



AN EFFICIENT AUTO REDUNDANT TECHNIQUE FOR ANALYSIS OF SINGLE LAYER GRID WITH CURVED MEMBERS

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This paper presents an Auto Redundant Technique for analysis of grid with curved members. This technique is based on the force method, but in the technique choice of the redundant is completely eliminated. The analysis technique is found very effective, accurate and programmable. A comprehensive C++ program has been developed to compute internal forces at the end of each member of the grid for different load cases and their combinations. Presently in this paper analysis of grid is carried out with fixed support when it is subjected to concentrated point load, twisting moment, bending moment, full/partial uniformly distributed load and full/partial uniformly varying load. In this technique, any number of load cases can be accommodated without creating any additional node(s) on the member. The power of the analysis procedure is effectively demonstrated through the solution of one benchmark problem. The results obtained through the program for complementary load cases are compared with the results from analysis software and are found to match.

Keywords: auto redundant, grid, curved members, member end reactions, member flexibility matrix, structure flexibility matrix

1. Introduction

A grid is a plane structure assumed to be lying in the horizontal plane and all the forces are normal to the plane i.e. acting in the vertical direction. All the moments and couples have their

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vectors in the plane of the grid. This orientation of loading, results in twisting moment, bending moment and shear force in the members (Gimena et al., 2009). Member end reactions or internal forces at a section are twisting moment, bending moment and shear force.

The following assumptions are adopted for the planar curved beam element (Yau and Yang, 2008): (1) The material is elastic and homogeneous; (2) The cross-section of the curved beam is uniform; (3) Every cross-section remains rigid, i.e. undistorted, during deformation (Kapania and Li, 2003); (4) The length and radius of the curved beam are large in comparison with the cross-sectional dimensions of the beam; (5) The shearing deformation on the curved beam is negligible.

The two basic approaches, force method and displacement method in matrix method of analysis are well known as flexibility method and stiffness method respectively (Weaver and Gere, 1986). In conventional flexibility method which stems from consistent deformation method, redundant are identified and removed to make the structure statically determinate. Such a structure is known as released structure. The released structure is also used for computation of displacements due to unit redundant (Atluri et al., 2001), which is known as single release system. However this procedure leads to difficulties in automation the computations. For the implementation of computer program member flexibility method with mixed released system is suitable (Shaw, 1972) i.e. different released structures for the formulation of [AMQ] and {AML}.

Four different approaches of the force method of structural analysis are topological force methods, algebraic force methods, mixed algebraic-combinatorial force methods, and integrated force method (Kaveh et al., 2007, Kaveh and Koohestani, 2008, Kaveh and Daei, 2009, Kaveh and Nasab, 2010). The Integrated Force Method (IFM) was proposed by Patnaik (1973) for the analysis of discrete and continuous systems (KrishnamRaju and Nagabhushanam, 2000).

Felippa et al. (1997) have presented a direct flexibility method for the solution of finite element equations. This method is based on a decomposition of the finite element model into substructures, which may be reduced to individual elements. Substructures are preprocessed by the direct stiffness method to generate free-free flexibility matrices for floating substructures. Kemp (2002) has proposed a mixed flexibility method of analysis of framed structures, in which the element end moments and independent modes of sway deflection are taken as unknowns. Sedagati (2005) has derived compatibility matrix directly by utilizing a displacement-deformation relationship and the Single Value Decomposition Technique. Gimena et al. (2008a, b, c) stiffness matrix and the equivalent load vector to determine the internal forces and displacements in a 3D-curved beam element defined by its parametric equations with varying cross-section area and generalized loads applied.

This paper presents an innovative and efficient Auto-Redundant technique for analysis of the single layer grid. In this technique, the actual grid structure is divided into sub-grids. The sub-

grid may be supported sub-grid or unsupported sub-grid. The sub-grids are processed by the flexibility method to generate structure flexibility matrices corresponding to members of the grid structure; and consequently structure flexibility matrix of whole grid structure is obtained.

The flexibility approach requires calculation of displacements corresponding to the redundant for released structure due to actual loading (Chen, 2003 and Dahlberg, 2004). Equivalent joint loads are needed for the analysis of the structures, either by classical method or matrix method (Pippard and Baker, 1957). Hansora et al., in press, have formulated equations in matrices form to evaluate fixed end actions for a beam curved in plan, subjected to different types of loads.

2. Formulation

The formulations of the Equations are based on the consideration that the curve runs in clockwise direction only and j-end & k-end are rear end & forward end respectively (Figure 1). Following are the basic Equations of flexibility method:

$$\{Q\} = [FS]^{-1}(\{DQ\} - \{DQL\}) \quad (1)$$

$$\{AM\} = [AMQ]\{Q\} + \{AML\} \quad (2)$$

The coefficients of [FS] and [DQL] are determined by unit load method. Figure 2 shows a typical grid structure of fixed ends. The grid is decomposed into the group of members called sub-grid. Thus contribution of a typical member i considering all sub-grids can be expressed as:

$$[FSm]_i = [AMQ]_i^T [FMg]_i [AMQ]_i \quad (3)$$

$$\{DQLm\}_i = [AMQ]_i^T [FMg]_i \{ \{AML\}_i + \{-AM_F\}_i \} \quad (4)$$

where $\{-AM_F\}$ represents the equivalent joint loads.

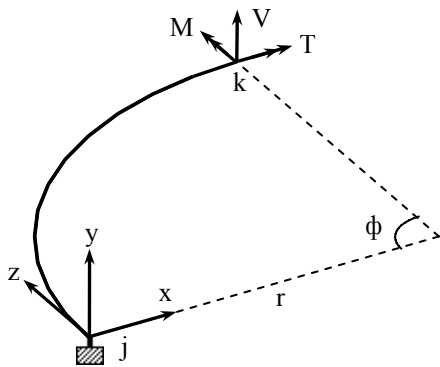


Figure 1. Basic curved member

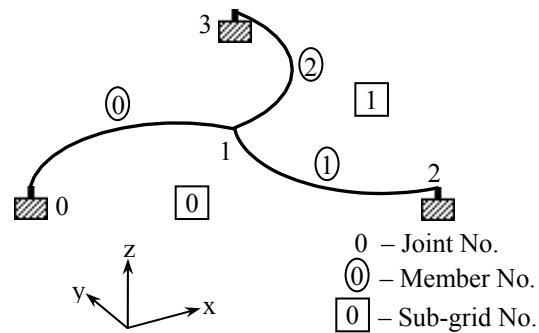


Figure 2. Typical grid

2.1. Formulation of [FMg]

Following is the member flexibility matrix in the member direction (Ghali and Neville, 1989):

$$[FM]_i = \begin{bmatrix} FM_{11} & FM_{12} & FM_{13} \\ FM_{21} & FM_{22} & FM_{23} \\ FM_{31} & FM_{32} & FM_{33} \end{bmatrix} \quad (5)$$

$$FM_{11} = r \left[\frac{1}{EI} \left(\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right) + \frac{1}{GJ} \left(\frac{\phi}{2} + \frac{\sin 2\phi}{4} \right) \right]; \quad FM_{12} = r \left[\frac{1}{EI} \left(-\frac{\sin^2 \phi}{2} \right) + \frac{1}{GJ} \left(\frac{\sin^2 \phi}{2} \right) \right]$$

$$FM_{13} = r^2 \left[\frac{1}{EI} \left(\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right) + \frac{1}{GJ} \left(\frac{\phi}{2} + \frac{\sin 2\phi}{4} - \sin \phi \right) \right]$$

$$FM_{21} = FM_{12}; \quad FM_{22} = r \left[\frac{1}{EI} \left(\frac{\phi}{2} + \frac{\sin 2\phi}{4} \right) + \frac{1}{GJ} \left(\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right) \right]$$

$$FM_{23} = r^2 \left[\frac{1}{EI} \left(-\frac{\sin^2 \phi}{2} \right) - \frac{1}{GJ} \left(1 - \frac{\sin^2 \phi}{2} - \cos \phi \right) \right]$$

$$FM_{31} = FM_{13}; \quad FM_{32} = FM_{23}; \quad FM_{33} = r^3 \left[\frac{1}{EI} \left(\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right) + \frac{1}{GJ} \left(\frac{3\phi}{2} + \frac{\sin 2\phi}{4} - 2 \sin \phi \right) \right]$$

