Nonlinear Vibration Analysis of a Cantilever Beam with Nonlinear Geometric

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Abstract: Analyzing the nonlinear vibration of beams is one of the important issues in structural engineering. According to this, an impressive analytical method which is called modified iteration perturbation method (MIPM) is used to obtain the behavior and frequency of a cantilever beam with geometric nonlinear. This new method is combined by the Mickens and iteration methods. Moreover, this method does not require small parameter in the equation which is difficult to be found for nonlinear oscillation. The accuracy of the solution that is obtained by the use of MIPM has been shown graphically and compared with exact solution. Comparison shows that good adaptation is obtained and MIPM as a powerful method for solving the vibrational behavior of structures analytically.

Keywords: Nonlinear Vibration, Cantilever Beam, Nonlinearity Geometric, Modified Iteration Perturbation Method, Total Loss Coefficient

1. Introduction

Investigation the dynamic response of the systems such as air plane wings is vital aspect in mechanical engineering. Because the skin panel of these structures may experience serious structural problems such as thermal buckling or panel flutters, studying about them should be very carefully without missing any parameter. In this paper, the air plane wings are assumed as a cantilever beam with nonlinear geometry.

Recently, researchers have analyzied free vibration of nonlinear beams. Rehfield [1] proposed an approximate method for nonlinear vibration problems with material nonlinear effects for various boundary conditions. Sathyamoorthy [2] developed the work on finite element method for nonlinear beams under static and dynamic loads and classical methods for the analysis of beams with material, geometry and other types of nonlinearities. Klein [3] used finite element approach and Rayleigh-Ritz for analyzing the vibration of the tapered beams. A dynamic discretization technique was applied to calculate the natural frequencies of a non-rotating double tapered beam based on both the Euler-Bernoulli and Timoshenko Beam Theories by Downs [4]. Sato [5], improved the Ritz method to study a linearly tapered beam with ends restrained elastically against rotation and subjected to an axial force. Lau [6], used the exact method for studying on the free vibration of tapered beam with end mass.

The Green's function method in Laplace transform domain was used to study the vi-

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bration of general elastically restrained tapered beams by Lee et al [7]. It obtain the approximate fundamental solution by using a number of stepped beams to represent the tapered beam. Kopmaz et al. [8] considered different approaches to describing the relationship between the bending moment and curvature of an Euler-Bernoulli beam undergoing a large deformation. Then, in the case of a cantilevered beam subjected to a single moment at its free end, the difference between the linear and the nonlinear theories based on both the mathematical curvature and the physical curvature was shown.

In this paper, the vibration equation of a cantilever beam is solved by using of modified iteration perturbation method. It shows that the solution is quickly convergent and doesn't need to complicated calculations. Also, the equation derived from this problem is nonlinear. Usually, solving equations with nonlinear components have been one of the most time-consuming and difficult affairs among engineers. In recent years, much attention has been done to new approximate methods for solving nonlinear oscillation [9, 10]. Some of them are, frequency amplitude formulation [11], variational iteration [12, 13], iteration perturbation method [14-16], homotopy-perturbation [17, 18], parametrized-perturbation [19], max-min [20, 21, 22], newton harmonic balance method [23, 24], differential transformation method [25], energy balance [26, 27] and etc.

2. Materials and methods

2.1. Basic idea of modified iteration perturbation method

The modified iteration perturbation method (MIPM) is composed of the Mickens and Iteration methods. For the first time, it

was used by Marinca and Herisanu [28].

