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On the multivariate variation control chart

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Abstract

Multivariate control charts such as Hotelling's T^2 and χ^2 are commonly used for monitoring several related quality characteristics. These control charts use correlation structure that exists between quality characteristics in an attempt to improve monitoring. The purpose of this article is to discuss some issues related to the *G* chart proposed by Levinson et al. [9] for detecting shifts in the process variance-covariance matrix. They use a *G* statistic which is distributed as a chi-square with $p(p+1)/2$ degrees of freedom where *p* denotes the number of variables under study. The authors show through simulation that the chi-square distribution only holds for certain cases. The results could be important to practitioners who use *G* chart for monitoring purposes.

Keywords: Statistical process control; T^2 chart; χ^2 chart; G chart; Goodness of fit test

1. Introduction

Multivariate control charts are widely used in practice to monitor the joint performance of several related quality characteristics. Many authors have proposed quality control procedures for several related variables and contributed to the development of multivariate quality control. Mason et al. [12] summarized many of these developments. Mason et al. [11] and Linna et al. [10] give more recent work. Many authors have contributed to the development of multivariate control procedures for monitoring shifts in the variance-covariance matrix. Work in this area includes that of Alt [1], Healy [4], Levinson et al. [9], Aparisi et al. [2] and Khoo [7]. The purpose of this article is to summarize the conditions and the consequences regarding the use of the *G* statistic considered by Levinson et al. [9] for detecting shifts in the covariance matrix of several quality characteristics. The authors show via simulation that the assumption of chi-square distribution for the *G* statistic is not always true.

Some basic concepts of the *G* chart along with hypothesis testing for covariance matrix are reviewed in Section 2. Section 3 presents several numerical examples. The concluding remarks are given in Section 4.

2. Evaluating equality of two covariance matrices

Suppose the output quality of a production process can be measured by the joint level of *p* correlated quality characteristics. Further suppose $\mathbf{X} = (X_1, X_2, \ldots, X_p)'$ is a $p \times 1$ random vector whose j^{th} element is the j^{th} quality characteristic of interest.

It is assumed that **X** follows a *p*-variate normal distribution with mean $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)'$ and covariance matrix $\boldsymbol{\Sigma}$.

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be a random sample of size *n* of these quality vectors. Hence, $\overline{\mathbf{X}} = (\overline{\mathbf{X}}_1, \overline{\mathbf{X}}_2, \ldots, \overline{\mathbf{X}}_n)'$ can be used as an estimate of the mean vector.

Two common estimators for the covariance matrix $\boldsymbol{\Sigma}$ are **S1** and **S2** which are calculated using the full data set and the mean square successive differences, respectively.

The following equations show the respective procedure for calculating **S1** and **S2**:

$$\mathbf{S1} = \frac{1}{n_1 - 1} \sum_{j=1}^{n_1} (\mathbf{X}_j - \overline{\mathbf{X}})(\mathbf{X}_j - \overline{\mathbf{X}})', \quad (1)$$

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where $\overline{\mathbf{X}} = \frac{1}{n_1} \sum_{j=1}^{n_1} \mathbf{X_j}$, and

$$\mathbf{S2} = \frac{1}{2(n_2 - 1)} \sum_{j=2}^{n_2} (\mathbf{X_j} - \mathbf{X_{j-1}})(\mathbf{X_j} - \mathbf{X_{j-1}})', \quad (2)$$

where n_1 and n_2 are sample sizes from populations 1 and 2, respectively. To test the equality of two covariance matrices, i.e. H$_0$: $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$ vs. H$_1$: $\mathbf{\Sigma}_1 \neq \mathbf{\Sigma}_2$ (assuming $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$), Kramer and Jensen [8] showed that under the null hypothesis, the *G* statistic defined as $2.3026mM$ follows a chi-square distribution with *p(p+1)/2* degrees of freedom, where:

$$M = (n_1 + n_2 - 2)\log_{10}|\mathbf{S}| - (n_1 - 1)\log_{10}|\mathbf{V}_1| - (n_2 - 1)\log_{10}|\mathbf{V}_2|, \quad (3)$$

$$m = 1 - [\frac{1}{n_1 - 1} + \frac{1}{n_2 - 1} - \frac{1}{n_1 + n_2 - 2}] \times [\frac{2p^2 + 3p - 1}{6(p+1)}], \quad (4)$$

$$\mathbf{S} = \frac{(n_1 - 1)\mathbf{V}_1 + (n_2 - 1)\mathbf{V}_2}{n_1 + n_2 - 2}, \quad (5)$$

where $\mathbf{V}_1$ and $\mathbf{V}_2$ are sample estimates for the population covariance matrix $\mathbf{\Sigma}$ obtained using **S1** in Equation (1). The null hypothesis is rejected when one of the following conditions are met:

