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The directional hybrid measure of efficiency in data envelopment analysis

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Abstract. The efficiency measurement is a subject of great interest. The majority of studies on DEA models have been carried out using radial or non-radial approaches regarding the application of DEA for the efficiency measurement. This paper, based on the directional distance function, proposes a new generalized hybrid measure of efficiency under generalized returns to scale with the existence of both radial and non-radial inputs and outputs. It extends the hybrid measure of efficiency from Tone (2004) to a more general case. The proposed model is not only flexible enough for the decision-maker to adjust the radial and non-radial inputs and outputs to attain the efficiency score but also avoids the computational and interpretive difficulties, thereby giving rise to an important clarification and understanding of the generalized DEA model. Furthermore, several frequently-used DEA models (such as the CCR, BCC, ERM and SBM models) which depend on the radial or non-radial approaches are derived while their results were compared to the ones obtained from this hybrid model. The empirical examples emphasize the consequence of the proposed measure.

Keywords: Data envelopment analysis, Directional distance function, Hybrid model, Efficiency score.

1. Introduction

Data Envelopment Analysis (DEA) is a non-parametric approach for evaluating the performance of a set of peer entities called Decision Making Units (DMUs), which provides

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a single efficiency score while simultaneously considering multiple inputs and multiple outputs. The first DEA model, as originally proposed in Charnes et.al (1978) which was characterized by the CCR model, construct on the earlier work of Farrell (1957) for assessing an educational center in USA. Following Charnes et.al (1978), it was Banker et al. (1984) who exerted an effort in extending the DEA. According to the related extensive literature, numerous studies have endeavored to describe the novel development, as well as substituting and supporting the DEA's notion and method (see Emrouznejad (2008), and Liu et.al (2013)).

It is well known that the DMUs' performances are appraised by the DEA in which the production relationship will be examined devoid of any functional requirement amid them. If a production technology is assumed wherein m inputs will be required for creating s outputs, the inputs and outputs will be depicted by x and y, respectively. Moreover, production possibility set (PPS) T will be identified by the subsequent equation: $T = \{(x, y) : y \text{ is produced by } x\}$. T involves the entire imaginable input/output combinations. Furthermore, the T's boundary points are referred to as the efficient frontier, or the production frontier (Charnes et al., 1978). It needs to be highlighted that the DMUs which be affiliated with this frontier can be counted as efficient while the remainders would be then inefficient.

Literature shows the availability of numerous DEA models to estimate efficiency scores. In terms of evaluating efficiency score, DEA models take either a radial approach or a non-radial approach which have unalike properties. In the radial approach, inputs and outputs are assumed to change proportionally. This approach is therefore prone to neglect non-radial input and output slacks. Because it does not detect input excesses and output shortfalls, radial models can only classify each DMU as weakly-efficient or inefficient. The radial approach is characterized by the CCR (Charnes et.al, 1978) and stocktickerBCC (Banker et al., 1984) models. In the history of DEA, there have been several investigations into the radial approaches for measuring the efficiency score. In line with this, the readers are referred to the works by Cook and Seiford (2009), and Cooper et al. (2011) for further meticulous evidences.

In contrast, the non-radial DEA models, referring to Koopmans (1951) and Russell (1985), directly deal with input excesses and output shortfalls, and thus, are capable of distinguishing efficient DMUs from inefficient ones. Although in the non-radial measures the optimal efficiency value accounts for the non-radial slacks, the projected DMU can lose the proportionality in the original. In history, several studies have attempted to explain the non-radial measures for the technical efficiency based on the performance evaluation (Charnes et al., 1985; Cooper et al., 1999; Pastor et al., 1999; Pastor et al., 1999; Cooper et al., 2000; Tone, 2001; Cooper et al., 2011).

In an effort to overcome the shortcomings of the radial and non-radial models, a hybrid measure technology was also proposed by Podinovski (2004) which lent its basis to the hypothesis that only certain inputs would proportionally alter with the outputs while the remainder inputs failed to do so. This model was indeed introduced in order to alleviate the aforementioned inadequacies associated with the radial and non-radial models in both approaches. Moreover, Podinovski (2004) was successful in overcoming the constraints associated with the full proportionality and non-proportionality between the input and output. This was fulfilled by integrating the settings for both the constant and variable return-to-scale in the DEA mathematical program.

Another hybrid measure technology termed as the hybrid DEA model was suggested by Tone (2004). In practice, both radial and non-radial measure approaches have been concurrently presented in the DEA mathematical program. It is necessary to underscore that in the hybrid DEA model, the constraint associated with full proportionality and

non-proportionality between the different inputs (outputs) has been successfully eradicated.
Based on the directional distance function, the objective of this research is to demonstrate a new generalized form of the hybrid measure initially introduced by Tone (2004) to assess the efficiency score of DMUs. Evidently, both radial and non-radial approaches can be linked based on a combination of the current theory and both radial and non-radial inputs and outputs. It is significant to note that several renowned DEA models (such as the CCR: Charnes et al. (1985); BCC: Banker et al. (1984); ERM: Pastor (1999) and SBM: Tone (2001)) which depend on the radial or non-radial approach are delivered and their results are compared to the results acquired from the proposed model. Motivated by this objective, we exhibit how separating inputs and outputs as radial and non-radial can be measured to suit the user's needs, and how it can be addressed to open the way for a more comprehensive and accurate measure of estimation to be used with the aim of attaining more pragmatic results for the decision-making units (DMUs).
The remainder of this paper is arranged as follows. In section 2, the preliminary DEA models representative of the radial and non-radial measures of efficiency are briefly described along with the objective. Section 3 lays out the theoretical research dimensions, followed by an alternative formulation with tools appropriate for linking the radial and non-radial measures. Section 4 describes the positive properties of the proposed model. Section 5 compares the proposed methodologies with alternative DEA models. The empirical evaluations and results are considered according to real data to provide further clarification on the proposed approach in section 6. The last section concludes the paper with a discussion.

2. Preliminaries

This section provides a brief overview of the preliminary DEA models as the radial and non-radial measures representative of efficiency. The objective is also presented.
Consider a set of J DMUs, where each DMUj $(j = 1, ..., J)$ consumes K inputs to produce S outputs. Suppose the existing input and output vectors of DMUj are $x_j = (x_{1j}, x_{2j}, ..., x_{Kj})^t$ and $y_j = (y_{1j}, y_{2j}, ..., y_{Sj})^t$, where $x_j \in \mathbf{R}^K$and $y_j \in \mathbf{R}^S$. It is supposed that all inputs and outputs are non-negative, such that at least one input and one output of each DMU is strictly positive. The input and output matrices $X\,(K \times J)$ and $Y\,(S \times J)$ can be represented as $X = [x_1, x_2, ..., x_j, ..., x_J]$ and $Y = [y_1, y_2, ..., y_j, ..., y_J]$. In addition, consistent with the standard suppositions and general returns to scale (GRS) supposition of technology, the PPS presented by the existing DMUs and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_J)^t \in \mathbf{R}^J$, is designated the following:

$$T_G = \left\{ (x, y) : x \geq \sum_{j=1}^{J} \lambda_j x_{kj}, y \leq \sum_{j=1}^{J} \lambda_j y_{sj}, \lambda \in \Omega, k = 1, ..., K, s = 1, ..., S, j = 1, ..., J \right\}.$$

