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Recognition by prime graph of the almost simple group PGL(2, 25)

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Abstract. Throughout this paper, every groups are finite. The prime graph of a group G is denoted by $\Gamma(G)$. Also G is called recognizable by prime graph if for every finite group H with $\Gamma(H) = \Gamma(G)$, we conclude that $G \cong H$. Until now, it is proved that if k is an odd number and p is an odd prime number, then $\mathrm{PGL}(2, p^k)$ is recognizable by prime graph. So if k is even, the recognition by prime graph of $\mathrm{PGL}(2, p^k)$, where p is an odd prime number, is an open problem. In this paper, we generalize this result and we prove that the almost simple group $\mathrm{PGL}(2, 25)$ is recognizable by prime graph.

Keywords: Linear group, almost simple group, prime graph, element order, Frobenius group.

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1. Introduction

Let $\mathbb{N}$ denotes the set of natural numbers. If $n \in \mathbb{N}$, then we denote by $\pi(n)$, the set of all prime divisors of n. Let G be a finite group. The set $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of element orders of G is denoted by $\pi_e(G)$. We denote by $\mu(S)$, the maximal numbers of $\pi_e(G)$ under the divisibility relation. The *prime graph* of G is a graph whose vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge (and we write $p \sim q$), whenever G contains an element of order pq. The prime graph of G is denoted by $\Gamma(G)$. A finite group G is called *recognizable by prime graph* if for every finite group H such that $\Gamma(H) = \Gamma(G)$, then we have $G \cong H$.

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In [10], it is proved that if p is a prime number which is not a Mersenne or Fermat prime and $p \neq 11$, 19 and $\Gamma(G) = \Gamma(\mathrm{PGL}(2,p))$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}(2,p)$ and if $p = 13$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}(2,13)$ or $\mathrm{PSL}(2,27)$. In [3], it is proved that if $q = p^\alpha$, where p is a prime and $\alpha > 1$, then $\mathrm{PGL}(2,q)$ is uniquely determined by its element orders. Also in [1], it is proved that if $q = p^\alpha$, where p is an odd prime and α is an odd natural number, then $\mathrm{PGL}(2,q)$ is uniquely determined by its prime graph. In this paper as the main result we consider the recognition by prime graph of almost simple group $\mathrm{PGL}(2,25)$.

2. Preliminary Results

Lemma 2.1 ([8]) Let G be a finite group and $|\pi(G)| \geqslant 3$. If there exist prime numbers r, s, $t \in \pi(G)$, such that $\{tr, ts, rs\} \cap \pi_e(G) = \emptyset$, then G is non-solvable.

Lemma 2.2 (see [20]) Let G be a Frobenius group with kernel F and complement C. Then every subgroup of C of order pq, with p, q (not necessarily distinct) primes, is cyclic. In particular, every Sylow subgroup of C of odd order is cyclic and a Sylow 2-subgroup of C is either cyclic or generalized quaternion group. If C is a non-solvable group, then C has a subgroup of index at most 2 isomorphic to $SL(2,5) \times M$, where M has cyclic Sylow p-subgroups and $(|M|, 30) = 1$.

Using [14, Theorem A], we have the following result:

Lemma 2.3 Let G be a finite group with $t(G) \geqslant 2$. Then one of the following holds:
(a) G is a Frobenius or 2-Frobenius group;
(b) there exists a nonabelian simple group S such that $S \leqslant \overline{G} := G/N \leqslant \mathrm{Aut}(S)$ for some nilpotent normal subgroup N of G.

Lemma 2.4 ([21]) Let $G = L_n^\varepsilon(q)$, $q = p^m$, be a simple group which acts absolutely irreducibly on a vector space W over a field of characteristic p. Denote $H = W \leftthreetimes G$. If $n = 2$ and q is odd then $2p \in \pi_e(H)$.

3. Main Results

Theorem 3.1 The almost simple group $PGL(2,25)$ is recognizable by prime graph.

Proof. Throughout this proof, we suppose that G is a finite group such that $\Gamma(G) = \Gamma(\mathrm{PGL}(2,25))$.

First of all, we remark that by [19, Lemma 7], we have $\mu(\mathrm{PGL}(2,25)) = \{5, 24, 26\}$. Therefore, the prime graph of $\mathrm{PGL}(2,25)$ has two connected components which are $\{5\}$ and $\pi(5^4-1)$. Also we conclude that the subsets $\{2,5\}$ and $\{3,5,13\}$ are two independent subsets of $\Gamma(G)$. In the sequel, we prove that G is neither a Frobenius nor a 2-Frobenius group.

Let $G = K : C$ be a Frobenius group with kernel K and complement C. By Lemma 2.2, we know that K is nilpotent and $\pi(C)$ is a connected component of the prime graph of G. Hence we conclude that $\pi(K) = \{5\}$ and $\pi(C) = \{2,3,13\}$, since 5 is an isolated vertex in $\Gamma(G)$.