To transform any relation either from member directions to structure directions or vice-versa, rotation matrix is required. Following are the rotation matrices of j-end and k-end respectively.

$$[Rj]_i = \begin{bmatrix} \cos \theta_j & \sin \theta_j & 0 \\ -\sin \theta_j & \cos \theta_j & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [Rk]_i = \begin{bmatrix} \cos \theta_k & \sin \theta_k & 0 \\ -\sin \theta_k & \cos \theta_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (6)$$

$$[FMg]_i = [Rk]_i^T [FM]_i [Rk]_i \quad (7)$$

2.2. Formulation of [AMQ]

A simple grid is shown in Figure 3 with control parameters in Figure 4. The redundant for the grid are chosen as twisting moment (T), bending moment (M) and vertical force (V) as shown in Figure 5. These redundant are applied one by one at the origin (0, 0) as shown in Figures 6-8 (Hansora et al., 2009 and Hansora et al., 2010). The member end reactions for each member are determined in the structure directions at the rear end (j-end) and forward end (k-end), due to application of unit value of redundant one by one. Following relationship of coordinates is assumed for formulation: $x_0 < x_1 > x_2 > x_3$ and $y_0 < y_1 < y_2 < y_3$

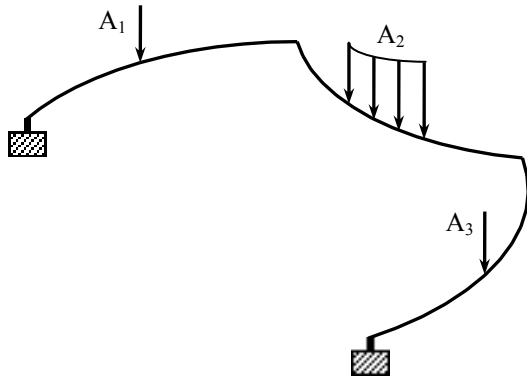


Figure 3. Typical grid

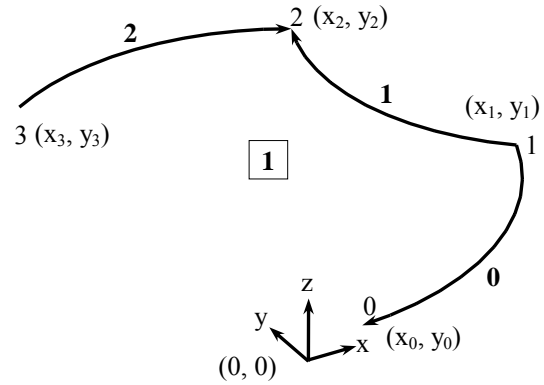


Figure 4. Control parameters

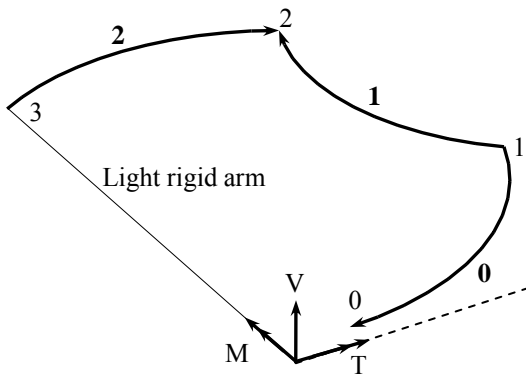


Figure 5. Redundant

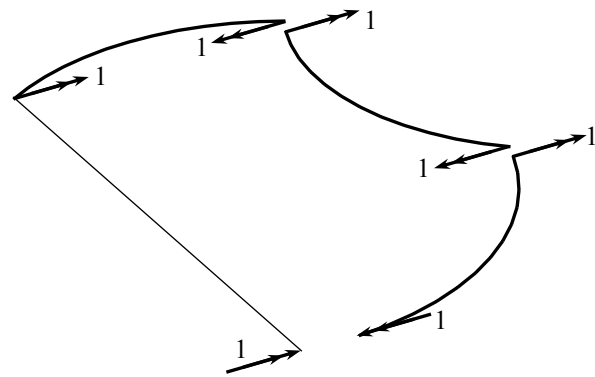


Figure 6. Redundant T=1

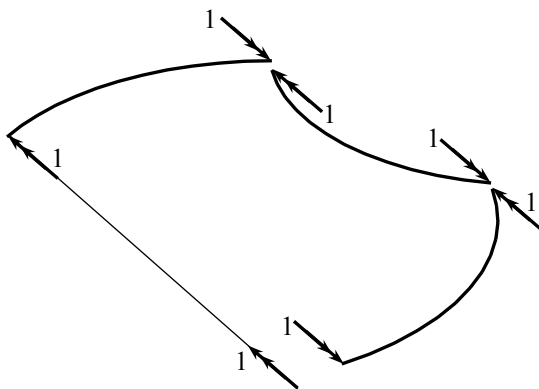


Figure 7. Redundant M=1

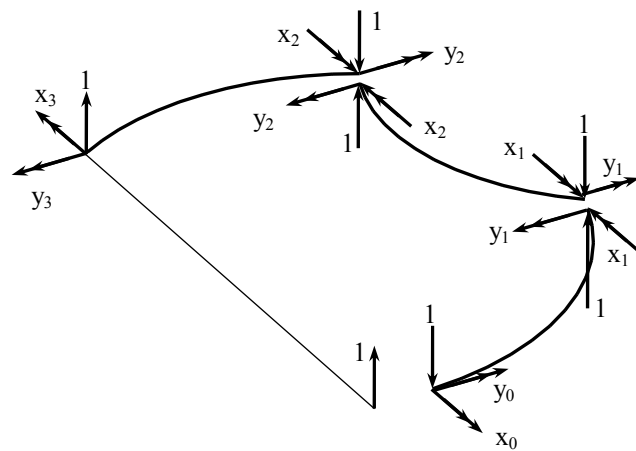


Figure 8. Redundant V=1

The member end reactions at forward end (k-end) of the member due to unit value of redundant can be expressed as per following generalized equation:

$$[AMQ]_i = [BF]_i = -c \begin{bmatrix} 1 & 0 & -yk \\ 0 & 1 & xk \\ 0 & 0 & 1 \end{bmatrix} \quad (8)$$

Similarly, for the rear end (j-end)

$$[AMQ]_j = [BR]_j = c \begin{bmatrix} 1 & 0 & -yj \\ 0 & 1 & xj \\ 0 & 0 & 1 \end{bmatrix} \quad (9)$$

The constant c depends on the direction of forward end with respect to the sub-grid. c=1, if the member arrow gives clockwise direction about sub-grid no. and vice-versa.