Set Ω is denoted as $\Omega = \left\{ \lambda : L \leq \sum_{j=1}^{J} \lambda_j \leq U,\, \lambda \geq 0 \right\}$, where two non-negative scalar parameters exist, namely L $(0 \leq L \leq 1)$ and U (≥ 1) for $\sum_{j=1}^{n} \lambda_j$. It should be noted that by allowing $L = 0$, $U = \infty$ and $L = 1$, $U = 1$, then T_G would change to T_C and T_V, which are referred to as the PPS pertinent to the constant returns to scale (CRS) (Charnes et al., 1985) and the variable returns to scale (VRS) (Banker et al., 1984) assumptions of technology, respectively. Besides, if it is assumed that $L = 1$, $U = \infty$ and $L = 0$, $U = 1$, then T_G can be altered to T_{ID} and T_{ND}, which are known as the PPS relating to the increasing returns to scale (IRS) (Fre and Grosskopf,

1985) and decreasing returns to scale (DRS) (Seiford and Thrall, 1990) assumptions of technology, respectively.
By considering subscript 'o' as the DMU under evaluation, an input-oriented DEA formulation representative of the radial models in relation to T_G can be expressed as:

$$Model(1) \qquad \theta^* = Min\ \theta \\ s.t.\ (\theta x_o, y_o) \in T_G\,.$$

Correspondingly, an output-oriented model can be:

$$Model(2) \qquad \varphi^* = Max\ \varphi \\ s.t.\ (x_o, \varphi y_o) \in T_G\,.$$

Definition 2.1 The optimal value $\theta^*(\varphi^*)$ of model (1) (model (2)) is the efficiency index of DMUo. When $\theta^* = 1(\varphi^* = 1)$, it can be stated that DMUo is (at least) weakly efficient.

Considering model (1), the optimal value of θ is limited to $0 < \theta^* \leq 1$; the constraints would necessitate activity $(\theta\, x_o, y_o)$ to be the property of T_G and the objective searches for the minimum θ that is able to decrease input vector x_o radially to θx_o while staying in T_G. At this instant, the input excesses $s^- \in \mathbf{R}^K$ besides the output shortfalls $s^+ \in \mathbf{R}^S$ are defined. Moreover, they are recognized as the 'slack' vectors by $s^- = \theta x_o - X\lambda$, $s^+ = Y\lambda - y_o$, where $s^- \geq 0$ and $s^+ \geq 0$ for any feasible solution (θ, λ) of model (1). It then becomes indispensable to solve the second-phase LP problem, because it is aimed at determining the potential input excesses and output shortfalls.

Definition 2.2 Radial Efficiency DMUo can be called "Radial Efficient" on the condition that it is possible for an optimal solution $(\theta^*, \lambda^*, s^{-^*}, s^{+^*})$ of the two phases to satisfy $\theta^* = 1$ as well as be zero-slack $(s^{-^*} = 0, s^{+^*} = 0)$.

Yet, it is observed that in terms of efficiency improvement, model (1) suffers from some structural inadequacies, among which model (1) neglects the non-radial slacks in reporting the efficiency score θ^*. Furthermore, numerous remaining non-radial slacks could be discovered in many cases. Consequently, the decision-making might be misled by the radial approach if it is utilized as the only criterion in appraising DMU performance. The condition is that the mentioned slacks need to be a central part in assessing managerial efficiency. Likewise, output-oriented model (2) could be analysed too.
With the purpose of estimating the efficiency of a DMU(x_o, y_o) and by taking into consideration the DMU's corresponding input and output vectors, a non-oriented DEA formulation representative of non-radial models with regards to T_G is given as:

$$Model(3) \qquad \rho^* = Min\ \frac{\frac{1}{K}\sum_{k=1}^{K}\theta_k}{\frac{1}{S}\sum_{s=1}^{S}\varphi_s} \\ s.t.\ (\theta_k\, x_o, \varphi_s y_o) \in T_G. \\ \theta_k \geq 0,\ k = 1, ..., K,\ \varphi_s \geq 0,\ s = 1, ..., S.$$

θ_k demonstrates the contraction rate related to input k while φ_s exhibits the extension rate of output s for the oth DMU. Also, ρ^* determines the optimal solution as the efficiency score. This model is the same as the ERM model (Russell, 1985) and set T_G is considered as PPS.

Definition 2.3 Non-Radial Efficiency DMUo can be named "Non-Radial Efficient" on the condition that it is possible for an optimal solution $(\theta^*, \varphi^*, \lambda^*)$ of model (3) to satisfy $\rho^* = 1$.

Either ratio optimization or fractional programming is employed to appraise model (3) via reducing the ratio of input efficiency to output efficiency. Similar to non-radial models, the supposition of proportionate contraction in the inputs is ignored by model (3), which is also meant to attain all-out decline rates in the inputs, while such inputs may disregard the fluctuating quantities of the original input resources.
In the past, the assumption was that non-radial models are advantageous in gauging performance measurement; nevertheless, the moment the radial inputs and outputs are submitted, validation is complete. Consequently, combining both radial and non-radial measures can facilitate accurate measurement, whereas radial and non-radial measures are mixed in the problem.

3. The Directional Hybrid Measure (DHM)

A hybrid measure technology was proposed by Tone (2004), known as a hybrid DEA model. The inputs and outputs were categorized into radial and non-radial groupings, each of which was grounded on its own properties to advance to the most efficient limit. To be precise, the hybrid model comprises radial (CCR) and non-radial (SBM) models, thus benefiting from their associated advantages and overcoming the inadequacies.
The hybrid DEA model of efficiency in relation to T_G, which is based on the directional distance function, is introduced in this section. This model is called the directional hybrid measure (DHM) and it bears several favorable characteristics. DHM is a compound of radial and non-radial models, which compensates for weaknesses and exploits their strengths.
The directional distance function first presented by Chambers et al. (Chambers et al., 1996; Chambers et al., 1998), may be regarded as a version of Luenberger's shortage function (Luenberger, 1992, 1995). The traditional Shephard distance function can be generalized by the mentioned function (Shepherd, 2015). Moreover, the directional distance function is appropriate for yielding a technical efficiency measure within the full input-output space. In practice, the abovementioned function centrifugally reflects a certain input-output vector (x, y) from itself toward the PPS boundary in a pre-apportioned direction vector $(-g^-,\, g^+) \in \left(-(\mathbf{R}_+^K),\, \mathbf{R}_+^S\right)$, which can also be defined as:

$$\vec{D}_T = \left(x, y; -g^-,\, g^+\right) = Max\left\{\theta : \left(x - \theta g^-,\, y + \theta g^+\right) \in T\right\}.$$