$$2.3026mM > \chi^2_{p(p+1)/2, 1-\alpha/2},$$

$$2.3026mM < \chi^2_{p(p+1)/2, \alpha/2},$$

indicating that the two samples come from two populations with different covariance matrices. The concept of the *G* statistic was first proposed by Box [3]. He shows that the *G* statistic approximately follows a chi square distribution when both $\mathbf{V}_1$ and $\mathbf{V}_2$ are independent estimates obtained for population covariance matrix $\mathbf{\Sigma}$ using Equation (1). Levinson et al. [9] expanded his concept to explore the question of whether or not the process covariance matrix changes over time. They use a control chart approach to investigate the stability of a process with respect to covariance matrix. This approach can be used in conjunction with a T^2 control chart to specify whether a signal generated by the T^2 control chart is due to a shift in the mean vector or covariance matrix. This procedure is analogous to the approach used in univariate control charting. For more information on T^2 control chart see Montgomery [13]. Based on their proposed procedure, a shift in the process covariance matrix has occurred if the *G* statistic based on $\mathbf{V}_1$ and $\mathbf{V}_2$ computed from Equation (1), fall outside the control limits of a chart constructed using quantiles of a chi-square distribution with *p(p+1)/2* degrees of freedom and a desired false alarm rate. The center line of the chart is the median or the 50th percentile of the chi-square distribution with *p(p+1)/2* degrees of freedom. The center line may be used in conjunction with run tests as another indication of process change. In an example, they used 40 samples of size 6 from a 5 variate normal distribution and considered Equation (1) to estimate the i^{th} sample covariance matrix denoted by $\mathbf{V}_{2,i}$ for *i=1,2,..., 40.* The average of $\mathbf{V}_{2,i}$ values denoted by $\mathbf{V}_1$ is replaced in the following equation to compute the *G* statistic for the i^{th} sample:

$$\mathbf{S} = \frac{(n_1 - 1)\mathbf{V}_1 + (n_2 - 1)\mathbf{V}_{2,i}}{n_1 + n_2 - 2}. \quad (6)$$

Since $\mathbf{V}_1$ and $\mathbf{V}_{2,i}$ in Equation (6) are not independent, it can be shown that the *G* statistic does not follow a chi square distribution with *p(p+1)/2* degrees of freedom and as a result leading to improper conclusions. Sullivan and Woodall [17] mathematically show that both **S1** and **S2** are unbiased estimators for the covariance matrix $\mathbf{\Sigma}$ and it can be shown that **S1** in comparison to **S2** has smaller mean square error. Hence, one should question if the results obtained for the *G* statistic computed using the procedures proposed by Kramer and Jensen [8] and Levinson et al. [9] yield the same results. This issue is investigated in the next section.

3. Distribution of the *G* statistic

In this section, the distribution of the *G* statistic is investigated through numerical simulation. The authors considered the following three different scenarios for computing the *G* statistic:

- Both $\mathbf{V}_1$ and $\mathbf{V}_2$ are the estimates for **S1** calculated from Equation (1), (henceforth referred to as **S1-S1**).
- $\mathbf{V}_1$ is calculated from Equation (1) and $\mathbf{V}_2$ is calculated from Equation (2), (henceforth referred to as **S1-S2**).
- Both $\mathbf{V}_1$ and $\mathbf{V}_2$ are calculated from Equation (2), (henceforth referred to as **S2-S2**).

The results are presented for $n_1 = n_2 = 3,6,10$. For each case, a subgroup of size n_i from a p-variate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ is generated and the G statistic is computed. This process is repeated 15000 times and the values for the G statistic are recorded. Based on the recorded data the distribution of the G statistic is evaluated using Anderson-Darling goodness of fit test.

For the sake of simplicity, the authors consider cases where p is equal to 2 [14], 3, 4 [16] and 5 [9]. For each case, the proper mean vectors and covariance matrices are used (Table 1).

The p-values corresponding to the Anderson-Darling statistic are shown in Table 2. Levinson et al. [9] assumed chi-square distribution for the G statistic when p and n possess values equal to 5 and 6, respectively. However, results in Table 2 do not approve such a thing. According to Table 2, in some cases of **S1-S1** such as *n=6, p=2* and *n=10, p=3,* the distribution of the G statistic is chi-square and for the other cases, namely **S1-S2** and **S2-S2**, the G statistic dose not have chi-square distribution in any case.

For the case of comparing variances of two populations when p is equal to 2, Pearson and Wilks [15] show that the exact distribution for the G statistic is given by:

$$P(G < g) = \frac{\Gamma(2n-3)}{\Gamma(n-1)\Gamma(n-2)2^{2n-6}}$$

$$\times\left\{ g^{2n-4}\log_{10}\frac{1+\sqrt{1-g^2}}{g} + \frac{1}{2}\int_0^{g^2} y^{n-3}(1-y)^{-\frac{1}{2}}dy \right\}.$$

These results indicate that the chi square approximation depends on the value of p and the sample size and cannot be extended to all combinations of these parameters.

Negative data in Table 2 indicate that some data generated by simulation are negative and we cannot fit the chi square distribution to them; Figures 1 and 2 are corroboration to this matter. In Table 3, it is shown in detail that for *p=2, G* statistics in **S1-S1** scenario is not chi-square distribution even in large samples. Figures 1 and 2 show the distribution of the G statistic under the above three scenarios for different p and n values. Again, we can infer from these figures that under certain conditions the chi square distribution holds.