2.3. Formulation of [AML] and [AM_F]

As described in Section 1, different released system can be chosen for the [AMQ] and {AML}. The member end reactions at forward end of the member are denoted by {BLF}.

$$\{AML\}_i = \{BLF\}_i \quad (10)$$

Similarly, the member end reactions at rear end of the member are denoted by {BLR}.

$$\{AML\}_j = \{BLR\}_j \quad (11)$$

The equivalent joint load at the forward end (k-end) in structure directions is denoted by {ELF}_i.

$$\{-AM_F\}_i = \{ELF\}_i \quad (12)$$

The combination of member end reactions in the released structure and equivalent joint load at the forward end (k-end) is denoted by {BLC}_i.

$$\{BLC\}_i = \{BLF\}_i + \{ELF\}_i \quad (13)$$

2.4. Formulation of [FS_m] and {DQL_m}

Substituting [AMQ] from Equation 8 into Equation 3; since the free end of the cantilever is taken as forward end, therefore,

$$[FS_m]_i = [BF]_i^T [FMg]_i [BF]_i \quad (14)$$

Substituting [AMQ] from Equation 8, {AML} from Equation 10 and {-AM_F} from Equation 12 into Equation 4:

$$\begin{aligned} \{DQL_m\}_i &= [BF]_i^T [FMg]_i \{ \{BLF\}_i + \{ELF\}_i \} \\ \therefore \{DQL_m\}_i &= [BF]_i^T [FMg]_i \{BLC\}_i \end{aligned} \quad (15)$$

2.5. Final Member End Reactions {AMR} and {AMF}

Final member end reactions at forward end of the member is obtained using Equation 2 as,

$$\{AMF\}_i = [Rk]_i \{ \{BF\}_i \{Qm\}_i + \{BLF\}_i \} \quad (16)$$

Similarly, final member end reactions at rear end of the member is:

$$\{AMR\}_i = [Rj]_i \{ \{BR\}_i \{Qm\}_i + \{BLR\}_i \} \quad (17)$$

Here, $\{Qm\}_i$, $\{BLR\}_i$ and $\{BLF\}_i$ are as per structure directions. To convert member end reactions in member directions, rotation matrices $[Rj]_i$ and $[Rk]_i$ are used in Equations 16-17.

3. Numerical Example

The member end reactions for the typical grid, shown in Figure 9 have been evaluated using Equations 5-17.

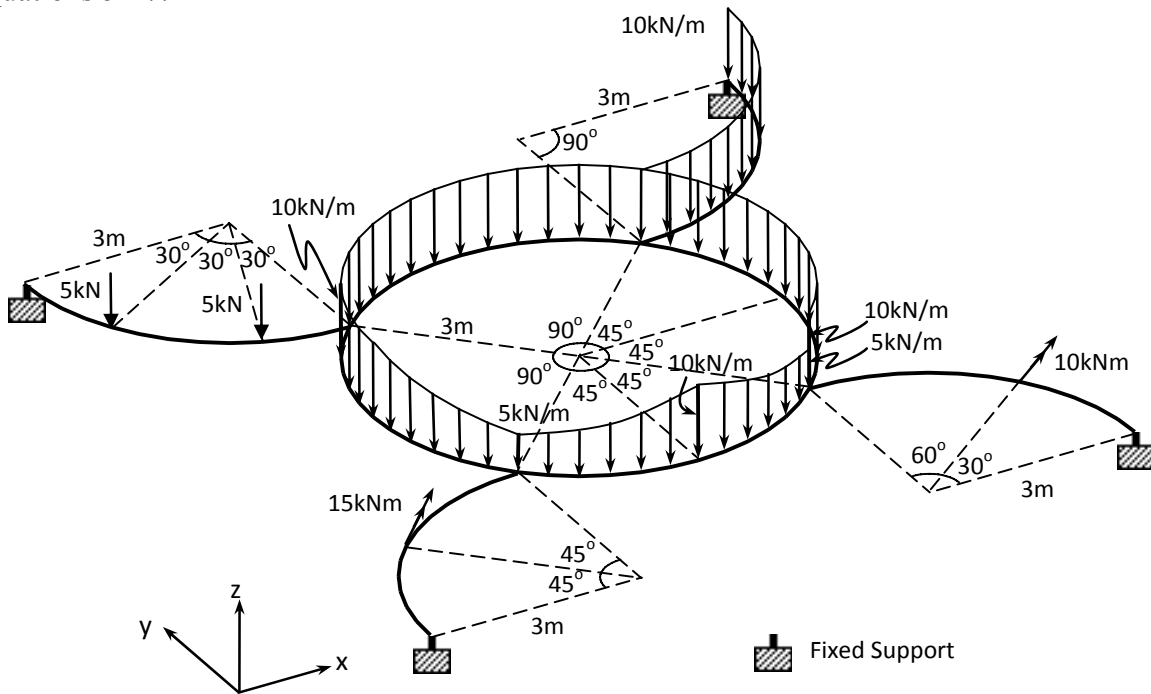


Figure 9. Typical grid

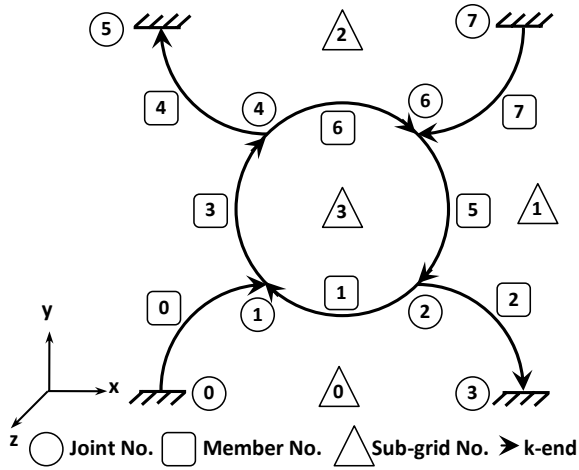


Figure 10. Control parameters

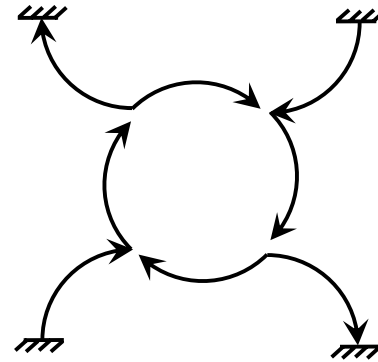


Figure 11. Released structures for {AML}

Table 1. Input data

Cross-section of beam for member no.1, 3, 5 & 6: width $b=0.230\text{m}$, depth $d=0.450\text{m}$
 Cross-section of beam for member no.0, 2, 4 & 7: width $b=0.300\text{m}$, depth $d=0.600\text{m}$
 $E=21.7185 \times 10^6 \text{ kN/m}^2$, $G=9.2812 \times 10^6 \text{ kN/m}^2$
 No. of Joint = 8, No. of Member = 8, No. of sub-grid = 4

Joint No.	x (m)	y (m)	Member No.	j-end (Rear End)	k-end (Forward End)	r (m)	ϕ	θ_j	θ_k	Nos. of load cases
0	0.000	0.000	0	0	1	3.0	90°	90°	0°	1
1	3.000	3.000	1	2	1	3.0	90°	225°	135°	2
2	7.243	3.000	2	2	3	3.0	90°	0°	270°	1
3	10.243	0.000	3	1	4	3.0	90°	135°	45°	1
4	3.000	7.243	4	4	5	3.0	90°	180°	90°	2
5	0.000	10.243	5	6	2	3.0	90°	315°	225°	1
6	7.243	7.243	6	4	6	3.0	90°	45°	315°	1
7	10.243	10.243	7	7	6	3.0	90°	270°	180°	1