By taking into account the DMUs with their conforming input and output vectors, let us assume that the K current inputs are disintegrated into n of N radial items $(X^R \in \mathbf{R}^{N\times J})$and m of M non-radial items $(X^{NR} \in \mathbf{R}^{M\times J})$, with $N + M = K$. Let us also assume that the S current outputs are decomposed into p of P radial items ($Y^{NR} \in \mathbf{R}^{P\times J}$) and q of Q non-radial items ($Y^{NR} \in \mathbf{R}^{Q\times J}$) with $P + Q = S$. Subsequently, the input and output matrices $X$and Y are $X = \begin{pmatrix} X^R \\ X^{NR} \end{pmatrix}$ and $Y = \begin{pmatrix} Y^R \\ Y^{NR} \end{pmatrix}$, where R and NR signify the radial and non-radial input or output variables, in that order. In fact, the inputs and outputs are categorized into two classes, namely radial and non-radial, and each one based on its own specifications enhances up to the efficiency limit.
At this instance, the DHM model proportional to T_G would assess the DMUo's efficiency

score by resolving the following mathematical program:

$$Model(4) \quad \psi^* = Min \ \frac{1-\frac{N}{K}(\alpha)-\frac{1}{K}\sum_{m=1}^{M}\theta_m^-}{1+\frac{P}{S}(\beta)+\frac{1}{S}\sum_{q=1}^{Q}\varphi_q^+}$$

$$s.t.\,(x_o^R-\alpha g_n^-, x_o^{NR}-\theta_m^- g_m^{'-}, y_o^R+\beta g_p^+, y_o^{NR}+\varphi_q^+ g_q^{'+})\in T_G.$$

$$\alpha\geq 0,\beta\geq 0,\theta_m^-\geq 0,\ m=1,...,M,\ \varphi_q^+\geq 0,\ q=1,...,Q.$$

The radial contraction rate and extension rate of input n and output p for the $o-$th DMU, reflected onto the efficient frontier of T_G in direction g are specified by α and β, respectively. Also, the non-radial contraction rate as well as the extension rate of input m and output q for the o-th DMU reflected onto the efficient frontier of T_G in direction g can be respectively identified by θ_m^- and φ_q^+. The direction vector $g=(-g^-, g^+)=(-g_n^-, -g_m^{'-}, g_p^+, g_q^{'+})$ was chosen so that for instance, it is possible to adopt the subsequent direction vectors:

$$\begin{aligned}&Max\{x_{nj}/\ g_n^-,\ n=1,...,N\}\leq 1,\\ &Max\{x_{mj}/\ g_m^{'-},\ m=1,...,M\}\leq 1,\ (j=1,...,J)\,.\end{aligned} \quad (1)$$

It is also possible to adopt the subsequent direction vectors:

$$\begin{aligned}&g_n^-=x_{no}\,(n=1,...,N),\ g_m^{'-}=x_{mo}(\,m=1,...,M),\\ &g_p^+=y_{Po}\,(p=1,...,P),\ g_q^{'+}=y_{qo}(\,q=1,...,Q).\end{aligned} \quad (2)$$

or

$$\begin{aligned}&g_n^-=Max\{x_{no}\,;n=1,...,N\},\ g_m^{'-}=Max\{x_{mo}:m=1,...,M\},\\ &g_p^+=Max\{y_{Po}\ :p=1,...,P\},\ g_q^{'+}=Max\{y_{qo}:\ q=1,...,Q\}.\end{aligned} \quad (3)$$

Additionally, the values of $\alpha,\theta_m^-\,(m=1,...,M)\,,\beta,\varphi_q^+\,(q=1,...,Q)$ will be increased together by the objective functions of model (4). In this approach, $\lambda_o=1,\ \lambda_j=0\ \ (j\neq o, j=1,...,J)\alpha=\theta_m^-=\beta=\varphi_q^+=0, m=1,...,M,\ q=1,...,Q$ can be considered a feasible expression.

At this point, model (4) can be transformed into the following model by defining T_G:

$$Model(5) \quad \psi^* = Min \ \frac{1 - \frac{N}{K}(\alpha) - \frac{1}{K}\sum_{m=1}^{M}\theta_m^-}{1 + \frac{P}{S}(\beta) + \frac{1}{S}\sum_{q=1}^{Q}\varphi_q^+}$$

$$s.t. \sum_{j=1}^{J}\lambda_j x_{nj} \leq \ x_{no} - \alpha g_n^- \qquad n = 1, ..., N. \tag{4}$$

$$\sum_{j=1}^{J}\lambda_j x_{mj} \leq \ x_{mo} - \ \theta_m^- g_m^{'-} \qquad m = 1, ..., M. \tag{5}$$

$$\sum_{j=1}^{J}\lambda_j y_{pj} \geq y_{po} + \beta g_p^+ \qquad p = 1, ..., P. \tag{6}$$

$$\sum_{j=1}^{J}\lambda_j y_{qj} \geq y_{qo} + \varphi_q^+ g_q^{'+} \qquad q = 1, ..., Q. \tag{7}$$

$$\lambda \in \Omega, \alpha \geq 0, \beta \geq 0 \tag{8}$$

$$\theta_m^- \geq 0, \ \varphi_q^+ \geq 0, \qquad m = 1, ..., M, \ q = 1, ..., Q. \tag{9}$$

Proposition 3.1 The optimal value of the objective function from the DHM model is $0 \leq \psi^* \leq 1$.

Proof. Primarily, $\alpha^* = \theta_m^{-*} = \beta^* = \varphi_q^{+*} = 0\,(m = 1, ..., M,\, q = 1, ..., Q)$ and $\lambda_o^* = 1, \lambda_j^* = 0 \ (j \neq o, j = 1, ..., J)$, are assigned in order to prove $\psi^* \leq 1$. Subsequently, $(\alpha^*, \theta_m^{-*}, \beta^*, \varphi_q^{+*}, \lambda^*)$ is considered a feasible solution for model (5). If the objective function value for the abovementioned solution is 1, then $\psi^* \leq 1$ would be obtained in terms of minimization. In addition, $0 \leq \psi^* \leq 1$ would be obtained because $x_o^R - \alpha g_n^- \geq 0, x_o^{NR} - \theta_m^- g_m^{'-} \geq 0$ according to (1). ■

It is then proposed that ψ^* be counted as an efficiency measure. In other words, on the condition that an optimal solution for model (5) is allowed to be $(\alpha^*, \theta_m^{-*}, \beta^*, \varphi_q^{+*}, \lambda^*)$, the optimal value of the objective functions from model (5), ψ^*, would therefore be the efficiency score of DMUo $(x_o^R, x_o^{NR}, y_o^R, y_o^{NR}) \in T_G$. Accordingly, a DMU is deemed DHM-efficient as presented below:

Definition 3.2 DHM- Efficiency A DMU is DHM-efficient if and only if $\psi^* = 1$.

This condition is equivalent to $\alpha^* = \theta_m^{-*} = \beta^* = \varphi_q^{+*} = 0\,(m = 1, ..., M,\, q = 1, ..., Q)$, for each optimal solution of model (5), i.e., there is no input inefficiency (waste) and no output inefficiency (shortfall) for all inputs and outputs in any optimal solution.
Having resolved model (5), for a DHM-inefficient DMU $(x_o^R, x_o^{NR}, y_o^R, y_o^{NR})$, i.e., $\psi^* < 1$ and by considering an optimal solution $(\alpha^*, \theta_m^{-*}, \beta^*, \varphi_q^{+*}, \lambda^*)$ the improved activity

$(x_o^{*R}, x_o^{*NR}, y_o^{*R}, y_o^{*NR})$ as DHM projection can be calculated as:

$$\begin{aligned} x_o^{*R} &\Leftarrow (x_{1o} - \alpha^* g_1^-, x_{2o} - \alpha^* g_2^-, ..., x_{No} - \alpha^* g_N^-), \\ x_o^{*NR} &\Leftarrow (x_{1o} - \theta_1^{-*} g_1^{'-}, x_{2o} - \theta_2^{-*} g_2^{'-}, ..., x_{Mo} - \theta_M^{-*} g_M^{'-}), \\ y_o^{*R} &\Leftarrow (y_{1o} + \beta^* g_1^+, y_{2o} + \beta^* g_2^+, ..., y_{Po} + \beta^* g_P^+), \\ y_o^{*NR} &\Leftarrow (y_{1o} + \varphi_1^{+*} g_1^{'+}, y_{2o} + \varphi_2^{+*} g_2^{'+}, ..., y_{Qo} + \varphi_Q^{+*} g_Q^{'+}).) \end{aligned} \quad (10)$$

Proposition 3.3 In each optimal solution of the DHM model, the entire non-radial input and output constraints will be binding.