By considering the following equation as the general nonlinear oscillation:

$$\ddot{u} + \omega^2 u = f(u,\dot{u},\ddot{u}), \quad u(0) = A, \quad \dot{u}(0) = 0 \quad (1)$$

Eq. (1) is rewritten in the following from:

$$\ddot{u} + \Omega^2 u = u(\Omega^2 - \omega^2 + \frac{f(u,\dot{u},\ddot{u})}{u}) := ug(u,\dot{u},\ddot{u}), \quad (2)$$

where Ω is a priori unknown frequency of the periodic solution, u(t) being sought. The proposed iteration scheme is:

$$\ddot{u}_{n+1} + \Omega^2 u_{n+1} = u_{n-1}[g(u_{n-1},\dot{u}_{n-1},\ddot{u}_{n-1}) +$$
$$g_u(u_{n-1},\dot{u}_{n-1},\ddot{u}_{n-1})(u_n - u_{n-1}) + ...$$
$$g_{\dot{u}}(u_{n-1},\dot{u}_{n-1},\ddot{u}_{n-1})(\dot{u}_n - \dot{u}_{n-1}) + \quad (3)$$
$$g_{\ddot{u}}(u_{n-1},\dot{u}_{n-1},\ddot{u}_{n-1})(\ddot{u}_n - \ddot{u}_{n-1})], \quad n = 0,1,2,$$

where the inputs of starting functions are (Lim and Wu, 2002):

$$u_{-1}(t) = u_0(t) = A\cos\Omega t. \quad (4)$$

It is further required that for each n, the solution to Eq. (3) is to satisfy the initial conditions.

$$u_n(0) = A, \quad \dot{u}_n(0) = 0. \quad n = 1,2,3,... \quad (5)$$

Note that for given $u_{n-1}(t)$ and $u_n(t)$, Eq. (3) is second order inhomogeneous differential equation for $u_{n+1}(t)$. The right hand side of Eq. (3) can be expand into the following Fourier series:

$$u_{n-1}[g(u_{n-1},\dot{u}_{n-1},\ddot{u}_{n-1}) +$$
$$g_u(u_{n-1},\dot{u}_{n-1},\ddot{u}_{n-1})(u_n - u_{n-1}) + ...$$
$$g_{\dot{u}}(u_{n-1},\dot{u}_{n-1},\ddot{u}_{n-1})(\dot{u}_n - \dot{u}_{n-1}) +$$
$$g_{\ddot{u}}(u_{n-1},\dot{u}_{n-1},\ddot{u}_{n-1})(\ddot{u}_n - \ddot{u}_{n-1})] =$$
$$a_1(A,\Omega,\omega)\cos\Omega t + ... \quad (6)$$
$$\sum_{n=2}^{N} a_n(A,\Omega,\omega)\cos n\Omega t + b_1(A,\Omega,\omega)\sin\Omega t +$$
$$\sum_{n=2}^{N} b_n(A,\Omega,\omega)\sin n\Omega t,$$

where the cofficients $a_n(A,\Omega,\omega)$ and $b_n(A,\Omega,\omega)$ are known, and the integer N depends upon the function $g(u,\dot{u},\ddot{u})$ on the right hand side of Eq. (2) in Eq. (6), the requirement of no secular term needs.

$$a_1(A,\Omega,\omega)=0, \quad b_1(A,\Omega,\omega)=0. \quad (7)$$

The solution of Eq. (3) with the initial conditions of Eq. (5) is given by:

$$u_{n+1}(t)=A\cos\Omega t -$$
$$\sum_{n=2}^{N}\frac{a_n(A,\Omega,\omega)}{(n^2-1)\Omega^2}(\cos n\Omega t -\cos\Omega t)- \quad (8)$$
$$\sum_{n=2}^{N}\frac{b_n(A,\Omega,\omega)}{(n^2-1)\Omega^2}(\sin n\Omega t -\sin\Omega t).$$

Eq. (7) allows the determination of the frequency Ω as a function of A and ω. This procedure can be performed to any desired iteration step n. As in the following example will be shown, an excellent approximate analytical representation to the exact solution is obtained. This representation valid for small vlues as well as large values of the oscillation amplitude.