Sub-Grid No.	Members	Sub-Grid 0	Sub-Grid 1	Sub-Grid 2	Sub-Grid 3
0	0, 1, 2	Member c	Member c	Member c	Member c
1	2, 5, 7	0 1	2 -1	4 1	1 1
2	4, 6, 7	1 -1	5 -1	6 -1	3 1
3	1, 3, 5, 6	2 1	7 -1	7 1	5 1 6 1

Member No.	Loading Detail*
0	$T=15\text{kNm}$, $\alpha=45^\circ$
1	$w1=5\text{kN/m}$, $w2=10\text{kN/m}$, $\alpha1=0^\circ$, $\alpha2=45^\circ$ & $w1=10\text{kN/m}$, $w2=5\text{kN/m}$, $\alpha1=45^\circ$, $\alpha2=90^\circ$
2	$M=10\text{kNm}$, $\alpha=30^\circ$
3	$w1=5\text{kN/m}$, $w2=10\text{kN/m}$, $\alpha1=0^\circ$, $\alpha2=90^\circ$
4	$P=5\text{kN}$, $\alpha=30^\circ$ & $P=5\text{kN}$, $\alpha=60^\circ$
5	$w1=10\text{kN/m}$, $w2=10\text{kN/m}$, $\alpha1=0^\circ$, $\alpha2=90^\circ$
6	$w1=10\text{kN/m}$, $w2=10\text{kN/m}$, $\alpha1=0^\circ$, $\alpha2=90^\circ$
7	$w1=10\text{kN/m}$, $w2=10\text{kN/m}$, $\alpha1=0^\circ$, $\alpha2=90^\circ$

* refer Figure 12 for notations

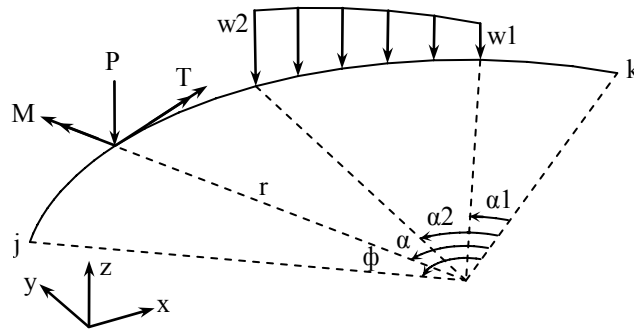


Figure 12. Loading notations

In this application, cross-section of the grid elements is rectangular. The torsional moment of inertia for the rectangular cross-section is given by an approximate formula as (Tufekci and Dogruer, 2006):

$$J = \frac{db^3}{3} \left[1 - 0.63 \frac{b}{d} \left(1 - \frac{b^4}{12d^4} \right) \right] \quad \text{for } d \geq b \quad (18)$$

Solution procedure is to be given in Appendix.

4. Validation

A comprehensive C++ program has been prepared to analyze grid subjected to combination of different type of loads. Results obtained through program are validated by comparing with the results from analysis software STAAD Pro.

Table 2. Load data

Member No.	Load: A (As per Table 1)	Load: B (Complementary to Load:A)	Load: C = Load: A + Load: B
0	T=15kNm, $\alpha=45^\circ$	---	T=15kNm, $\alpha=45^\circ$
1	w1=5kN/m, w2=10kN/m, $\alpha1=0^\circ$, $\alpha2=45^\circ$ & w1=10kN/m, w2=5kN/m, $\alpha1=45^\circ$, $\alpha2=90^\circ$	w1=5kN/m, w2=0kN/m, $\alpha1=0^\circ$, $\alpha2=45^\circ$ & w1=0kN/m, w2=5kN/m, $\alpha1=45^\circ$, $\alpha2=90^\circ$	w1=10kN/m, w2=10kN/m, $\alpha1=0^\circ$, $\alpha2=45^\circ$ & w1=10kN/m, w2=10kN/m, $\alpha1=45^\circ$, $\alpha2=90^\circ$
2	M=10kNm, $\alpha=30^\circ$	---	M=10kNm, $\alpha=30^\circ$
3	w1=5kN/m, w2=10kN/m, $\alpha1=0^\circ$, $\alpha2=90^\circ$	w1=5kN/m, w2=0kN/m, $\alpha1=0^\circ$, $\alpha2=90^\circ$	w1=10kN/m, w2=10kN/m, $\alpha1=0^\circ$, $\alpha2=90^\circ$
4	P=5kN, $\alpha=30^\circ$ & P=5kN, $\alpha=60^\circ$	---	P=5kN, $\alpha=30^\circ$ & P=5kN, $\alpha=60^\circ$
5	w1=10kN/m, w2=10kN/m, $\alpha1=0^\circ$, $\alpha2=90^\circ$	---	w1=10kN/m, w2=10kN/m, $\alpha1=0^\circ$, $\alpha2=90^\circ$
6	w1=10kN/m, w2=10kN/m, $\alpha1=0^\circ$, $\alpha2=90^\circ$	---	w1=10kN/m, w2=10kN/m, $\alpha1=0^\circ$, $\alpha2=90^\circ$
7	w1=10kN/m, w2=10kN/m, $\alpha1=0^\circ$, $\alpha2=90^\circ$	---	w1=10kN/m, w2=10kN/m, $\alpha1=0^\circ$, $\alpha2=90^\circ$

Note: All other data as per Table 1

Table 3. Comparison of results

Member End Actions	Member No.	Results of the Program		Sum of Results of	Results from	Member No.	Results of the Program		Sum of Results of	Results from
		Load: A	Load: B	Load: A & B	STAAD Load: C		Load: A	Load: B	Load: A & B	STAAD Load: C
T _j	0	-82.390	-20.209	-102.599	-102.597	1	29.233	4.614	33.847	33.849
M _j		-98.135	-29.976	-128.111	-128.109		11.760	0.443	12.203	12.205
V _j		36.071	11.303	47.374	47.374		17.213	7.227	24.440	24.440
T _k		-0.530	3.934	3.404	3.406		26.103	8.049	34.151	34.153
M _k		-36.429	-13.700	-50.129	-50.131		-12.139	-5.028	-17.167	-17.169
V _k		-36.071	-11.303	-47.374	-47.374		18.130	4.554	22.684	22.684
T _j		2	-0.059	4.177	4.118		4.121	3	-0.718	4.420
M _j	44.172		4.701	48.873	48.875	-13.995	-1.878		-15.872	-15.871
V _j	-45.184		-6.553	-51.737	-51.737	17.941	6.749		24.690	24.690
T _k	-86.381		-14.956	-101.338	-101.336	7.447	-0.621		6.826	6.827
M _k	126.952		15.481	142.433	142.426	8.248	3.980		12.228	12.227
V _k	45.184		6.553	51.737	51.737	17.402	5.032		22.434	22.434
T _j	4		14.655	-2.992	11.663	11.665	5		13.026	-1.951
M _j		40.709	5.590	46.299	46.300	-4.126		-0.360	-4.486	-4.485
V _j		-44.097	-4.456	-48.554	-48.554	19.153		0.674	19.827	19.827
T _k		-101.093	-7.779	-108.872	-108.868	1.959		1.664	3.623	3.625
M _k		138.128	16.360	154.488	154.486	19.516		-0.072	19.444	19.442
V _k		54.097	4.456	58.554	58.554	27.971		-0.674	27.297	27.297
T _j		6	31.701	2.458	34.159	34.160		7	-115.082	-4.103
M _j	10.175		2.088	12.263	12.264	-190.357	-5.646		-196.003	-195.999
V _j	26.695		-0.576	26.120	26.120	86.705	1.250		87.955	87.955
T _k	38.889		0.361	39.250	39.251	18.386	-1.896		16.490	16.492
M _k	-21.787		-0.731	-22.518	-22.519	-55.033	0.353		-54.680	-54.682
V _k	20.429		0.576	21.004	21.004	-39.581	-1.250		-40.831	-40.832

T & M in kNm and V in kN

A comprehensive C++ program developed for the technique is giving accurate results of member end reactions. The very small amount of difference is only due to truncation error.