Proof. If an optimal solution of the DHM model is assumed to be $(\alpha^*, \theta_m^{-*}, \beta^*, \varphi_q^{+*}, \lambda^*)$, it is presumed that the theorem cannot be true as it yields an incongruity. Without the generality loss, it is possible to assume the following for non-radial output constraints: $\exists\ t \in \{1, 2, ..., Q\}$, $\sum_{j=1}^{J} \lambda_j^* y_{tj} > y_{to} + g_t^{'+} \varphi_t^{+*}$ and subsequently $\exists\, \widehat{\varphi_t^+}$, $\sum_{j=1}^{J} \lambda_j^* y_{tj} = y_{to} + g_t^{'+}\, \widehat{\varphi_t^+}$ and $\widehat{\varphi_t^+} > \varphi_t^{+*}$. Bearing in mind $\varphi_q^{+*} = \widehat{\varphi_q^+} (q = 1, 2, ..., Q,\ q \neq t)$, $\frac{\sum_{j=1}^{J} \lambda_j^* y_{to} - y_{to}}{g_t^{'+}} = \widehat{\varphi_t^+}$ in the company of $\lambda_j^* = \widehat{\lambda}_j, (j = 1, 2, ..., J)$,$\theta_m^{-*} = \widehat{\theta_m^-}$ $(m = 1, 2, ..., M)$, $\alpha^* = \widehat{\alpha}$ and$\beta^* = \widehat{\beta}$, a feasible solution to model as $(\widehat{\lambda},\ \widehat{\alpha},\ \widehat{\beta},\ \widehat{\theta^-},\ \widehat{\varphi^+})$ is certainly obtainable. Hereafter, the following is obtained: $\frac{1 - \frac{N}{K}(\widehat{\alpha}) - \frac{1}{K}\sum_{m=1}^{M} \widehat{\theta_m^-}}{1 + \frac{P}{S}(\widehat{\beta}) + \frac{1}{S}\sum_{q=1}^{Q} \widehat{\varphi_q^+}} < \frac{1 - \frac{N}{K}(\alpha^*) - \frac{1}{K}\sum_{m=1}^{M} \theta_m^{-*}}{1 + \frac{P}{S}(\beta^*) + \frac{1}{S}\sum_{q=1}^{Q} \varphi_q^{+*}}$, which is lower than the optimum solution of the model, thus leading to a contradiction. ■

Proposition 3.4 In each optimal solution of the DHM model, at least one of the radial input and output constraints will be binding.

Proof. If an optimal solution of the DHM model is assumed to be $(\alpha^*, \theta_m^{-*}, \beta^*, \varphi_q^{+*}, \lambda^*)$, it is presumed that the theorem cannot be true, as it leads to an incongruity. Without the generality loss, it is possible to assume the following for all radial output constraints:$\forall\, n \in \{1, 2, ..., N\}$, and$\sum_{j=1}^{J} \lambda_j^* x_{nj} < x_{no} - \alpha^* g_n^-$, subsequently $\alpha^* < \frac{x_{no} - \sum_{j=1}^{J} \lambda_j^* x_{nj}}{g_n^-}$. By situating $\forall\ n \in \{1, 2, ..., N\}$,$\frac{x_{no} - \sum_{j=1}^{J} \lambda_j^* x_{nj}}{g_n^-} = \widehat{\alpha}$ in the radial input constraint, $\sum_{j=1}^{J} \lambda_j^* x_{nj} \leq x_{no} - \widehat{\alpha} g_n^-$ is obtained, that is, $\widehat{\alpha}$ is certainly a feasible solution that is higher than α^*. Then, $\alpha^* < \widehat{\alpha}$ is obtained, which leads to a contradiction. ■

Proposition 3.5 The improved activity$(x_o^{*R}, x_o^{*NR}, y_o^{*R}, y_o^{*NR})$as a DHM-projection is Pareto efficient.

Proof. If an optimal solution of the DHM model is assumed to be $(\alpha^*, \theta_m^{-*}, \beta^*, \varphi_q^{+*}, \lambda^*)$, it is supposed the theorem cannot be true because this leads to an incongruity. Thus, because all non-radial constraints in the model are binding at optimality, there is a vector

$\widehat{\lambda}$ satisfying $\widehat{\lambda} \in \Omega$ that defines an improved activity for non-radial input and output as:

$$\widehat{x_{mo}} = \sum_{j=1}^{J} \widehat{\lambda}_j x_{mj} \leq x^*_{mo} \qquad m = 1, ..., M.$$

$$\widehat{y_{qo}} = \sum_{j=1}^{J} \widehat{\lambda}_j y_{qj} \geq y^*_{qo} \qquad q = 1, ..., Q.$$

Such that the aforementioned inequity will be strict for at least one input or one output. Without the generality loss, it is possible to assume $\widehat{x_{mo}} = \sum_{j=1}^{J} \widehat{\lambda}_j x_{mj} \leq x^*_{mo}$, $(m = 1, ..., M)$, so we can define $\exists\, t \in \{1, 2, ..., Q\}$, $\widehat{\theta_t^-} = \frac{x_{to}-(x^*_{to}-\widehat{x_{to}})}{g'^-_t} < \frac{x_{to}-x^*_{to}}{g'^-_t} = \theta_t^{-*}$, and for the rest $\widehat{\theta_m^-} = \frac{x_{mo}-(x^*_{mo}-\widehat{x_{mo}})}{g'^-_m} \leq \frac{x_{mo}-x^*_{mo}}{g'^-_m} = \theta_m^{-*}\,(m = 1, ..., M, m \neq t)$. By considering $\lambda^*_j = \widehat{\lambda}_j (j = 1, 2, ..., J)$,$\varphi_q^{-*} = \widehat{\varphi_q^-}\,(q = 1, 2, ..., Q)$, $\alpha^* = \widehat{\alpha}$ and$\beta^* = \widehat{\beta}$, a feasible solution to the model is certainly obtainable as $(\widehat{\lambda},\ \widehat{\alpha},\ \widehat{\beta},\ \widehat{\theta^-},\ \widehat{\varphi^+})$.
Hereafter, $\frac{1-\frac{N}{K}(\widehat{\alpha})-\frac{1}{K}\sum_{m=1}^{M}\widehat{\theta_m^-}}{1+\frac{P}{S}(\widehat{\beta})+\frac{1}{S}\sum_{q=1}^{Q}\widehat{\varphi_q^+}} < \frac{1-\frac{N}{K}(\alpha^*)-\frac{1}{K}\sum_{m=1}^{M}\theta_m^{-*}}{1+\frac{P}{S}(\beta^*)+\frac{1}{S}\sum_{q=1}^{Q}\varphi_q^{+*}}$, which is lower than the optimum model solution, thus leading to a contradiction. Nonetheless, since at least one of the radial input and output constraints will be binding, the proof for this problem resembles the last case. The improved activity $(x_o^{*R}, x_o^{*NR}, y_o^{*R}, y_o^{*NR})$ as a DHM projection is thus Pareto efficient. ■

4. DHM Properties

The DHM model exhibits numerous favorable properties when selecting an appropriate direction vector.