If C is non-solvable, then by Lemma 2.2, C consists a subgroup isomorphic to $SL(2,5)$. This implies that $5 \in \pi(SL(2,5)) \subseteq \pi(C)$, which is a contradiction since $\pi(C) = \{2,3,13\}$. Therefore, C is solvable and so it contains a $\{3,13\}$-Hall subgroup, say H. Since K is a normal subgroup of G, KH is a subgroup of G. Also we have $\pi(KH) = \{3, 5, 13\}$. Thus KH is a subgroup of odd order and so it is a solvable subgroup of G. On the other hand, in the prime graph of G, the subset $\{3, 5, 13\}$ is independent. Hence KH is a solvable subgroup of G such that its prime graph contains no edge, which contradicts to Lemma 2.1. Therefore, we get that G is a Frobenius group.

Let G be a 2-Frobenius group with the normal series $1 \lhd H \lhd K \lhd G$, where K is a Frobenius group with kernel H and G/H is a Frobenius group with kernel K/H. We know that G is a solvable group. This implies that G contains a $\{3, 5, 13\}$-Hall subgroup, say T. Again similar to the previous discussion, we get a contraction.

By the above argument, the finite group G is neither Frobenius nor 2-Frobenius. So by Lemma 2.3, we conclude that there exists a nonabelian simple group S such that:

$$S \leqslant \overline{G} := \frac{G}{K} \leqslant \mathrm{Aut}(S)$$

in which K is the Fitting subgroup of G. We know that $\pi(S) \subseteq \pi(G)$. Since $\pi(G) = \{2, 3, 5, 13\}$, so by [13, Table 8], we get that S is isomorphic to one of the simple group A_5, A_6, $\mathrm{PSU}_4(2)$, ${}^2F_4(2)'$, $\mathrm{PSU}_3(4)$, $\mathrm{PSL}_3(3)$, $S_4(5)$, or $\mathrm{PSL}_2(25)$. Now we consider each possibility for the simple group S, step by step.

Step 1. Let S be isomorphic to the alternating group A_5 or A_6. Since $\pi(S) \cup \pi(\mathrm{Out}(S)) = \{2, 3, 5\}$, we conclude that $13 \in \pi(K)$. We know that the alternating groups A_5 and A_6 consist a Frobenius subgroup $2^2 : 3$. Hence since $13 \in \pi(K)$, by [17, Lemma 3.1], we deduce that $13 \sim 3$, which is a contradiction.

Step 3.2. Let S be isomorphic to the simple group $\mathrm{PSU}_4(2)$. By [5], the finite group S contains a Frobenius group $2^2 : 3$, so similar to the above argument we get a contradiction.

Step 3.3. Let S be isomorphic to the simple group ${}^2F_4(2)'$. By [4], in the prime graph of the simple group S, 5 is not an isolated vertex.

Step 3.4. Let S be isomorphic to the simple group $\mathrm{PSU}_3(4)$. Again by [4], in the prime graph of the simple group S, 5 is not an isolated vertex.

Step 3.5. Let S be isomorphic to the simple group $\mathrm{PSL}_3(3)$. Since $\pi(\mathrm{PSL}_3(3)) = \{2, 3, 13\}$, we get that $5 \in \pi(K)$. On the other hand Sylow 3-subgroups of $\mathrm{PSL}_3(3)$ are not cyclic. Hence $5 \notin \pi(K)$, since 5 and 3 are nonadjacent in $\Gamma(G)$.

Step 3.5. Let S be isomorphic to the simple group $S_4(5)$. Again by [4], in the prime graph of the simple group S, 5 is not an isolated vertex.

Step 3.4. Let S be isomorphic to $\mathrm{PSL}_2(25)$. Hence $\mathrm{PSL}_2(25) \leqslant \bar{G} \leqslant \mathrm{Aut}(\mathrm{PSL}_2(25))$.

Let $\pi(K)$ contains a prime r such that $r \neq 5$. Since K is nilpotent, we may assume that K is a vector space over a field with r elements. Hence the prime graph of the semidirect product $K \rtimes \mathrm{PSL}_2(25)$ is a subgraph of $\Gamma(G)$. Let B be a Sylow 5-subgroup of $\mathrm{PSL}_2(25)$. We know that B is not cyclic. On the other hand $K \rtimes B$ is a Frobenius group such that $\pi(K \rtimes B) = \{r, 5\}$. Hence B should be cyclic which is a contradiction.

Let $\pi(K) = \{5\}$. In this case, by Lemma 2.4, we get that there is an edge between 2 and 5 in the prime graph of G which is a contradiction. Therefore, by the above discussion, we deduce that $K = 1$. Also this implies that $\mathrm{PSL}_2(25) \leqslant G \leqslant \mathrm{Aut}(\mathrm{PSL}_2(25))$.

We know that $\mathrm{Aut}(\mathrm{PSL}_2(25)) \cong Z_2 \times Z_2$. Since in the prime graph of $\mathrm{PSL}_2(25)$ there is not any edge between 13 and 2, we get that $G \ncong \mathrm{PSL}_2(25)$. Also if $G = \mathrm{PSL}_2(25) : \langle\theta\rangle$, where θ is a field automorphism, then we get that 2 and 5 are adjacent in G, which is a contradiction. If $G = \mathrm{PSL}_2(25) : \langle\gamma\rangle$, where γ is a diagonal-field automorphism, then we get that G does not contain any element with order $2 \cdot 13$ (see [3, Lemm 12]), which is contradiction, since in $\Gamma(G)$, $2 \sim 13$. This argument shows that $G \cong \mathrm{PGL}_2(25)$, which completes the proof. ∎

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