5. Conclusion

In Auto-Redundant Technique, there is no need to choose redundant to generate {AMQ} as in the flexibility method. Also, there is no need to construct flexibility matrices of the individual members of the structure in a block diagonal form i.e. unassembled flexibility matrix. A mechanized member oriented computation is possible due to the mixed released system. Therefore, the mixed release system has been found definitely superior to single release system. The technique, for the analysis of grid, is applicable for most of practical load cases and its combination and fixed support condition. The results, obtained using program, are validated by comparing results obtained through analysis software. The algebraic sum of results obtained for two complementary uniformly varying load cases are validated by analysis software. The technique can be extended for other support conditions. The technique can be further extended to grid structure consisting of straight and curved members.

Notations

E	: Young's modulus of elasticity of material
G	: modulus of rigidity of material
I	: moment of inertia
J	: torsional moment of inertia
r	: radius of curved member of grid
ϕ	: angle between the j-end and k-end of the member
θ_j	: angle between the global x-y axis and local x-y axis at j-end
θ_k	: angle between the global x-y axis and local x-y axis at k-end
{Q}	: unknown redundant ($nrd \times 1$)
{AM}	: member end reactions in actual structure
{AM _F }	: fixed end reactions due to given loading for the member
[FS]	: structure flexibility matrix ($nrd \times nrd$)
{DQ}	: given displacement corresponding to redundant in given structure ($nrd \times 1$)
{DQL}	: displacements corresponding to the redundant due to actual loading in released structure ($nrd \times 1$)
[FM] _i	: member flexibility matrix (in member direction) for member i (3×3)
[FMg] _i	: member flexibility matrix (in structure direction) for member i (3×3)
[FSm] _i	: structure flexibility matrix corresponding to the member i, i.e. the displacements corresponding to the redundant due to unit value of the redundant in released structure for member i ($nrdm \times nrdm$)
{DQLm} _i	: displacements corresponding to the redundant due to actual loading in released structure for the member i ($nrdm \times 1$)
[AMQ] _i	: member end actions in release structure due to unit value of redundant {Q} for the member i
{AML} _i	: member end reactions due to loads for member i
{Qm} _i	: redundant corresponding to the member i ($nrdm \times 1$)
[R _j] _i & [R _k] _i	: rotation matrices of j-end and k-end for member i, respectively (3×3)
[BR] _i & [BF] _i	: coordinate matrices of j-end and k-end, respectively ($3 \times 3nrm$)

- $\{ELF\}_i$: fixed end actions (equivalent joint loads) at forward end for the member i (3×1)
- $\{BLF\}_i$ & $\{BLR\}_i$: member end reactions at forward end & rear end respectively, in released structure for member i (3×1)
- $\{BLC\}_i$: combination of member end reactions in the released structure and equivalent joint load at forward end of member i (3×1)
- $\{AMF\}_i$ & $\{AMR\}_i$: member end reactions at rear end (T_j, M_j & V_j) and forward end (T_k, M_k & V_k) respectively, for member i (3×1)
- T_j, M_j & V_j : Twisting moment, bending moment and shear force at j -end, respectively
- T_k, M_k & V_k : Twisting moment, bending moment and shear force at k -end, respectively
- nrd : total nos. of redundant
- nrdm : nos. of redundant corresponding to the member, considering all sub-grid
- nrm : no. of time the member repeated, considering all sub-grid

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Appendix

Redundant nos. for sub-grid 0 = 0, 1, 2

Redundant nos. for sub-grid 1 = 3, 4, 5

Redundant nos. for sub-grid 2 = 6, 7, 8

Redundant nos. for sub-grid 3 = 9, 10, 11

Calculation for member No. 0

Redundant nos. corresponding to the member 0 = 0, 1, 2

$$\begin{aligned}
 [FM]_0 &= 10^{-6} \begin{bmatrix} 88.556 & 30.797 & 4.148 \\ 30.797 & 88.556 & -169.13 \\ 4.148 & -169.13 & 460.269 \end{bmatrix} [Rj]_0 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} [Rk]_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 [FMg]_0 &= 10^{-6} \begin{bmatrix} 88.556 & 30.797 & 4.148 \\ 30.797 & 88.556 & -169.13 \\ 4.148 & -169.13 & 460.269 \end{bmatrix} [BR]_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [BF]_0 = \begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix} \\
 [FSm]_0 &= 10^{-6} \begin{bmatrix} 88.556 & 30.797 & -169.13 \\ 30.797 & 88.556 & 4.148 \\ -169.13 & 4.148 & 460.269 \end{bmatrix} \{BLR\}_0 = \begin{bmatrix} 179.090 \\ -96.150 \\ 35.343 \end{bmatrix} \\
 \{BLF\}_0 &= \begin{bmatrix} -83.668 \\ -20.486 \\ -35.343 \end{bmatrix} \{ELF\}_0 = \begin{bmatrix} 2.476 \\ 8.131 \\ 0 \end{bmatrix} \{BLC\}_1 = \begin{bmatrix} -81.193 \\ -12.355 \\ -35.343 \end{bmatrix} \{DQLm\}_0 = 10^{-6} \begin{bmatrix} 7717.195 \\ -2382.983 \\ -15786.02 \end{bmatrix}
 \end{aligned}$$

Calculation for member No. 1

Redundant nos. corresponding to the member 1 = 0, 1, 2, 9, 10, 11

$$\begin{aligned}
 [FM]_1 &= 10^{-6} \begin{bmatrix} 266.723 & 90.713 & 18.625 \\ 90.713 & 266.723 & -509.403 \\ 18.625 & -509.403 & 1394.18 \end{bmatrix} [Rj]_1 = \begin{bmatrix} -0.707 & -0.707 & 0 \\ 0.707 & -0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} [Rk]_1 = \begin{bmatrix} -0.707 & 0.707 & 0 \\ -0.707 & -0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 [FMg]_1 &= 10^{-6} \begin{bmatrix} 357.436 & 0 & 347.032 \\ 0 & 176.009 & 373.372 \\ 347.032 & 373.372 & 1394.18 \end{bmatrix} \\
 [BR]_1 &= \begin{bmatrix} -1 & 0 & 3 & 1 & 0 & -3 \\ 0 & -1 & -7.243 & 0 & 1 & 7.243 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} [BF]_1 = \begin{bmatrix} 1 & 0 & -3 & -1 & 0 & 3 \\ 0 & 1 & 3 & 0 & -1 & -3 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \\
 [FSm]_1 &= 10^{-6} \begin{bmatrix} 357.436 & 0 & -725.276 & -357.436 & 0 & 725.276 \\ 0 & 176.009 & 901.399 & 0 & -176.009 & -901.399 \\ -725.276 & 901.399 & 6353.221 & 725.276 & -901.399 & -6353.221 \\ -357.436 & 0 & 725.276 & 357.436 & 0 & -725.276 \\ 0 & -176.009 & -901.399 & 0 & 176.009 & 901.399 \\ 725.276 & -901.399 & -6353.221 & -725.276 & 901.399 & 6353.221 \end{bmatrix} \{BLR\}_1 = \begin{bmatrix} -22.229 \\ 74.974 \\ 35.343 \end{bmatrix} \\
 \{BLF\}_1 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \{ELF\}_1 = \begin{bmatrix} 11.115 \\ 13.095 \\ -17.672 \end{bmatrix} \{BLC\}_1 = \begin{bmatrix} 11.115 \\ 13.095 \\ -17.672 \end{bmatrix} \{DQLm\}_1 = 10^{-6} \begin{bmatrix} -2159.84 \\ -4293.214 \\ -22290.98 \\ 2159.84 \\ 4293.214 \\ 22290.98 \end{bmatrix}
 \end{aligned}$$

