac{1}{(x_1 - x_2)^2r^3}x_1x_2z_1z_2\big(& - x_1x_2y_1y_2z_1^2 - x_1x_2y_1^2z_1z_2 - x_1^2y_1y_2z_1z_2 \\ & - 2x_1x_2y_1y_2z_1z_2 - x_2^2y_1y_2z_1z_2 - x_1x_2y_2^2z_1z_2 \\ & - x_1x_2y_1y_2z_2^2 + x_1y_1z_1r + x_2y_1z_1r \\ & + x_1y_2z_1r + x_2y_2z_1r + x_1y_1z_2r + x_2y_1z_2r \\ & + x_1y_2z_2r + x_2y_2z_2r\big) \end{aligned}$$

and

$$C = \frac{-r(y_1 + y_2) + x_1y_1y_2(z_1 + z_2)}{(x_1 - x_2)y_1y_2}.$$

For simplicity, we denote $T^{-1}s_iT$ by b_i for $1 \leqslant i \leqslant 3$. Suppose, to get contradiction, that the representation is reducible. That is, there exists a non trivial proper invariant subspace M of $\mathbb{C}^2$ of dimension 1. The subspace M has to be one of the following subspaces $< e_1 >$ or $< e_2 >$.

Case1 $S =< e_1 >$. Since $e_1 \in M$, it follows that $b_3e_1 = (A, r) \in M$. This implies that $r = 0$, a contradiction.

Case2 $S =< e_2 >$. Since $e_2 \in M$, it follows that $b_2e_2 = (w, P) \in M$. By Lemma 4.1, we have that w is different from zero, which is a contradiction. Therefore the representation is irreducible. ■

We now present a lemma concerning the number B used in defining $T^{-1}s_3T$ in Proposition 4.2.

Lemma 4.3 The complex number B equals zero in each of the following cases:

(1) $x_1y_2z_2 = x_2y_1z_1$
(2) $x_1y_1z_2 = x_2y_2z_1$
(3) $x_1y_2z_1 = x_2y_1z_2$
(4) $x_1y_1z_1 = x_2y_2z_2$

Proof. We verify that $B = 0$ in case (1). Suppose that $x_1y_2z_2 = x_2y_1z_1$ then

$$
\begin{aligned}
B = \frac{1}{(x_1-x_2)^2r^3}x_1x_2z_1z_2\Big(& -\frac{x_2y_1z_1}{y_2z_2}x_2y_1y_2z_1^2 - \frac{x_2y_1z_1}{y_2z_2}x_2y_1^2z_1z_2 - \frac{x_2^2y_1^2z_1^2}{y_2^2z_2^2}y_1y_2z_1z_2 \\
& - 2\frac{x_2y_1z_1}{y_2z_2}x_2y_1y_2z_1z_2 - x_2^2y_1y_2z_1z_2 - \frac{x_2y_1z_1}{y_2z_2}x_2y_2^2z_1z_2 \\
& - \frac{x_2y_1z_1}{y_2z_2}x_2y_1y_2z_2^2 + \frac{x_2y_1z_1}{y_2z_2}y_1z_1x_2y_1z_1 + x_2y_1z_1x_2y_1z_1 \\
& + \frac{x_2y_1z_1}{y_2z_2}y_2z_1x_2y_1z_1 + x_2y_2z_1x_2y_1z_1 + \frac{x_2y_1z_1}{y_2z_2}y_1z_2x_2y_1z_1 \\
& + x_2y_1z_2x_2y_1z_1 + \frac{x_2y_1z_1}{y_2z_2}y_2z_2x_2y_1z_1 + x_2y_2z_2x_2y_1z_1\Big) \\
= \frac{1}{(x_1-x_2)^2r^3}x_1x_2z_1z_2\Big(& -\frac{x_2^2y_1^2z_1^3}{z_2} - \frac{x_2^2y_1^3z_1^2}{y_2} - \frac{x_2^2y_1^3z_1^3}{y_2z_2} \\
& - 2x_2^2y_1^2z_1^2 - x_2^2y_1y_2z_1z_2 - x_2^2y_1y_2z_1^2 - x_2^2y_1^2z_1z_2 + \frac{x_2^2y_1^3z_1^3}{y_2z_2} \\
& + x_2^2y_1^2z_1^2 + \frac{x_2^2y_1^2z_1^3}{z_2} + x_2^2y_1y_2z_1^2 + \frac{x_2^2y_1^3z_1^2}{y_2} + x_2^2y_1^2z_1z_2 \\
& + x_2^2y_1^2z_1^2 + x_2^2y_1y_2z_1z_2\Big) = 0.
\end{aligned}
$$

Likewise, we show that $B = 0$ under each of the other conditions. ■

We now present a necessary condition for irreducibility.

Proposition 4.4 The representation is reducible in each of the following cases:

(1) $x_1y_2z_2 = x_2y_1z_1$
(2) $x_1y_1z_2 = x_2y_2z_1$
(3) $x_1y_2z_1 = x_2y_1z_2$
(4) $x_1y_1z_1 = x_2y_2z_2$

Proof. Assume that we have either one of the following conditions holds true:

$$x_1y_2z_2 = x_2y_1z_1,\quad x_1y_1z_2 = x_2y_2z_1,\quad x_1y_2z_1 = x_2y_1z_2 \text{ or } x_1y_1z_1 = x_2y_2z_2.$$

Let S be the one dimensional subspace generated by e_2. If

$$s_1(1,2) = \frac{y_1+y_2}{y_1y_2} - \frac{x_2(z_1+z_2)}{r} = 0,$$

then w, as defined in section 4, equals $(x_1-x_2)^2y_1^2y_2^2z_1z_2$. This implies that $w \neq 0$. By Lemma 4.1, we get a contradiction. Therefore, without loss of generality, we assume that $s_1(1,2) \neq 0$. We then conjugate the representation by the invertible matrix T. Recall that $b_i = T^{-1}s_iT$ $(i = 1,2,3)$. We then have that $b_2e_2 = (w,P) = (0,P)$ by Lemma 4.1, and $b_3e_2 = (B,C) = (0,C)$ by Lemma 4.3. It follows that S is invariant under this representation. ■

This leads us to state a necessary and sufficient condition for the irreducibility of the representation.

Theorem 4.5 The representation is irreducible if and only if $x_1y_2z_2 \neq x_2y_1z_1$, $x_1y_1z_2 \neq x_2y_2z_1$, $x_1y_2z_1 \neq x_2y_1z_2$ and $x_1y_1z_1 \neq x_2y_2z_2$.

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