2.2. Application of the modified iteration perturbation method

The vibration equation of a cantilever beam with nonlinear geometry can be assumed as:

$$\ddot{u}+\varepsilon_1(u^2\ddot{u}+u\dot{u}^2)+u+\varepsilon_2 u^3=0 \quad (9)$$

where initial conditions are

$$u(0)=A, \quad \dot{u}(0)=0.$$

Eq. (9) is rewritten as follows:

$$\ddot{u}+\Omega^2 u =u(\Omega^2-1-\varepsilon_1(u\ddot{u}+\dot{u}^2)-\varepsilon_2 u^2) \quad (10)$$

The inputs of the starting function are $u_{-1}(t)=u_0(t)=A\cos\Omega t$ and

$$g(u,\dot{u},\ddot{u},t)=\Omega^2-1-\varepsilon_1(u\ddot{u}+\dot{u}^2)-\varepsilon_2 u^2.$$

So, the first iteration is given by the equation

$$\ddot{u}_1+\Omega^2 u_1 =A(\Omega^2-1+\frac{\varepsilon_1 A^2\Omega^2}{2}-\frac{3}{4}\varepsilon_2 A^2)\cos(\Omega t)+$$
$$(\frac{\varepsilon_1 A^3\Omega^2}{2}-\frac{\varepsilon_2 A^3}{4})\cos(3\Omega t) \quad (11)$$

In order to ensure that no secular terms appear in the next iteration, the resonance must be avoided. Therefore, the coefficient of $\cos\Omega t$ in Eq. (11) requires to be zero.

$$\Omega_1^2=\frac{4+3\varepsilon_2 A^2}{4+2\varepsilon_1 A^2} \quad (12)$$

So, from Eq. (11), with initial conditions in Eq. (5), the following first-order approximate solution will be obtained:

$$u_1(t)=A\cos(\Omega_1 t)+\frac{A^3(2\varepsilon_1\Omega_1^2-\varepsilon_2)}{32\Omega_1^2}(\cos(\Omega_1 t)-\cos(3\Omega_1 t)) \quad (13)$$

For n=1 into Eq. (3), with the initial functions in Eq. (4) and u_1 given by Eq. (13), the following differential equation is obtained for u_2:

$$\ddot{u}_2+\Omega^2 u_2 =A(\Omega^2-1+\frac{\varepsilon_1 A^2\Omega^2}{2}-\frac{3}{4}\varepsilon_2 A^2+$$
$$\frac{A^4(2\varepsilon_1^2\Omega^4-5\varepsilon_1\varepsilon_2\Omega^2+2\varepsilon_2^2)}{64\Omega^2}-...$$
$$\frac{A^4(2\varepsilon_1^2\Omega^2-\varepsilon_1\varepsilon_2)}{16})\cos(\Omega t)+$$
$$(\frac{\varepsilon_1 A^3\Omega^2}{2}-\frac{\varepsilon_2 A^3}{4}-\frac{A^5(2\varepsilon_1^2\Omega^4-5\varepsilon_1\varepsilon_2\Omega^2+2\varepsilon_2^2)}{128\Omega^2}-... \quad (14)$$
$$\frac{19A^5(2\varepsilon_1^2\Omega^2-\varepsilon_1\varepsilon_2)}{128})\cos(3\Omega t)+$$
$$(-\frac{A^5(2\varepsilon_1^2\Omega^4-5\varepsilon_1\varepsilon_2\Omega^2+2\varepsilon_2^2)}{128\Omega^2}-$$
$$\frac{5A^5(2\varepsilon_1^2\Omega^2-\varepsilon_1\varepsilon_2)}{128})\cos(5\Omega t)$$