4.1 *Monotonicity*

The proposed measure apparently diminishes monotonically for any upsurge in input usage or any decrease in output production. In reality, it intensely monotonously declines for each$\alpha, \theta_m^-\,(m = 1, ..., M)\,, \beta$,$\varphi_q^+\,(q = 1, ..., Q)$.

4.2 *Decomposition of Inefficiency*

Considering an optimal solution $(\alpha^*, \theta_m^{-*}, \beta^*, \varphi_q^{+*}, \lambda^*)$, ψ^*can be shown as$\psi^* = \frac{1-a^*}{1+b^*}$, where a^* delineates the input inefficiencies as $a^* = a_1^* + a_2^*$, where a_1^* and a_2^* specify the radial and non-radial input inefficiencies as $a_1^* = \frac{N}{K}(\alpha^*)$ and $a_2^* = \frac{1}{K}\sum_{m=1}^{M}\theta_m^{-*}$ respectively. Furthermore, the output inefficiencies are defined by b^* as $b^* = b_1^* + b_2^*$, wherein b_1^* and b_2^* determine the radial and non-radial output inefficiencies as follows: $b_1^* = \frac{P}{S}(\beta^*)$ and $b_2^* = \frac{1}{S}\sum_{q=1}^{Q}\varphi_q^{+*}$, respectively. This expression is advantageous for detecting the sources of inefficiency in addition to determining the magnitude of their effects on the efficiency score ψ^*.

4.3 *Computational Aspect*

It is possible to convert model (5) to a distinctive case of the hybrid model (Tone, 2004) through a change in variables: $\theta_m^- = \frac{s_m^-}{x_{mo}}$ $(m = 1, ..., M)$, $\varphi_q^+ = \frac{s_q^+}{y_{qo}}$ $(q = 1, ..., Q)$. Indeed, stating model (5) with respect to the total slacks is simple. In connection with (2), the consequence is a novel problem that produces an alternative expression of DHM efficiency as:

$$Model(6) \quad \psi^* = Min \frac{1 - \frac{N}{K}(\alpha) - \frac{1}{K}\sum_{m=1}^{M} \frac{s_m^-}{x_{mo}}}{1 + \frac{P}{S}(\beta) + \frac{1}{S}\sum_{q=1}^{Q} \frac{s_q^+}{y_{qo}}}$$

$$s.t. \sum_{j=1}^{J} \lambda_j x_{nj} \leq \; x_{no}(1-\alpha) \qquad n = 1, ..., N. \tag{11}$$

$$\sum_{j=1}^{J} \lambda_j x_{mj} + s_m^- = x_{mo} \qquad m = 1, ..., M. \tag{12}$$

$$\sum_{j=1}^{J} \lambda_j y_{pj} \geq y_{po}(1+\beta) \qquad p = 1, ..., P. \tag{13}$$

$$\sum_{j=1}^{J} \lambda_j y_{qj} - s_q^+ = y_{qo} \qquad q = 1, ..., Q. \tag{14}$$

$$\lambda \in \Omega, \; \alpha \geq 0, \; \beta \geq 0, \tag{15}$$

$$s_m^- \geq 0, \; s_q^+ \geq 0, \qquad m = 1, ..., M, \; q = 1, ..., Q. \tag{16}$$

By employing the Charnes-Cooper transformation (Charnes and Cooper, 1962), the following can be assumed:

$$\begin{aligned}
&\delta^{-1} = \left(\frac{1 - \frac{N}{K}(\alpha) - \frac{1}{K}\sum_{m=1}^{M} \frac{s_m^-}{x_{mo}}}{1 + \frac{P}{S}(\beta) + \frac{1}{S}\sum_{q=1}^{Q} \frac{s_q^+}{y_{qo}}} \right), \\
&\sigma = \delta\alpha, \; \tau = \delta\beta, \\
&t_m^- = \delta s_m^-, \quad m = 1, ..., M, \\
&t_q^+ = \delta s_q^+, \quad q = 1, ..., Q, \\
&\mu_j = \delta\lambda_j, \quad j = 1, ..., J.
\end{aligned} \tag{17}$$

Here, model (6) is converted to a linear program like so:

$$Model(7) \quad \gamma^* = Min\ \delta - \frac{N}{K}(\sigma) - \frac{1}{K}\sum_{m=1}^{M}\frac{t_m^-}{x_{mo}}$$

$$s.t.\ \delta + \frac{P}{S}(\tau) + \frac{1}{S}\sum_{q=1}^{Q}\frac{t_q^+}{y_{qo}} = 1 \tag{18}$$

$$\sum_{j=1}^{J}\mu_j x_{nj} \leq x_{no}(\delta - \sigma) \qquad n = 1, ..., N. \tag{19}$$

$$\sum_{j=1}^{J}\mu_j x_{mj} + t_m^- = \delta x_{mo} \qquad m = 1, ..., M. \tag{20}$$

$$\sum_{j=1}^{J}\mu_j y_{pj} \geq y_{po}(\delta + \tau) \qquad p = 1, ..., P. \tag{21}$$

$$\sum_{j=1}^{J}\mu_j y_{qj} - t_q^+ = \delta y_{qo} \qquad q = 1, ..., Q. \tag{22}$$

$$\mu \in \Omega,\ \delta \geq 0,\ \sigma \geq 0, \tag{23}$$

$$t_m^- \geq 0,\ t_q^+ \geq 0, \qquad m = 1, ..., M,\ q = 1, ..., Q. \tag{24}$$

At this time, it is feasible to resolve linear model (7) with the aim of solving the DHM model. Having detected the optimal solution of model (7), the optimal DHM model solution can be obtained through changing the given variable.

4.3.1 Completeness

Such a model can be considered complete, because it is non-oriented and differs from oriented ones. Moreover, this kind of model takes into account all inefficiencies concomitant with the non-zero slacks that might be recognized through the model.

4.3.2 Unit invariance

If a direction vector is chosen so that the n-th component of $g_n^-(n = 1, ..., N)$ the m-th component of $g_m^{'-}$ $(m = 1, ..., M)$,the p-th component of $g_p^+(p = 1, ..., P)$ and the q-th component of $g_q^{'+}$ $(q = 1, ..., Q)$, contain similar measurement units as the n-th radial input, m-th non-radial input, p-th radial output and q-th radial output, respectively, the DHM model will be accordingly unit invariant; for example, the considered condition is met by vectors (2) and (3).