Calculation for member No. 2

Redundant nos. corresponding to the member 2 = 0, 1, 2, 3, 4, 5

$$\begin{aligned}
 [FM]_2 &= [FM]_0 [Rj]_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [Rk]_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} [FMg]_2 = 10^{-6} \begin{bmatrix} 88.556 & -30.797 & -169.13 \\ -30.797 & 88.556 & -4.148 \\ -169.13 & -4.148 & 460.269 \end{bmatrix} \\
 [BR]_2 &= \begin{bmatrix} 1 & 0 & -3 & -1 & 0 & 3 \\ 0 & 1 & 7.243 & 0 & -1 & -7.243 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} [BF]_2 = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -10.243 & 0 & 1 & 10.243 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \\
 [FSm]_2 &= 10^{-6} \begin{bmatrix} 88.556 & -30.797 & -484.570 & -88.556 & 30.797 & 484.570 \\ -30.797 & 88.556 & 902.901 & 30.797 & -88.556 & -902.901 \\ -484.570 & 902.901 & 9665.866 & 484.570 & -902.901 & -9665.866 \\ -88.556 & 30.797 & 484.570 & 88.556 & -30.797 & -484.570 \\ 30.797 & -88.556 & -902.901 & -30.797 & 88.556 & 902.901 \\ 484.570 & -902.901 & -9665.866 & -484.570 & 902.901 & 9665.866 \end{bmatrix} \{BLR\}_2 = \begin{Bmatrix} 22.229 \\ -74.974 \\ -35.343 \end{Bmatrix} \\
 \{BLF\}_2 &= \begin{Bmatrix} 75.14 \\ 176.002 \\ 35.343 \end{Bmatrix} \{ELF\}_2 = \begin{Bmatrix} 0.357 \\ 0.310 \\ -2.893 \end{Bmatrix} \{BLC\}_2 = \begin{Bmatrix} 75.496 \\ 176.313 \\ 32.450 \end{Bmatrix} \{DQLm\}_2 = 10^{-6} \begin{Bmatrix} 4232.509 \\ -13153.91 \\ -136166.6 \\ -4232.509 \\ 13153.91 \\ 136166.6 \end{Bmatrix}
 \end{aligned}$$

Calculation for member No. 3

Redundant nos. corresponding to the member 3 = 9, 10, 11

$$\begin{aligned}
 [FM]_3 &= [FM]_I [Rj]_3 = \begin{bmatrix} -0.707 & 0.707 & 0 \\ -0.707 & -0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} [Rk]_3 = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 [FMg]_3 &= 10^{-6} \begin{bmatrix} 176.009 & 0 & 373.372 \\ 0 & 357.436 & -347.032 \\ 373.372 & -347.032 & 1394.18 \end{bmatrix} [BR]_3 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} [BF]_3 = \begin{bmatrix} -1 & 0 & 7.243 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix} \\
 [FSm]_3 &= 10^{-6} \begin{bmatrix} 176.009 & 0 & -901.399 \\ 0 & 357.436 & 725.276 \\ -901.399 & 725.276 & 6353.22 \end{bmatrix} \\
 \{BLR\}_3 &= \begin{Bmatrix} 83.668 \\ 20.486 \\ 35.343 \end{Bmatrix} \{BLF\}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \{ELF\}_3 = \begin{Bmatrix} 12.724 \\ -10.941 \\ -20.061 \end{Bmatrix} \{BLC\}_3 = \begin{Bmatrix} 12.724 \\ -10.941 \\ -20.061 \end{Bmatrix} \{DQLm\}_3 = 10^{-6} \begin{Bmatrix} 5250.76 \\ -3051.177 \\ -27761.63 \end{Bmatrix}
 \end{aligned}$$

Calculation for member No. 4

Redundant nos. corresponding to the member 4 = 6, 7, 8

$$[FM]_4 = [FM]_0 [Rj]_4 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [Rk]_4 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} [FMg]_4 = 10^{-6} \begin{bmatrix} 88.556 & -30.797 & 169.13 \\ -30.797 & 88.556 & 4.148 \\ 169.13 & 4.148 & 460.269 \end{bmatrix}$$

$$[BR]_4 = \begin{bmatrix} 1 & 0 & -7.243 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} [BF]_4 = \begin{bmatrix} -1 & 0 & 10.243 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix} [FSm]_4 = 10^{-6} \begin{bmatrix} 88.556 & 30.797 & -737.919 \\ -30.797 & 88.556 & 319.589 \\ -737.919 & 319.589 & 6286.175 \end{bmatrix}$$

$$\{BLR\}_4 = \begin{Bmatrix} -27.314 \\ 99.965 \\ -47.124 \end{Bmatrix} \{BLF\}_4 = \begin{Bmatrix} -134.548 \\ -250.846 \\ 57.124 \end{Bmatrix} \{ELF\}_4 = \begin{Bmatrix} 6.019 \\ 0.528 \\ -5 \end{Bmatrix} \{BLC\}_4 = \begin{Bmatrix} -128.529 \\ -250.318 \\ 52.124 \end{Bmatrix}$$

$$\{DQLm\}_4 = 10^{-6} \begin{Bmatrix} -5142.644 \\ 17992.7 \\ 51459.75 \end{Bmatrix}$$

Calculation for member No. 5

Redundant nos. corresponding to the member 5 = 3, 4, 5, 9, 10, 11

$$[FM]_5 = [FM]_I [Rj]_5 = \begin{bmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} [Rk]_5 = \begin{bmatrix} -0.707 & -0.707 & 0 \\ 0.707 & -0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[FMg]_5 = 10^{-6} \begin{bmatrix} 176.009 & 0 & -373.372 \\ 0 & 357.436 & 347.032 \\ -373.372 & 347.032 & 1394.18 \end{bmatrix}$$

$$[BR]_5 = \begin{bmatrix} -1 & 0 & 7.243 & 1 & 0 & -7.243 \\ 0 & -1 & -7.243 & 0 & 1 & 7.243 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} [BF]_5 = \begin{bmatrix} 1 & 0 & -3 & -1 & 0 & 3 \\ 0 & 1 & 7.243 & 0 & -1 & -7.243 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

$$[FSm]_5 = 10^{-6} \begin{bmatrix} 176.009 & 0 & -901.399 & -176.009 & 0 & 901.399 \\ 0 & 357.436 & 2935.812 & 0 & -357.436 & -2935.812 \\ -901.399 & 2935.812 & 28994.95 & 901.399 & -2935.812 & -28994.95 \\ -176.009 & 0 & 901.399 & 176.009 & 0 & -901.399 \\ 0 & -357.436 & -2935.812 & 0 & 357.436 & 2935.812 \\ 901.399 & -2935.812 & -28994.95 & -901.399 & 2935.812 & 28994.95 \end{bmatrix}$$

$$\{BLR\}_5 = \begin{Bmatrix} -99.965 \\ -27.314 \\ 47.124 \end{Bmatrix} \{BLF\}_5 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \{ELF\}_5 = \begin{Bmatrix} -16.002 \\ 13.657 \\ -23.562 \end{Bmatrix} \{BLC\}_5 = \begin{Bmatrix} -16.002 \\ 13.657 \\ -23.562 \end{Bmatrix} \{DQLm\}_5 = 10^{-6} \begin{Bmatrix} 5980.814 \\ -3295.192 \\ -63943.64 \\ -5980.814 \\ 3295.192 \\ 63943.64 \end{Bmatrix}$$

