Numerical Analysis of Supersonic Flows in UCS Using Motionless Viewing Window Technique

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Abstract: A fundamental topic in computational fluid dynamics is the role of coordinate particularly for searching optimized coordinates. The objective of this study is the numerical modeling of supersonic flows using a new coordinate system i.e. Unified Coordinate System (UCS) uses moving mesh with motionless viewing window technique. UCS has an advantage over traditional coordinate systems (Eulerian and Lagrangian) in supersonic flows especially in discontinuous regions (shocks and expansions). Moreover most of the difficulties of the traditional coordinate system may be removed by using UCS. Unified coordinate system benefits from both the eulerian and lagrangian systems. This is why a unified coordinate system as an optimization system is introduced. The researchers briefly review how to apply boundary conditions, how to calculate fluxes and strang dimensional splitting. The researchers also explain the Riemann problem which is the basic step to solve the equations in this coordinate and integrated system forms. In this article the researchers use new numerical method that has more advantages than other numerical methods such as slip line and shock resolution and low numerical dissipation. The important fact is that in using the UCS, there is no need to generate a body-fitted mesh prior to computing flow past a body; the mesh is automatically generated by the flow. These can be seen in the numerical results and comparing with exact solutions and also in quick numerical computations.

Keywords: Moving Mesh, Riemann Problem, Strang Method, Shock Wave, Unified Coordinate.

1. Introduction

In computational fluid dynamics lots of numerical methods are used for solving the fluid flow problems. It has been known since its onset that the numerical solution to a given flow depends on the relation between the flow and the coordinates (mesh) used to compute it.

Lagrangian and eulerian coordinates are two traditional coordinate systems that are used in many numerical works, in this article the researchers used a new coordinate system to calculate supersonic 2-D flows and shock capturing. In this new numerical method for decreasing overall calculation time, increasing shock resolution and to prevent calculation break down the researchers use motionless viewing window technique and non-iterative methods of riemann problem solution.

Each of the two known coordinate systems to analyze the fluid flow (eulerian and lagrangian coordinates) have their own benefits and drawbacks.

Eulerian method is relatively simple but has some disadvantages as follows:

a. It smears contact discontinuities badly.

b. It needs generating a body-fitted mesh prior to computing flow past a body.

Lagrangian method, by contrast, resolves contact discontinuities (including material interfaces and free surfaces) sharply, but it has drawbacks too:

a. The gas dynamics equations could not be written in conservative partial differential equations (PDE) form, rendering numerical computation complicated.

b. It breaks down due to the cell deformation. The objective of this article is to review and calculate the fluid flow in two-dimensional coordinate system by using moving mesh with motionless viewing window technique which was previously extended by W.H. Hui and his

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colleagues [1, 2]. To put it in perspective the researchers shall first comment on the relative merits of the existing coordinate systems, mainly, eulerian, lagrangian, Arbitrary-Lagrangian- Eulerian (ALE), and the moving mesh (coordinate).

1.1. Theoretical discussions

For more than 200 years, two coordinates systems for describe the fluid flow were available: eulerian system is fixed in space, while the lagrangian system follows the fluid. One important point is the question of the equivalence of these two coordinate systems. The question has been asked by many researchers in the field of fluid mechanics and generally a positive answer has been considered. Surprisingly, the first mathematical proof of equivalency, meaning the existence of a one-to-one map between the two sets of weak solutions, obtained by using the two systems, was given as lately as 1987 by Wagner [3] and holds only for one dimensional flow. For two- and three-dimensional flows, W.H. Hui *et al.* [1, 4] showed that theoretically they are not equivalent to each other. There will be more explanations in the next sections [1].

1.2. Computational discussions

In computation view eulerian and lagrangian systems are not equivalent even for the one-dimensional flow. For one-dimensional flow, lagrangian system along with a compatible way with godunov shock wave [5] is better than the eulerian system. The situation in 2-D and 3-D flows is more complicated:
Each of the eulerian and lagrangian systems has advantages and drawbacks. Generally, the eulerian method is relatively simple because the gas dynamics equations can be written in conservation PDE form, which provides the theoretical foundation for shock-capturing computation.