Avoiding the presence of a secular term requires:

$$\Omega_2=\frac{\sqrt{A_1\times(A_2+\sqrt{A_3})}}{A_4}$$
$$A_1=32+16\varepsilon_1 A^2-3\varepsilon_1^2 A^4$$
$$A_2=64+48\varepsilon_2 A^2+\varepsilon_1\varepsilon_2 A^4 \quad (15)$$
$$A_3=4096+6144\varepsilon_2 A^2+128\varepsilon_1\varepsilon_2 A^4+1792\varepsilon_2^2 A^4-$$
$$160\varepsilon_1\varepsilon_2^2 A^6+49\varepsilon_1^2\varepsilon_2^2 A^8$$
$$A_4=64+32\varepsilon_1 A^2-6\varepsilon_1^2 A^4$$

Solving the Eq. (14) with the initial conditions in Eq. (5), the following equation will be achieved:

$$u_2(t) = A\cos(\Omega_2 t) + (-\frac{34}{768}\varepsilon_1^2 A^5 +$$

$$\frac{21}{768\Omega_2^2}\varepsilon_1\varepsilon_2 A^4 - \frac{24}{768\Omega_2^2}\varepsilon_2 A^2 - \frac{2}{768\Omega_2^4}\varepsilon_2^2 A^4 + ...$$

$$\frac{48}{768}\varepsilon_1 A^2)\cos(\Omega_2 t) + \frac{A^5(2\varepsilon_1\Omega_2^2 - \varepsilon_2)}{512\Omega_2^4} \qquad (16)$$

$$\times (10\varepsilon_1 A^2\Omega_2^2 - 16\Omega_2^2 - \varepsilon_2 A^2)\cos(3\Omega_2 t) + ...$$

$$\frac{A^5(4\varepsilon_1\Omega^2 - \varepsilon_2)}{3840\Omega_2^4}\cos(5\Omega_2 t)$$

3. Results and discussions

To illustrate and verify the accuracy of MIPM, it compared with published data and exact solution. The exact frequency for a cantilever beam with considering nonlinear geometry can be derived as follows [29]:

$$\omega_{Exact} = \frac{2\pi}{4\sqrt{2}A\int_0^{\pi/2} \frac{\sqrt{1+\varepsilon_1 A^2\cos^2 t}\sin t}{\sqrt{A^2(1-\cos^2 t)(\varepsilon_2 A^2\cos^2 t + \varepsilon_2 A^2 + 2)}}dt}$$
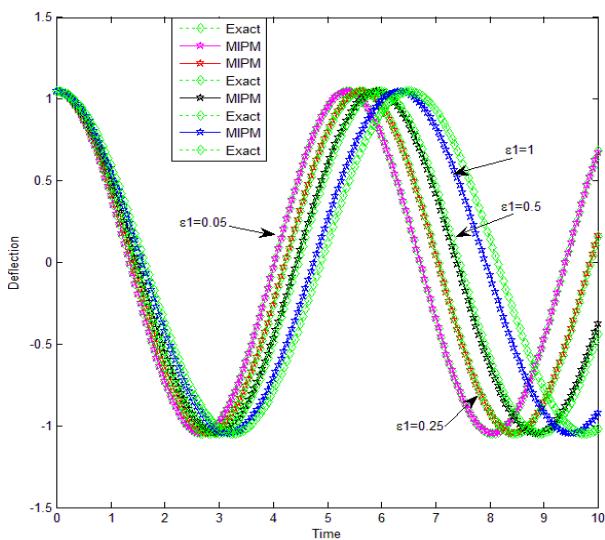
$$\qquad (17)$$

Figs. 1 and 2, show effects of constant parameters, ε_1 and ε_2, on deflection of the cantilever beam considering nonlinearity geometric. It has been shown that the results are in very good adaptation with those obtained by the exact solution.

In addition, stability has been investigated in Fig. 3 and the phase diagram has been presented. By using of this method the treatment of the system investigation and study the stability and response based on constant parameters is very easy. In addition, by changing the constant parameters, It doesn't need to change the solution or solve the problem again.

Also, The effect of small parameters ε_1 and ε_2 on the frequency corresponding to various initial amplitudes has been studied in Figs.4 and 5.