Calculation for member No. 6

Redundant nos. corresponding to the member 6 = 6, 7, 8, 9, 10, 11

$$[FM]_6 = [FM]_I [Rj]_6 = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} [Rk]_6 = \begin{bmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[FMg]_6 = 10^{-6} \begin{bmatrix} 357.436 & 0 & -347.032 \\ 0 & 176.009 & -373.372 \\ -347.032 & -373.372 & 1394.18 \end{bmatrix}$$

$$[BR]_6 = \begin{bmatrix} -1 & 0 & 7.243 & 1 & 0 & -7.243 \\ 0 & -1 & -3 & 0 & 1 & 3 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \quad [BF]_6 = \begin{bmatrix} 1 & 0 & -7.243 & -1 & 0 & 7.243 \\ 0 & 1 & 7.243 & 0 & -1 & -7.243 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

$$[FSm]_6 = 10^{-6} \begin{bmatrix} 357.436 & 0 & -2935.812 & -357.436 & 0 & 2935.812 \\ 0 & 176.009 & 901.399 & 0 & -176.009 & -901.399 \\ -2935.812 & 901.399 & 28994.95 & 2935.812 & -901.399 & -28994.95 \\ -357.436 & 0 & 2935.812 & 357.436 & 0 & -2935.812 \\ 0 & -176.009 & -901.399 & 0 & 176.009 & 901.399 \\ 2935.812 & -901.399 & -28994.95 & -2935.812 & 901.399 & 28994.95 \end{bmatrix}$$

$$\{BLR\}_6 = \begin{Bmatrix} 27.314 \\ -99.965 \\ 47.124 \end{Bmatrix} \quad \{BLF\}_6 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \{ELF\}_6 = \begin{Bmatrix} -13.657 \\ -16.002 \\ -23.562 \end{Bmatrix} \quad \{BLC\}_6 = \begin{Bmatrix} -13.657 \\ -16.002 \\ -23.562 \end{Bmatrix} \quad \{DQLm\}_6 = 10^{-6} \begin{Bmatrix} 3295.192 \\ 5980.814 \\ -2684.313 \\ -3295.192 \\ -5980.814 \\ 2684.313 \end{Bmatrix}$$

Calculation for member No. 7

Redundant nos. corresponding to the member 7 = 3, 4, 5, 6, 7, 8

$$[FM]_7 = [FM]_0 [Rj]_7 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} [Rk]_7 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [FMg]_7 = 10^{-6} \begin{bmatrix} 88.556 & 30.797 & -4.148 \\ 30.797 & 88.556 & 169.13 \\ -4.148 & 169.13 & 460.269 \end{bmatrix}$$

$$[BR]_7 = \begin{bmatrix} -1 & 0 & 10.243 & 1 & 0 & -10.243 \\ 0 & -1 & -10.243 & 0 & 1 & 10.243 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \quad [BF]_7 = \begin{bmatrix} 1 & 0 & -7.243 & -1 & 0 & 7.243 \\ 0 & 1 & 7.243 & 0 & -1 & -7.243 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

$$[FSm]_7 = 10^{-6} \begin{bmatrix} 88.556 & 30.797 & -422.479 & -88.556 & -30.797 & 422.479 \\ 30.797 & 88.556 & 587.604 & -30.797 & -88.556 & -587.604 \\ -422.479 & 587.604 & 9029.888 & 422.479 & -587.604 & -9029.888 \\ -88.556 & -30.797 & 422.479 & 88.556 & 30.797 & -422.479 \\ -30.797 & -88.556 & -587.604 & 30.797 & 88.556 & 587.604 \\ 422.479 & -587.604 & -9029.888 & -422.479 & 587.604 & 9029.888 \end{bmatrix} \quad \{BLR\}_7 = \begin{Bmatrix} -33.337 \\ 165.429 \\ 94.248 \end{Bmatrix}$$

$$\{BLF\}_7 = \begin{Bmatrix} 99.965 \\ 27.314 \\ -47.124 \end{Bmatrix} \quad \{ELF\}_7 = \begin{Bmatrix} -1.678 \\ 20.992 \\ -23.562 \end{Bmatrix} \quad \{BLC\}_7 = \begin{Bmatrix} 98.287 \\ 48.306 \\ -70.686 \end{Bmatrix} \quad \{DQLm\}_7 = 10^{-6} \begin{Bmatrix} 10484.84 \\ -4650.341 \\ -134390.9 \\ -10484.84 \\ 4650.341 \\ 134390.9 \end{Bmatrix}$$

Now, $[FS] = \Sigma [FSm]_i$ and $\{DQL\} = \Sigma \{DQLm\}_i$

Column No. 0 to 5

$$[FS] = 10^{-6} \begin{bmatrix} 534.55 & 0 & -1378.98 & -88.56 & 30.8 & 484.57 \\ 0 & 353.12 & 1808.45 & 30.8 & -88.56 & -902.9 \\ -1378.98 & 1808.45 & 16479.36 & 484.57 & -902.9 & -9665.87 \\ -88.56 & 30.8 & 484.57 & 353.12 & 0 & -1808.45 \\ 30.8 & -88.56 & -902.9 & 0 & 534.55 & 4426.17 \\ 484.57 & -902.9 & -9665.87 & -1808.45 & 4426.17 & 47960.7 \\ 0 & 0 & 0 & -88.56 & -30.8 & 422.48 \\ 0 & 0 & 0 & -30.8 & -88.56 & -587.46 \\ 0 & 0 & 0 & 422.48 & -587.46 & -9029.89 \\ -357.44 & 0 & 725.28 & -176.01 & 0 & 901.4 \\ 0 & -176.01 & -901.4 & 0 & -357.44 & -2935.81 \\ 725.28 & -901.4 & -6353.22 & 901.4 & -2935.81 & -28994.95 \end{bmatrix}$$

Column No. 6 to 11

$$[FS] = 10^{-6} \begin{bmatrix} 0 & 0 & 0 & -357.44 & 0 & 725.28 \\ 0 & 0 & 0 & 0 & -176.01 & -901.4 \\ 0 & 0 & 0 & 725.28 & -901.4 & -6353.22 \\ -88.56 & -30.8 & 422.48 & -176.01 & 0 & 901.4 \\ -30.8 & -88.56 & -587.46 & 0 & -357.44 & -2935.81 \\ 422.48 & -587.46 & -9029.89 & 901.4 & -2935.81 & -28994.95 \\ 534.55 & 0 & -4096.21 & -357.44 & 0 & 2935.81 \\ 0 & 353.12 & 1808.45 & 0 & -176.01 & -901.4 \\ -4096.21 & 1808.45 & 44311.01 & 2935.81 & -901.4 & -28994.95 \\ -357.44 & 0 & 2935.81 & 1066.89 & 0 & -5463.89 \\ 0 & -176.01 & -901.4 & 0 & 1066.89 & 5463.89 \\ 2935.81 & -901.4 & -28994.95 & -5463.89 & 5463.89 & 70696.34 \end{bmatrix}$$