4.3.3 Extension (Oriented DHM models)

By disregarding the denominator (numerator) of the objective function of DHM, the input (output)-oriented DHM model can be defined. Consequently, the efficiency values

ψ_I^* and ψ_o^* are obtained as follows:

$$Model(8) \quad \psi_I^* = Min\ 1 - \frac{N}{K}(\alpha) - \frac{1}{K}\sum_{m=1}^{M}\theta_m^- \qquad (DHM-I)$$

$$s.t.\ \sum_{j=1}^{J}\lambda_j x_{nj} \leq\ x_{no} - \alpha g_n^- \qquad n = 1,...,N. \qquad (25)$$

$$\sum_{j=1}^{J}\lambda_j x_{mj} \leq\ x_{mo} -\ \theta_m^- g_m^{'-} \qquad m = 1,...,M. \qquad (26)$$

$$\sum_{j=1}^{J}\lambda_j y_{pj} \geq y_{po} \qquad p = 1,...,P. \qquad (27)$$

$$\sum_{j=1}^{J}\lambda_j y_{qj} \geq y_{qo} \qquad q = 1,...,Q. \qquad (28)$$

$$\lambda \in \Omega, \alpha \geq 0, \theta_m^- \geq 0 \qquad m = 1,...,M. \qquad (29)$$

$$Model(9) \quad \psi_O^* = Min\ \frac{1}{1 + \frac{P}{S}(\beta) + \frac{1}{S}\sum_{q=1}^{Q}\varphi_q^+} \qquad (DHM-O)$$

$$s.t.\ \sum_{j=1}^{J}\lambda_j x_{nj} \leq\ x_{no} \qquad n = 1,...,N. \qquad (30)$$

$$\sum_{j=1}^{J}\lambda_j x_{mj} \leq\ x_{mo} \qquad m = 1,...,M. \qquad (31)$$

$$\sum_{j=1}^{J}\lambda_j y_{pj} \geq y_{po} + \beta g_p^+ \qquad p = 1,...,P. \qquad (32)$$

$$\sum_{j=1}^{J}\lambda_j y_{qj} \geq y_{qo} + \varphi_q^+ g_q^{'+} \qquad q = 1,...,Q. \qquad (33)$$

$$\lambda \in \Omega, \beta \geq 0,\ \varphi_q^+ \geq 0 \qquad q = 1,...,\ Q. \qquad (34)$$

It is noted that these lead to $\psi_O^* \geq \psi^*$ and $\psi_I^* \geq \psi^*$.

4.3.4 Integrating the DM's preference Knowledge

It is compulsory to consider a DM's judgments or priori knowledge in some practical cases on the condition that the DM does not similarly choose the efficient units for obtaining appropriate benchmarks. In practice, the vector g can be adapted compliantly in keeping with the input/output preference orders provided by the DM. Actually, the amounts of adapted direction vector f components designate the relative significance of the inputs/outputs generated by the DM. If it is supposed the non-zero weights $w_k(k = 1,...,K)$ and $w_s(s = 1,...,S)$ are concomitant with the priorities yielded by the DM to the inputs and outputs respectively, then if $w_k(w_s)$ is higher, the k-th input (s-th output) will be accordingly more significant. Subsequently, an extension to the DHM model can be proposed. In fact, by decomposing these weights into $w_n^-(n = 1, ..., N)$ and $w_p^+(p = 1, ..., P)$ for radial and non-radial inputs respectively, as well as $w_m^{'-}$ $(m = 1, ..., M)$ and $w_q^{'+}$ $(q = 1, ..., Q)$ for radial and non-radial outputs, respectively, these weights will be the coefficients of variables $\alpha^*, \beta^*, \theta^{*-}$ and φ^{*+} in the objective function. The components of the adapted direction vector, f, appear as follows:

$$f_n^- = \frac{1}{w_n^-} g_n^- \,(n = 1, ..., N), \;\; f_m^- = \frac{1}{w_m^{'-}} g_m^{'-} \;(m = 1, ..., M),$$
$$f_p^+ = \frac{1}{w_p^+} g_p^+ \,(p = 1, ..., P), \;\; f_q^+ = \frac{1}{w_q^{'+}} g_q^{'+} \;(q = 1, ..., Q).$$

The above indicate that if an input (output) bears greater significance, such input (output) needs to be linked to a greater weight or equivalently the component of the small direction. Taking into account (4.1), $w_n^- \leq 1\,(n = 1, ..., N)$, $w_m^{'-} \leq 1\,(m = 1, ..., M)$ becomes possible. In case the provided weights fail to meet the specified conditions, their normalized forms would do so.

4.4 *Comparisons of the Models with DHM*

Several renowned DEA models, such as CCR, BCC, ERM and SBM, which depend on the radial or non-radial approaches, are derived from the DHM model.
Initially, by setting all inputs and outputs as radial, the DHM model reduces to a revised fractional form of the radial models in direction g under the GRS assumption of technology as follows:

$$Model(10) \qquad \psi_R^* = Min \; \frac{1-\alpha}{1+\beta}$$
$$s.t. \; \sum_{j=1}^{J} \lambda_j x_{kj} \leq \; x_{ko} - \alpha g_k^- \qquad k = 1, ..., K. \qquad (35)$$
$$\sum_{j=1}^{J} \lambda_j y_{sj} \geq y_{so} + \beta g_s^+ \qquad s = 1, ..., S. \qquad (36)$$
$$\lambda \in \Omega, \alpha \geq 0, \beta \geq 0. \qquad (37)$$

Considering $L = 0$, $U = \infty$ and $L = 1$, $U = 1$, model (10) can be transformed into the directional CCR and BCC models, respectively (i.e., DCCR and DBCC). Also, the CCR and BCC models are special cases of model (10) under the CRS and VRS assumptions of technology which can be easily produced by allocating direction vector (2).
Secondly, by setting all inputs and outputs as non-radial, the directional slack-based measure (DSBM) model under the GRS assumption of technology as a special case of

the DHM model can be expressed as follows:

$$Model(11)\ \ \psi^*_{NR} = Min\ \frac{1-\frac{1}{K}\sum_{k=1}^{K}\theta_k^-}{1+\frac{1}{S}\sum_{s=1}^{S}\varphi_s^+}$$

$$s.t.\ \sum_{j=1}^{J}\lambda_j x_{kj} \leq\ x_{ko} - \theta_k^- g_k^- \qquad k = 1, ..., K \tag{38}$$

$$\sum_{j=1}^{J}\lambda_j y_{sj} \geq y_{so} + \varphi_s^+ g_s^+ \qquad s = 1, ..., S. \tag{39}$$

$$\lambda \in \Omega,\ \theta_k^- \geq 0,\ \ \varphi_s^+ \geq 0,\ k = 1, ..., K,\ s = 1, ..., S. \tag{40}$$

SBM and ERM models are distinctive cases of model (11) under the GRS assumption of technology, which can be easily produced by allocating direction vector (2), too.
Correspondingly, it is possible to define the oriented DCCR, DBCC or DSBM models, i.e., DCCR-I, DCCR-0 as designated in section 4.6 by disregarding the output or input efficiencies. Among the optimal solution obtained from these models, the following relationships are satisfied: $\psi^*_{DCCR-I} = \psi^*_{DCCR} = \psi^*_{DCCR-O}$; $\psi^*_{DBCC-I} \geq \psi^*_{DBCC}$ and $\psi^*_{DBCC-O} \geq \psi^*_{DBCC}$; $\psi^*_{DSBM-I} \geq \psi^*_{DSBM}$ and $\psi^*_{DSBM-O} \geq \psi^*_{DSBM}$.
Considering $L = 1$ and $U = 1$ for the DBCC, DSBM and DHM models, the following relationships are fulfilled:

$$\psi^*_{DSBM} \leq \psi^* \leq \psi^*_{DBCC}. \tag{41}$$

$$\psi^*_{DSBM-I} \leq \psi^*_I \leq \psi^*_{DBCC-I}. \tag{42}$$

$$\psi^*_{DSBM-O} \leq \psi^*_O \leq \psi^*_{DBCC-O}. \tag{43}$$

Also, considering $L = 0$ and $U = \infty$ for the DCCR, DSBM and DHM models, the relationships are similar to the terms of (41), (42) and (43).
Lastly, it is perceived from the DHM model that a single radial input (output) case diminishes to a non-radial input (output) model as model (11) . Nevertheless, it is impossible to decline the single non-radial input (output) to a radial model as model (10).