However, this method has two drawbacks:

a. It smears contact discontinuities badly

b. It needs the mesh generation a body-fitted mesh prior to computing flow past a body, but mesh generation is tedious, time-consuming and requires specialized training.

In contrast, a lagrangian method carefully resolves discontinuity of the contact discontinuities (including material interfaces and free surfaces) sharply, because they are adopted on to the lagrangian coordinates. This method has the disadvantages

as follows:

a. It may break down due to cell deformation, because a lagrangian computational cell is literally a fluid particle with finite – though small – size and hence deforms with the fluid and,

b. The gas dynamics equations could not be written in the conservation form of the partial differential equations (PDE) form, rendering complicated numerical computation [1, 6].

2. The optimal coordinate system

Is it possible to provide coordinate systems that benefits eulerian and lagrangian systems without having their disadvantages ? Such a system in some ways has advantages (being or not being optimized system depends on the rules that are quite logical).
In special cases, we require a system that has the following properties to calculate compressible flow:

• Conservation PDE form exists, as in eulerian;

• Contact discontinuities are sharply resolved, as in lagrangian;

• Mesh can be automatically generated to fit the given body shapes;

• Mesh is orthogonal;

• Mesh is uniform.

It can be seen in the next sections that the unified coordinate system has these characteristics. So we can say that unified coordinates system can be the optimized coordinates.

3. Gas dynamics equations in the unified coordinates

Euler equations for unsteady two-dimensional flows are:

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \tag{1}$$

where:

$$E = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{pmatrix} \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho u(e + \frac{p}{\rho}) \end{pmatrix} \quad G = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho v(e + \frac{p}{\rho}) \end{pmatrix}$$

$$e = \frac{1}{2}(u^2 + v^2) + \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Here, the unified coordinates $(\lambda, \xi, \eta, \varsigma)$ that is obtained from a transformation from the cartesian coordinates is introduced [1,7]:

$$dt = d\lambda,$$
$$dx = hud\lambda + Ad\xi + Ld\eta \qquad (2)$$
$$dy = hvd\lambda + Bd\xi + Md\eta$$

The equations governing the unified coordinates are introduced as:

$$\frac{\partial E}{\partial \lambda} + \frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta} = 0 \qquad (3)$$

Unified coordinates equations E, F and G are written this way:

$$E = \begin{pmatrix} \rho\,\Delta \\ \rho\,\Delta u \\ \rho\,\Delta v \\ \rho\,\Delta e \\ A \\ B \\ L \\ M \end{pmatrix} \qquad F = \begin{pmatrix} \rho\,(1-h)I \\ \rho\,(1-h)Iu + pM \\ \rho\,(1-h)Iv - pL \\ \rho\,(1-h)Ie + pI \\ -hu \\ -hv \\ 0 \\ 0 \end{pmatrix}$$

$$G = \begin{pmatrix} \rho\,(1-h)J \\ \rho\,(1-h)Ju - pB \\ \rho\,(1-h)Jv + pA \\ \rho\,(1-h)Je + pJ \\ 0 \\ 0 \\ -hu \\ -hv \end{pmatrix}$$

where:

$$\Delta = AM - BL$$
$$I = uM - vL$$
$$J = Av - Bu$$
$$e = \frac{1}{2}(u^2 + v^2) + \frac{1}{\gamma-1}\frac{p}{\rho}$$

So In this section we got familiar with the governing equations in a unified coordinates system. In the next parts of the article we will check how to solve this equation for supersonic flows.