Column No. 0 to 5

$$[FS]^{-1} = \begin{bmatrix} 19619.65 & -8964.12 & 3579.77 & 17676.34 & -4765.44 & 2881.21 \\ -8964.12 & 12137.53 & -2394.77 & -6007.67 & 2383.97 & -1073.59 \\ 3579.77 & -2394.77 & 886.21 & 2881.21 & -401.24 & 562.59 \\ 17676.34 & -6007.67 & 2881.21 & 35352.7 & -29511.14 & 5762.42 \\ -4765.44 & 2383.97 & -401.24 & -29511.14 & 48683.99 & -5762.42 \\ 2881.21 & -1073.59 & 562.59 & 5762.42 & -5762.42 & 1125.18 \\ 17167.62 & -5601.63 & 3452.47 & 17676.38 & 655.53 & 2881.21 \\ -4062.16 & 1784.31 & -943.49 & -4988.77 & 2384.0 & -1073.6 \\ 2722.08 & -943.49 & 602.83 & 2881.22 & -401.26 & 562.59 \\ 17732.84 & -6021.14 & 3165.64 & 17676.4 & -3121.41 & 2881.22 \\ -5343.16 & 2602.13 & -949.91 & -5731.46 & 3683.65 & -1119.14 \\ 2899.71 & -1031.26 & 599.64 & 2881.22 & -609.49 & 562.59 \end{bmatrix}$$

Column No. 6 to 11

$$[FS]^{-1} = \begin{bmatrix} 17167.62 & -4062.16 & 2722.08 & 17732.84 & -5343.16 & 2899.71 \\ -5601.63 & 1784.31 & -943.49 & -6021.14 & 2602.13 & -1031.26 \\ 3452.47 & -943.49 & 602.83 & 3165.64 & -949.91 & 599.64 \\ 17676.38 & -4988.77 & 2881.22 & 17676.4 & -5731.46 & 2881.22 \\ 655.53 & 2384.0 & -401.26 & 3121.41 & 3683.65 & -609.49 \\ 2881.22 & -1073.6 & 562.59 & 2881.22 & -1119.14 & 562.59 \\ 39260.57 & -15564.61 & 5497.34 & 18516.59 & -4386.44 & 3242.15 \\ -15564.61 & 12137.52 & -2394.77 & -4541.64 & 2602.13 & -1031.26 \\ 5497.34 & -2394.77 & 886.21 & 2976.22 & -949.91 & 599.64 \\ 18516.59 & -4541.64 & 2976.22 & 20351.13 & -6800.09 & 3344.7 \\ -4386.44 & 2602.13 & -949.91 & -6800.09 & 5145.36 & -1327.8 \\ 3242.16 & -1031.26 & 599.64 & 3344.71 & -1327.8 & 653.09 \end{bmatrix}$$

$$\{DQL\} = 10^{-6} \begin{Bmatrix} 9789.86 \\ -19830.11 \\ -174243.55 \\ 12233.15 \\ 5208.38 \\ -62167.99 \\ -12332.29 \\ 28623.85 \\ 183166.34 \\ -1865.41 \\ -1443.59 \\ 61157.31 \end{Bmatrix} \{Q\} = \begin{Bmatrix} -80.955 \\ 13.760 \\ 0.728 \\ -29.142 \\ -176.664 \\ 10.569 \\ 34.580 \\ -149.753 \\ 3.027 \\ -125.471 \\ 41.109 \\ -17.402 \end{Bmatrix}$$

$$\{Qm\}_0 = \begin{Bmatrix} -80.955 \\ 13.760 \\ 0.728 \end{Bmatrix} \{Qm\}_1 = \begin{Bmatrix} -80.955 \\ 13.760 \\ 0.728 \\ -125.471 \\ 41.109 \\ -17.402 \end{Bmatrix} \{Qm\}_2 = \begin{Bmatrix} -80.955 \\ 13.760 \\ 0.728 \\ -29.142 \\ -176.664 \\ 10.569 \end{Bmatrix} \{Qm\}_3 = \begin{Bmatrix} -125.471 \\ 41.109 \\ -17.402 \end{Bmatrix}$$

$$\{Qm\}_4 = \begin{Bmatrix} 34.580 \\ -149.753 \\ 3.027 \end{Bmatrix} \{Qm\}_5 = \begin{Bmatrix} -29.142 \\ -176.664 \\ 10.569 \\ -125.471 \\ 41.109 \\ -17.402 \end{Bmatrix} \{Qm\}_6 = \begin{Bmatrix} 34.580 \\ -149.753 \\ 3.027 \\ -125.471 \\ 41.109 \\ -17.402 \end{Bmatrix} \{Qm\}_7 = \begin{Bmatrix} -29.142 \\ -176.664 \\ 10.569 \\ 34.580 \\ -149.753 \\ 3.027 \end{Bmatrix}$$

$$\{AMR\}_0 = \begin{Bmatrix} -82.390 \\ -98.135 \\ 36.071 \end{Bmatrix} \{AMF\}_0 = \begin{Bmatrix} -0.530 \\ -36.429 \\ -36.071 \end{Bmatrix} \{AMR\}_1 = \begin{Bmatrix} 29.233 \\ 11.760 \\ 17.213 \end{Bmatrix} \{AMF\}_1 = \begin{Bmatrix} 26.103 \\ -12.139 \\ 18.130 \end{Bmatrix}$$

$$\begin{aligned} \{AMR\}_2 &= \begin{Bmatrix} -0.059 \\ 44.172 \\ -45.184 \end{Bmatrix} \{AMF\}_2 = \begin{Bmatrix} -86.381 \\ 126.952 \\ 45.184 \end{Bmatrix} \{AMR\}_3 = \begin{Bmatrix} -0.718 \\ -13.995 \\ 17.941 \end{Bmatrix} \{AMF\}_3 = \begin{Bmatrix} 7.447 \\ 8.248 \\ 17.402 \end{Bmatrix} \\ \{AMR\}_4 &= \begin{Bmatrix} 14.655 \\ 40.709 \\ -44.097 \end{Bmatrix} \{AMF\}_4 = \begin{Bmatrix} -101.093 \\ 138.128 \\ 54.097 \end{Bmatrix} \{AMR\}_5 = \begin{Bmatrix} 13.026 \\ -4.126 \\ 19.153 \end{Bmatrix} \{AMF\}_5 = \begin{Bmatrix} 1.959 \\ 19.516 \\ 27.971 \end{Bmatrix} \\ \{AMR\}_6 &= \begin{Bmatrix} 31.701 \\ 10.175 \\ 26.695 \end{Bmatrix} \{AMF\}_6 = \begin{Bmatrix} 38.889 \\ -21.787 \\ 20.429 \end{Bmatrix} \{AMR\}_7 = \begin{Bmatrix} -115.082 \\ -190.357 \\ 86.705 \end{Bmatrix} \{AMF\}_7 = \begin{Bmatrix} 18.386 \\ -55.033 \\ -39.581 \end{Bmatrix} \end{aligned}$$