5. Empirical Evaluation

The aim of this section is to elaborate on the DHM model using an applied example wherein the efficiency score is measured in terms of radial and non-radial data. The efficiency scores obtained from the DBCC and DSBM models are compared with the DHM model though this example as well. The results demonstrate important differences between the scores of the three methods used.

5.1 *Data Recourse*

If the aim is to examine the methods to more specifically expose their competencies, it is helpful to presume an empirical example of die press machines in the press division of a motorcycle parts manufacturing company (Lertworasirikul et al., 2011).
The motorcycle parts are produced by the die press division using die press machines.

Table 1. Data Related to the Die Press Separation of the Motorcycle Parts Manufacturing Company

DMU	Inputs						Outputs		
	Radial					Non-Radial	Non-Radial		
	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3
1	18	8	93	140	110	5	861	1.27	98
2	18	7	87	120	80	5	947	1.27	96
3	18	7	83	110	70	5	906	1.29	96
4	24	14	163	260	210	15	672	1.27	96
5	21	5	62	120	90	5	974	1.27	98
6	21	7	79	150	110	20	867	1.28	100
7	18	9	102	210	150	20	806	1.27	98
8	18	8	95	110	80	20	869	1.29	97
9	24	14	136	320	240	15	696	1.28	97
10	24	9	98	220	170	20	817	1.28	100
11	24	8	94	110	120	15	913	1.23	100
12	24	6	82	170	120	15	989	1.27	99
13	18	8	97	130	170	15	823	1.29	95
14	18	7	89	90	130	15	862	1.32	97
15	21	7	79	180	140	15	888	1.31	98
16	21	7	87	110	130	10	847	1.34	97
17	21	9	113	140	80	10	793	1.27	98
18	21	6	74	130	130	10	957	1.29	99
19	21	7	79	110	110	10	942	1.32	99
20	21	8	95	200	170	10	839	1.28	97
21	18	7	74	170	110	15	887	1.26	98
22	21	7	76	90	120	5	898	1.31	98
23	24	16	187	290	280	15	614	1.29	98
24	24	13	175	210	260	15	735	1.28	95
25	18	8	91	130	120	20	885	1.31	96

The die press division has a set of 25 die press machines of 80-ton pressing. In the current research, the die press machines are assumed to be the DMUs. Table 1 specifies that the mentioned company utilized 9 variables from the dataset having 6 inputs and 3 outputs. These inputs and outputs are categorized into 5 radial inputs, 1 non-radial input and 3 non-radial outputs. Table 1 was presented by Lertworasirikul et al. (2011) and is repeated here for ease of reference. It should be highlighted that the 7-th input was removed from the dataset to better illustrate the proposed models. The inputs and outputs for evaluating the DMUs' efficiency scores are specified below and their descriptive statistics are indicated in Table 2.

Radial inputs

Input 1(x_1): Overtime hours of direct labor (h)

Input 2 (x_2): Number of stopping times to change the die and adjust the die press machine in a month

Input 3 (x_3): Number of testing presses before a real press (stroke)

Input 4 (x_4): Time the die moves from the forklift (min)

Input 5 (x_5): Time for repair and adjustment (labor hours)

Non-Radial input

Input 6 (x_6): Time for preventive maintenance (h)

Non-Radial Outputs

Output 1 (y_1): Number of total presses resulting in good parts (strokes)

Output 2 (y_2): Process capability ratio (C_p)

Output 3 (y_3): Percent of on-schedule presses (%)

Table 2. Descriptive Statistics for the 25 DMUs

	Mean	Std. Deviation	Minimum	Maximum
Input 1	20.7600	2.43721	18.00	24.00
Input 2	8.4800	2.77068	5.00	16.00
Input 3	99.6000	32.03123	62.00	187.00
Input 4	160.8000	62.04300	90.00	320.00
Input 5	13.0000	5.20416	5.00	20.00
Input 6	140.0000	56.05057	70.00	280.00
Output 1	812.4000	190.81295	8.00	989.00
Output 2	1.2856	0.02311	1.23	1.34
Output 3	97.6000	1.44338	95.00	100.00

5.2 *Efficiency Analysis*

First of all, the DHM model is applied in the current research in order to measure the efficiency scores of DMUs and to systemically account for the existence of radial and non-radial inputs and outputs in a unified framework. Because the slacks of inputs 1 to inputs 5 are freely disposable, these items are categorized as radial. Besides, input 6 and all outputs are categorized as non-radial.

To ratify whether a linear correlation exists among radial and non-radial inputs, Pearson product-moment correlation coefficient analysis is employed in this study to investigate the empirical dataset presented in Table 1.

According to Table 3, for the correlation between input variables 1, 2, 3, 4, and 5, the p-values are less than 0.01, so the null hypothesis "there is no correlation between variables" would be rejected. This means there are significant correlation between inputs 1, 2, 3, 4, and 5. The correlation coefficients between inputs 1, 2, 3, 4, and 5 are greater than 0.518. For the correlation between input variable 6 and the remaining inputs, the p-values are greater than 0.05 and the null hypothesis "there is no correlation between variables" cannot be rejected.

This means there is no significant correlation between input 6 and the other inputs. The results specify there is a greater degree of linear correlation (i.e., proportionate relationship) between the five radial inputs and a lesser degree of linear correlation (i.e., non-proportionate relationship) between the radial inputs and non-radial input. Consequently, the radial inputs and non-radial input of the empirical dataset assessed utilizing the DHM model is sensible.

According to Table 4, the p-values for the correlation between outputs are greater than 0.05, the null hypothesis "there is no correlation between variables" cannot be rejected. This means there is no significant correlation between outputs. Therefore, outputs 1, 2 and 3 are accordingly supposed to be non-radial and there is no radial output to be utilized in the empirical evaluation.

To obtain the efficiency scores of DMUs, assume that $L = 1$ and $U = 1$ (which leads to the variable returns to scale (VRS) assumption of technology) and the direction vector is specified by (2). At this stage, the DSBM, DBCC and DHM models are resolved for

Table 3. Results of the Pearson Product-Moment Correlation Coefficient Analysis Related to the Inputs

	Input 1	**Input 2**	**Input 3**	**Input 4**	**Input 5**	**Input 6**
Input 1	1.000					
	——					
Input 2	0.518	1.000				
	(0.008)	——				
Input 3	0.518	0.963	1.000			
	(0.008)	(0.000)	——			
Input 4	0.572	0.824	0.731	1.000		
	(0.003)	(0.000)	(0.000)	——		
Input 5	0.604	0.856	0.840	0.818	1.000	
	(0.001)	(0.000)	(0.000)	(0.000)	——	
Input 6	0.158	0.301	0.264	0.373	0.336	1.000
	(0.452)	(0.144)	(0.203)	(0.066)	(0.101)	——

Note: The values in brackets are the P-value.