4. Boundary conditions, slip line and shock resolution

In this section we will refer to some of the advantages in the field of unified coordinates system as following and each of them is briefly explained:

4.1. Boundary conditions and solid boundaries

Consider a time-independent solid boundary (this includes steady flow as a special case):

$$S: \ B(x, y, z) = 0 \qquad (4)$$

Boundary conditions on S are:

$$q.\nabla B = 0 \qquad (5)$$

Then on S can be written:

$$hq.\nabla B = 0 \qquad (6)$$

Eq. (5) implies that fluid particles move on *S*, whereas Eq. (6) implies that pseudo-particles also move on *S*. Therefore, *S* is a material function of the pseudo-particles. Consequently, *B(x,y,z)* can be taken to correspond to one of the coordinates, ξ_0 say. In other words, a coordinate surface in the unified coordinate system can be taken to represent a time-independent solid surface and there is no need for a grid generation prior to flow computation, as is needed if eulerian coordinates are used [1,7].

4.2. Slip line resolution

In steady flow, path lines are identical with streamlines. Hence a slip line coincides with the streamline of a fluid particle and, therefore, also with the streamline of a pseudo-particle. Consequently, it can be taken to correspond to one of the coordinates, ξ^* say, thus avoiding the godunov averaging across it. Hence, in the unified coordinate system a slip line can be sharply resolved. This is in direct contrast to the eulerian coordinates where a slip line does not coincide with a coordinate line and, as a result, the godunov averaging across a slip line in a computational cell will forever smear it.

For unsteady flow, pathlines are in general distinct from the streamlines. While a slip line still coincides with the pathline of a fluid particle, it does not always coincide with a streamline. Hence, a slip line does not always coincide with a coordinate line in the unified coordinate system. In this regard, numerical experiments clearly indicate the trend that slip line resolution increases with increasing *h* from *h=0* (eulerian) to *h=1* (lagrangian) and the unified coordinates using grid-angle preserving *h*, yield better slip line resolution than the eulerian coordinates. Furthermore, if a steady flow is computed as an asymptotic state of unsteady flow for large time, sharp resolution of slip lines is achieved when *h* is determined by, which at the same time avoids severe grid deformation.

4.3. Shock Resolution

In using the unified coordinate system for flow computation, once the grid is set initially it is subsequently generated by the pseudo-particles motion. In this regard, it is interesting to note that pseudo-particles, which move parallel to the fluid particles, tend to crowd together when compressed, resulting in automatic refinement of the grid in the compression region. Consequently, shock resolution is improved in the unified coordinates over the eulerian. Moreover, the improvements increase with increasing shock strength.

5. Strang dimensional splitting method and solution of the governing equations in unified coordinates

5.1. The time step-wise eulerian approximation

The essence of Time Step- Wise Eulerian (TSE) is that while solving the physical conservation laws (the first four equations) for the flow variables $Q = (\rho, p, u, v)^T$ in time step are, λ from λ^k to λ^{k+1}.

The geometric variables $K = (A, B, L, M)^T$ and h in the ratio of λ are assumed to be fixed but are the function of ξ and η also to solve the stable physical four equations we use $\Omega^k(\lambda) \cdot \lambda^k < \lambda < \lambda^{k+1}$ from $K = K(\lambda^k, \xi, \eta)$ and $h = h(\lambda^k, \xi, \eta)$.

After determining Q we update to a new time step the geometric conservation laws (the last four equations of) to get $K(\lambda^{k+1}, \xi, \eta)$ and after solving the equation of preserving mesh angle, we calculate the value of $h(\lambda^{k+1}, \xi, \eta)$. In this way the effects of the flow on the cell shapes are taken into account. This completes the advancing of solution for one time step from $\lambda = \lambda^k$ to $\lambda = \lambda^{k+1}$ and the process can be repeated to advance the solution for the next time step.