Finaly, The effect of different parameters $\varepsilon_1, \varepsilon_2$ and A are studied in Figs. 6 and 7 simultaneously.
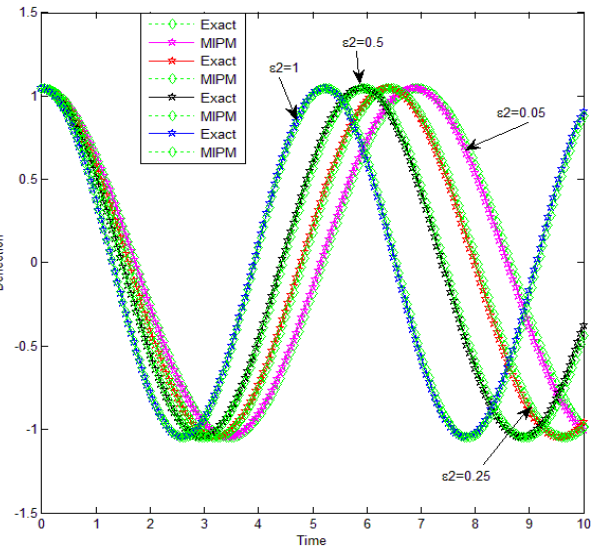

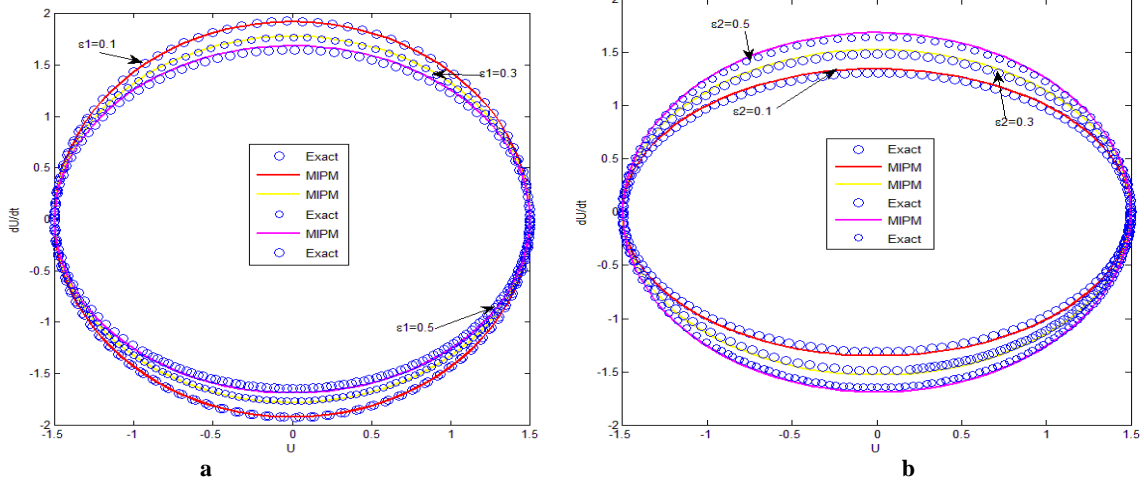**Fig. 1. The effect of ε_1 on position for $\varepsilon_2 = 1$ and $A = \frac{\pi}{3}$.**

Fig. 2. The effect of ε_2 on position for $\varepsilon_1 = 1$ and $A = \frac{\pi}{3}$.