Table 4. Results of the Pearson Product-Moment Correlation Coefficient Analysis Related to the Outputs

	Output 1	**Output 2**	**Output 3**
Output 1	1.000		
	——		
Output 2	0.218	1.000	
	(0.295)	——	
Output 3	0.341	0.230	1.000
	(0.095)	(0.269)	——

Note: The values in brackets are the P-value.

Table 5. Results Obtained From the Comparisons between the Models and DHM

DMU	**Efficiency Score**		
	DSBM	**DHM**	**DBCC**
1	1	1	1
2	1	1	1
3	1	1	1
4	0.41	0.60	0.94
5	1	1	1
6	1	1	1
7	1	1	1
8	1	1	1
9	0.42	0.62	0.95
10	0.78	0.81	1
11	1	1	1
12	1	1	1
13	1	1	1
14	1	1	1
15	0.77	0.89	0.99
16	1	1	1
17	1	1	1
18	1	1	1
19	1	1	1
20	0.64	0.80	0.96
21	1	1	1
22	1	1	1
23	0.38	0.64	0.97
24	0.44	0.62	0.93
25	1	1	1

the DMUs with the intention of attaining the DMUs' efficiency scores.
According to Table 5, although the 18 DMUs can efficiently operate with a score of 1, they fail to contain input and output slacks. There are also 7 DMUs that perform inefficiently with efficiency scores below 1. This means that these 7 DMUs have input excesses or output shortfalls against the 18 DMUs.

Table 6. Results Obtained from Decomposing the Inefficiency Using DHM

DMU	Input inefficiency			Output inefficiency			Efficiency score
	a^*	a_1^*	a_2^*	b^*	b_1^*	b_2^*	ψ^*
4	0.32	0.21	0.11	0.14	0.00	0.14	0.60
9	0.32	0.20	0.12	0.09	0.00	0.09	0.62
15	0.11	0.00	0.11	0.00	0.00	0.00	0.89
20	0.18	0.10	0.08	0.02	0.00	0.02	0.80
23	0.27	0.16	0.11	0.14	0.00	0.14	0.64
24	0.32	0.21	0.11	0.09	0.01	0.09	0.62

Table 5 depicts the obtained results, indicating remarkable dissimilarities in the efficiency scores related to the three models employed based on non-zero slacks. For example, $\psi^*_{DBCC} = 0.93$ was detected as the radial score for DMU 24; yet, the efficiency scores of the two DEA models are $\psi^*_{DSBM} = 0.44$ and $\psi^* = 0.62$. It can be interpreted that the non-zero slacks are disregarded throughout radial efficiency measurement. Furthermore, the radial specification was disregarded in the non-radial models. The radial variables proportionally enhance while the non-radial variables enhance differently in achieving the maximum efficiency in the DHM model. The DHM efficiency score may well be equivalent to one of the DSBM or DBCC models in some DMUs due to the absence of radial inefficiency or non-radial inefficiency.
Accordingly, it can be claimed that this model can find significant application, as it is able to estimate the slack inputs and outputs as well identify their types. Owing to this particular capability, the DHM model yields more precise evaluation and a score between DBCC and DSBM, and stands accountable for the non-radial excesses and shortfalls, thus leading to (41).
To better illuminate the introduced model, it is endeavored to support it in this study through decomposing the efficiency using expression 4.2. In Table 6, the results of DMU efficiency scores evaluated from the DHM model for inefficient DMUs are represented under the heading 'Efficiency Score'. In this table, the inefficiency indicators of radial and non-radial inputs are measured by means of a^* and stated under the heading 'Input Inefficiency'. Furthermore, the inefficiency indicators of radial and non-radial outputs are measured by means of b^* and stated under the heading 'Output Inefficiency'.

Having explored the dataset, five among the inefficient DMUs have higher inefficiency indicators caused by radial inputs compared with inefficiency caused by non-radial inputs. For example, if DMU 4 is taken into account, it is observed that the inefficiency associated with this DMU is related to input and output inefficiency ($a^*= 0.32$ and $b^*= 0.14$). In practice, this inefficiency is a result of radial input inefficiency (${a_1}^* = 0.21$) in addition to non-radial input inefficiency (${a_2}^*= 0.11$) plus non-radial output inefficiency (${b_2}^*= 0.14$). In contrast, one DMU has a higher inefficiency indicator caused by non-radial input rather than radial inputs. Evidently, the inefficiency associated with DMU 10 is related to input and output inefficiency ($a^*= 0.22$ and $b^*= 0.01$). In practice, this inefficiency is a result of radial input inefficiency (${a_1}^*= 0.10$) in addition to non-radial input inefficiency (${a_2}^*= 0.12$) plus non-radial output inefficiency (${b_2}^*= 0.01$).
Moreover, the input inefficiency ($a^*= 0.11$) gives rise to the inefficiency of DMU 15. This input inefficiency is completely caused by non-radial input inefficiency (${a_2}^*= 0.11$), with no inefficiency indicator from radial inputs and non-radial outputs. According to the above description, the empirical evaluation verifies that separating inputs and outputs is an important factor in evaluating the efficiency score of DMUs.

5.3 *Discussion and Concluding Remarks*

The current study is primarily focused on measuring efficiency from a DEA perspective. In line with this, a generalized form of the hybrid model originally introduced by Tone (2004) was developed with the intention of relating the two basic methods for radial and non-radial efficiency measurement, using the directional distance function notion. The proposed model is practical for calculating DMU efficiency when both radial and non-radial inputs (outputs) are integrated in the problem.
In summary, an efficiency measure was defined in this paper, which was shown to perform well. The reasons for this claim, in addition to the abovementioned reasons, include:

- This efficiency measure is well-defined, as it is the optimum value of a mathematical programming problem that is also calculated and inferred simply.
- While the DHM efficiency score never surpasses the radial efficiency score values, the DHM efficiency score exceeds the non-radial efficiency score values. The DHM efficiency score is positioned between DCCR (the easiest) and DSBM (the hardest), which is an indicator of the slacks' partial incorporation.
- It is possible to observe the sources of inefficiency besides the magnitude of their impact on the score by employing the proposed formula meant for decomposing the efficiency score into radial input (output) and non-radial input (outputs) inefficiencies.
- It is also feasible to derive the hybrid model along with numerous renowned DEA models such as CCR, BCC, ERM and SBM from the proposed model.

It should be highlighted that the model recommended in this research is flexible compared to the hybrid DEA model. In addition, it is a more direct combination of the radial and non-radial approaches, which therefore better elucidates the efficiency of the DMU to be assessed. It is thus possible to promptly understand the proposed model. In brief, DEA researchers and practitioners would be able to match the resources' assessed contractions as well as output expansions in a certain production system in order to obtain more accurate efficiency measurements. It is also suggested for the input (output) to be taken as non-radial, provided that the slacks for an input (output) are regarded as significant for calculating efficiency. In the meantime, the item would be considered radial on the condition that the slacks are disposable without restrictions.
Traditional DEA methods apply exact data for both inputs and outputs. However, in some organizations, the defined data are sometimes imprecise or vague. Thus, fuzzy and ordinal procedures for dealing with ambiguity and impreciseness of the proposed model introduced in this paper should be considered in future studies. In extending the mentioned model into a model with super efficiency, generating a rank for each DMU would also be a challenge to overcome.

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