5.2. Strang dimensional splitting method

Presently the dimensional splitting technique to find an approximate solution to the Riemann problem in multi-dimensional flows widely has been studied and enjoys broad applications. This technique converts a multi-dimensional problem into several one-dimensional problems. Godunov and strang dimensional splitting [5] repeatedly have been used in solving various problems. In theory, if the time accuracy of a solution for one-dimensional problem is the first grade, these two techniques both will be the first grade in time accuracy. But two-dimensional numerical problem (two-dimensional

riemann problem) that we have solved in this article and also in "W.H. Hui" works was specified that by using the dimensional splitting strang method the results are obtained more accurately. Therefore, in this article we have applied the strang method.

If $L_{\Delta\lambda}^{\xi}$ is the exact solution operator of 1-D euler equation of the $\lambda - \xi$ plane, we have:

$$Q^{k+1} = L_{\frac{\Delta\lambda}{2}}^{\xi} L_{\Delta\lambda}^{\eta} L_{\frac{\Delta\lambda}{2}}^{\xi} Q^k \qquad (7)$$

where:

$$\Delta\lambda = \lambda^{k+1} - \lambda^k$$

To continue we will briefly explain about problem solving $L_{\Delta\lambda}^{\xi}$ and Riemann problem with variable coefficients in governing equations on the $\lambda - \xi$ plane.

5.3. Drive of the final equations using strang dimensional splitting method

The key step in solving the one dimensional Riemann problem in the time step $\Omega^k(\lambda)$; $\lambda^k < \lambda < \lambda^{k+1}$; obtained from strang dimensional splitting and the TSE. Here riemann solution for the coordinate of $\lambda - \xi$ and the solution of Q in $\xi = 0$ for $\lambda \in \Omega^k(\lambda)$ will be stated. In Eq. (3) the time step (to make it simple we put it at zero) the next equation from a one-dimensional physical conservation equation obtained from the strang dimensional splitting that can be:

$$\frac{\partial E_p}{\partial \lambda} + \frac{\partial F_p}{\partial \xi} = 0 \qquad (8)$$

In which:

$$\lambda \in \Omega(\lambda) : 0 < \lambda < \Delta\lambda$$

$$E_p = \begin{pmatrix} \rho\,\Delta \\ \rho\,\Delta u \\ \rho\,\Delta v \\ \rho\,\Delta e \end{pmatrix} \quad F_p = \begin{pmatrix} \rho\,(1-h)I \\ \rho\,(1-h)Iu + pM \\ \rho\,(1-h)Iv - pL \\ \rho\,(1-h)Ie + pI \end{pmatrix}$$

In Eq. (8) physical variables $Q = (\rho, p, u, v)^T$ are related to unknown functions of λ and ξ while the geometric variables are $K = (A, B, L, M)^T$ and h that appear in equations coefficients and are independent from λ :

$$K = K(0, \xi) \qquad h = h(0, \xi) \qquad (9)$$

η in Eq. (8) acts as a parameter. In order to use Godunov method for advancing the solution from

$\lambda = 0$ to $\lambda = \Delta\lambda$, we consider the initial conditions for the neighboring cells (i, j) and $(i+1, j)$ to Riemann data that are fixed which lies:

$$Q\big|^{\lambda=0} = \begin{cases} Q_l(= Q_{i,j}^{\lambda=0}), \xi < 0 \\ Q_r(= Q_{i+1,j}^{\lambda=0}), \xi > 0 \end{cases} \tag{10}$$

(for simplicity we take the cell interface between these two cells to be located at $\xi = 0$).

At the same time, based on the TSE approximation, the coefficients in Eq. (8) are:

$$(K, h)\big|^{\lambda\in\Omega(\lambda)} = \begin{cases} (K, h)_l^{\lambda=0}(= (K, h)_{i,j}^{\lambda=0}), \xi < 0 \\ (K, h)_r^{\lambda=0}(= (K, h)_{i+1,j}^{\lambda=0}), \xi > 0 \end{cases} \tag{11}$$

We note that these coefficients are constants separately for $\xi > 0$ and $\xi < 0$, but in general are not equal to each other [6].