a   **b**

Fig. 3. The phase plane to show the stability.
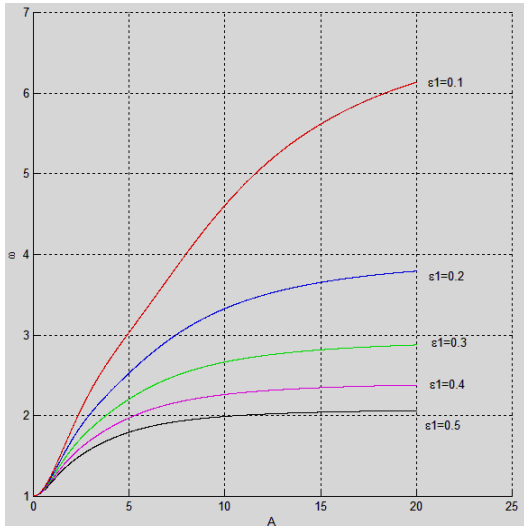
Fig. 4. Nonlinear frequency versus amplitude for various ε_1, **at** $\varepsilon_2 = 1$.
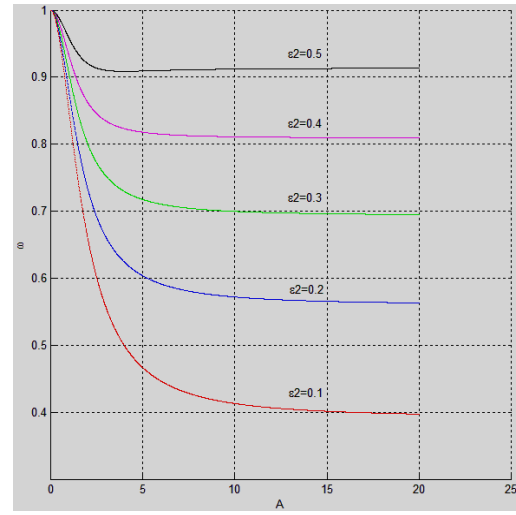
Fig. 5. **Nonlinear frequency versus amplitude for various** ε_2 , **at** $\varepsilon_1 = 1$.
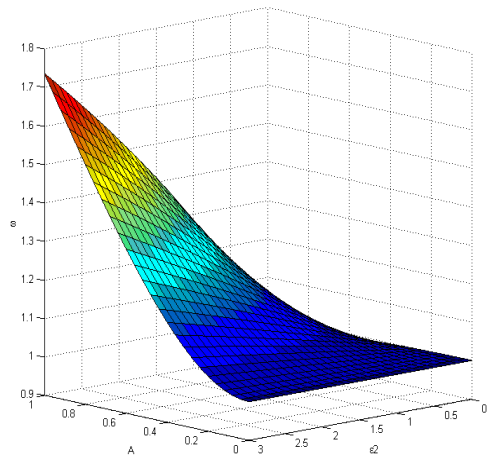

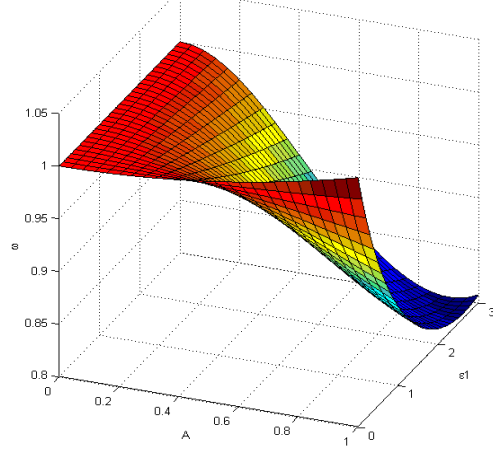**Fig. 6**. **Sensitivity analysis of frequency for** $\varepsilon_1 = 0.1$.

Fig. 7. Sensitivity analysis of frequency for $\varepsilon_2 = 0.1$.

4. Conclusions

In this study, MIPM is given to analyse a cantilevered beam with nonlinearity geometric. The accurate results with few iterations are obtained, that reduce the computational cost. The excellent agreement of the MIPM solutions with exact solution shows the reliability and the efficiency of the method. This method accelerated the convergence of the solution. It is predicted, that MIPM may find wide applications in engineering problems. However, it needs more research to know the optimized application of this method on engineering problems, especially mechanical affairs.

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