To put the Riemann problem in the $\lambda - \xi$ plane more explicitly in one-dimensional form, we note that the normal direction of the plane $\xi = $ constant is:

$$n = \frac{\nabla\xi}{|\nabla\xi|} = (M, -L)/S \tag{12}$$

And project the flow velocity q into the normal direction n and the tangential direction t to get

$$\begin{cases} \omega = q.n = (uM - vL)/S \\ \tau = q.t = (uL + vM)/S \end{cases} \tag{13}$$

We also make the following replacement:

$$\begin{cases} S = \sqrt{L^2 + M^2} \\ \tan\psi = M/L \end{cases} \tag{14}$$

Now, we should convert equations (8) for $\xi < 0$ and $\xi > 0$ separately. For $\xi < 0$, $(K, h) = (K, h)_l$ are constant values and $\Delta = \Delta_l, S = S_l, \psi = \psi_l$ has constant variables thus Eq. (8) can be written as:

$$\frac{\partial E_l'}{\partial\lambda} + \frac{\partial F_l'}{\partial\xi} = 0 \tag{15}$$

In which:

$\lambda \in \Omega(\lambda), \xi < 0$

$$E_l' = \Delta_l \begin{pmatrix} \rho \\ \rho\omega \\ \rho\tau \\ \rho e \end{pmatrix},$$

$$F_l' = S_l \begin{pmatrix} \rho(1 - h_l)\omega \\ \rho(1 - h_l)\omega^2 + p \\ \rho(1 - h_l)\omega\tau \\ \rho(1 - h_l)\omega e + \omega p \end{pmatrix}$$

and for $\xi > 0$ the Eq. (8) can be written as:

$$\frac{\partial E_r'}{\partial\lambda} + \frac{\partial F_r'}{\partial\xi} = 0 \tag{16}$$

In which:

$\lambda \in \Omega(\lambda), \xi > 0$

$$E_r' = \Delta_r \begin{pmatrix} \rho \\ \rho\omega \\ \rho\tau \\ \rho e \end{pmatrix}$$

$$F_r' = S_r \begin{pmatrix} \rho(1 - h_r)\omega \\ \rho(1 - h_r)\omega^2 + p \\ \rho(1 - h_r)\omega\tau \\ \rho(1 - h_r)\omega e + \omega p \end{pmatrix}$$

Here we see that Eqs. (15) and (16) are similar to the following equation:

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho ue + up \end{pmatrix} = 0 \tag{17}$$

In Eqs. (15) and (16) if we replace ω with u and τ with v we will get to Euler equations in the cartesian coordinate that can be solved in the same way as the method the Euler Eqs.(1)-(3).

Regarding Eqs. (15), (16) and (17) it can be said that all coefficients are fixed and the variable v(τ) can be separated. Also, we should remind that the riemann problem including Eqs. (15) and (16) we have a series of new problems. However, the coefficients are constant, but are generally different for $\xi < 0$ and $\xi > 0$ [7].

6. Numerical results

In this section, we consider an example of a steady two-dimensional flow riemann problem generated by two uniform parallel flows as:

$$(p, \rho, M, \theta) = \begin{cases} (0.25, 0.5, 7.0, 0) \\ (1.0, 1.0, 2.4, 0) \end{cases} \tag{18}$$

Here M represents the Mach number and θ indicates the flow angle which is equal to $\tan^{-1}(v/u)$.

It includes a shock wave, a slip line and an expansion wave as can be seen in Fig. 1.

Pressure and density distribution are shown in Figs. 2 and 3. The resolution of shock, expansion fan and slip line can be seen in these figures. Comparisons of 2-D riemann problem calculation

results in UCS with the exact solution are shown in Fig. 4 and those in the eulerian system are shown in Fig. 5. There are low shock resolution and high numerical dissipation in eulerian method but, these drawbacks do not exist in my numerical method. In Fig. 6 (density contour) a shock, a slip line and an expansion fan can be seen obviously. This contour is results of density calculation use UCS and motionless viewing window technique.
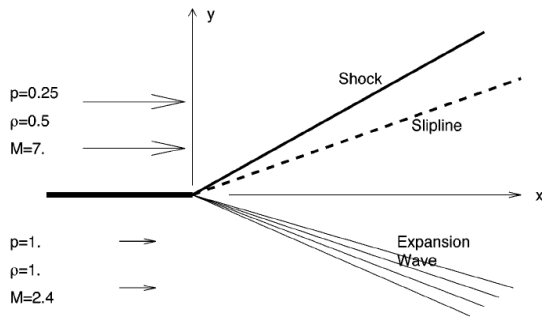
Solving this problem by the stationary slip line method which is sensitive to the dissipative property of the numerical methods and grading position is difficult. This issue is mentioned in [2, 5]. The density variations around the slip line, which dissipation and low resolution of the results it could be the reason of kinetic energy loss in the average stage of the computing cells.
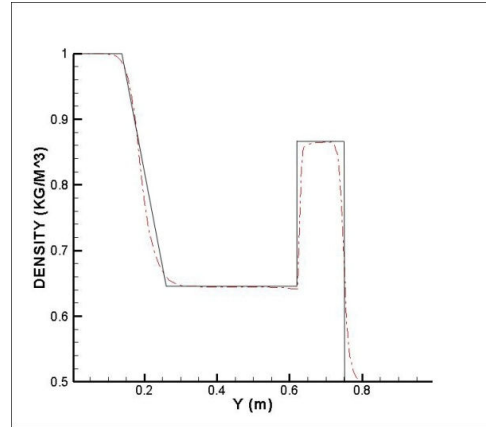

Fig. 1. View of a steady two-dimensional riemann problem [5]

Fig. 2. Pressure distribution [7]

Fig. 3. Density distribution [7]

Fig. 4. Comparison of calculation results with the exact solution (continuous line: exact solution, dash lines: Unified coordinates calculation) [7]
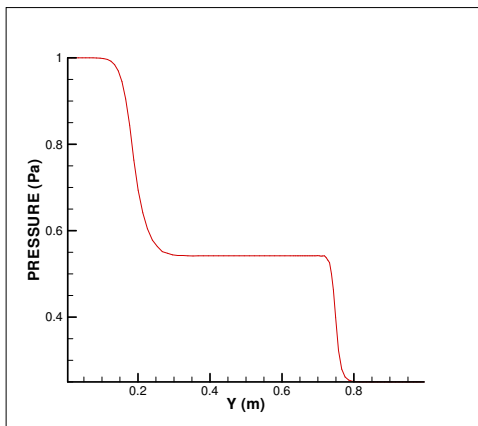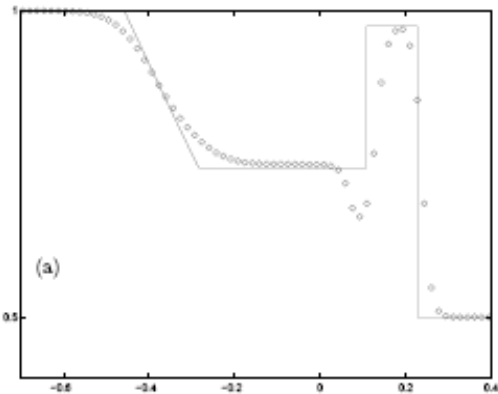

Fig. 5. Density distribution (points: Eulerian system, continuous line: exact solution)
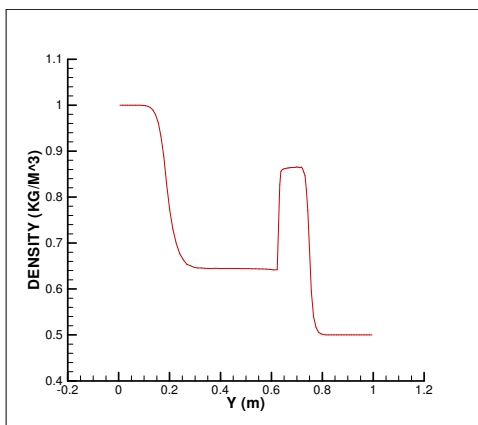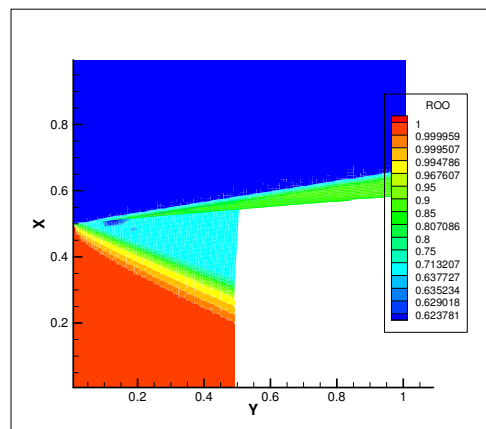

Fig. 6. Density contour

In calculations of 2-D riemann problem, the physical domain follow the pseudo-particles (velocity of particles= $h.q$). If we follow the computational cells (pseudo-particles), they will move out of the initial physical domain, and it would be difficult to have a steady state flow in the original physical domain. To avoid this, a special technique called the "motionless viewing window" is applied as in the classical lagrangian method. Accordingly, the column of cells which have moved out of the original physical domain to the right is deleted, while a new column of cells is added at the input flow boundary on the left. In this example we have considered $\gamma = 1/4$.

7. Conclusions

In this article non-iterative solution of the rie-mann problem has been used instead of other iterative solution of riemann problem and achieved the conclusion that although non-iterative methods of solving riemann problem requires more time than iterative methods, the solution time of overall solution using the unified coordinates system will be less. Also, using the iterative methods to solve riemann problem the answers are more accurate and with less error.

And finally we can refer to the superiority of unified coordinate system in comparison with the eulerian and lagrangian coordinate systems that can be seen in the results of calculations absolutely. In the future works unified coordinate system can be used to solve two phase flows, two-dimensional viscid flows, the supersonic aerodynamics flows and fluid mechanics problems. In such problems the advantage of this coordinates system has been proved by W. H. Hui [8] also more complex problems as well as the three-dimensional flows presented by W. H. Hui [9].

References

[1] Hui, W. H. ; Li, P. Y.; Li, Z.W.; "A Unified Coordinate System for solving the two-dimensional Euler equations", J Comput. Phys., 1999, 153, 596.

[2] Jia, P.; Jiang, S.; Zhao, G. P., "Two-dimensional compressible multi-material flow calculations in a Unified Coordinate System", Comput. Fluids, 2006, 35, 168.

[3] Wagner, D. H., "Equivalence of Euler and Lagrangian equations of gas dynamics for weak solutions", J. Differ. Equations, 1987, 68, 118.

[4] Hui, W. H.; Kudriakov, S., "A Unified Coordinate System for solving the three-dimensional Euler equations", J. Comput. Phys., 2001, 172, 235.

[5] Lepage, C. Y.; Hui, W. H., "A shock-adaptive Godunov scheme based on the generalized Lagrangian formulation", J. Comput. Phys., 1995, 122, 291.

[6] Hui, W. H.; Kudriakov, S., "Computation of shallow water waves using the unified coordinates,SIAM", J. Sci. Comput., 2002, 23, 1615.

[7] Zareyan, B., "Numerical analysis of 2-D flows use unified coordinate system", Msc thesis, Aerospace Engineering Department, Khaje Nasire Toosi University of Technology, 2009.

[8] Hui, W. H.; Koudriakov, S., "Role of coordinates in the computation of discontinuities in one-dimensional flow", Comput. Fluid Dyn. J., 2000, 8, 495.

[9] Hui, W. H.; Koudriakov, S., "On contact overheating and other computational difficulties of shock-capturing methods", Comput. Fluid Dyn. J., 2001, 10, 192.