When variances of the two populations are compared using the first scenario, that is **S1-S1**, for certain combinations of p and sample size the distribution of the G statistic follows a chi square distribution. Henceforth, the authors analyze only the first scenario since in the other scenarios, the G statistic dose not hold chi-square distribution in any case. Table 3 provides simulation results and shows under what combinations of these two parameters the chi square distribution with *p(p+1)/2* degrees of freedom holds. According to this table, as the value for p increases a larger sample size is required in order for the G statistic to have a chi square distribution.

Table 3 shows that in bivariate G-statistic is $\chi^2_{2(2+1)/2}$ only for *n=6,7,13*; and for *p=3,4,5*, G-statistic is $\chi^2_{p(p+1)/2}$ if $n \geq 9,11,13$ respectively. In other words, for all p's except *p=2*, G-statistic is chi-square distribution at large sample size. The results summarized in Table 4 that shows the minimum sample size for G-statistic being $\chi^2_{p(p+1)/2}$ in numerical examples.

4. Conclusion

The G statistic proposed originally by Box [3] can be used to detect shifts in the covariance matrix when monitoring several quality characteristics. However, the distribution of this statistic is chi square when there is no shift in the covariance matrix; p is greater than 2 and sample size is relatively large. In this paper, the authors used simulation and showed numerically that the conditions considered by Levinson et al. [9] to detect shifts in the covariance matrix does not meet the required conditions for the G statistic to hold a chi square distribution. This issue is important to practitioners who want to use variation chart to monitor covariance matrix to detect changes in the covariance structure of several quality characteristics.

Table 1. The mean vectors and covariance matrices for multivariate normal distributions.

P	μ	Σ
2	$\begin{bmatrix} 264 \\ 470 \end{bmatrix}$	$\begin{bmatrix} 100 & 66 \\ 66 & 121 \end{bmatrix}$
3	$\begin{bmatrix} 15.4 \\ 15.4 \\ 15.3 \end{bmatrix}$	$\begin{bmatrix} 7.82 & 7.93 & 7.98 \\ 7.93 & 9.38 & 8.87 \\ 7.98 & 8.87 & 9.79 \end{bmatrix}$
4	$\begin{bmatrix} 26.5 \\ 32.8 \\ 31.5 \\ 28.8 \end{bmatrix}$	$\begin{bmatrix} 30.6 & 12.7 & 59.4 & 53.9 \\ 12.7 & 59.4 & 5 & -12.1 \\ 55.1 & 5 & 186 & 147.3 \\ 53.9 & -12.1 & 147.3 & 149.5 \end{bmatrix}$
5	$\begin{bmatrix} 6.682 \\ 13.889 \\ 21.231 \\ 21.249 \\ 14.88 \end{bmatrix}$	$\begin{bmatrix} 1.5199 & 0.2456 & 1.4244 & 1.3339 & 0.785 \\ 0.2456 & 1.1068 & 0.5057 & 1.1152 & 0.0479 \\ 1.4244 & 0.5057 & 3.7531 & 3.2788 & 0.6744 \\ 1.3339 & 1.1152 & 3.2788 & 4.9943 & 0.4659 \\ 0.785 & 0.0479 & 0.6744 & 0.4659 & 1.5038 \end{bmatrix}$

Table 2. Goodness of fit test [*p*-values (Anderson-Darling statistic)].

N	P	S1-S1	S1-S2	S2-S2
3	**2**	<0.001 (18)	<0.001 (38)	<0.001 (58)
	3	<0.001 (35001)	<0.001 (105416)	<0.001 (105692)
	4	Negative Data	Negative Data	Negative Data
	5	Negative Data	Negative Data	Negative Data
6	**2**	**0.113 (1.63)**	<0.001 (240)	<0.001 (902)
	3	<0.001 (11)	<0.001 (768)	<0.001 (2120)
	4	<0.001 (38)	<0.001 (1925)	<0.001 (4585)
	5	<0.001 (110)	<0.001 (6875)	<0.001 (12256)
10	**2**	0.011 (3.92)	<0.001 (395)	<0.001 (1202)
	3	**>0.250 (0.70)**	<0.001 (879)	<0.001 (2835)
	4	<0.001 (6.3)	<0.001 (1888)	<0.001 (5284)
	5	<0.001 (13)	<0.001 (2939)	<0.001 (8890)

Table 3. Results of Goodness of fit test [P-values for the Anderson-Darling statistic].

$$H_0 : F_G = F_{\chi^2_{p(p+1)/2}} \quad vs \quad H_1 : F_G \neq F_{\chi^2_{p(p+1)/2}}$$

P-Value

p \ n	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	<.001	<.001	<.001	**.113**	**>.25**	.01	.006	.01	.002	.045	**.067**	.006	<.001
3	<.001	<.001	<.001	<.001	<.001	<.001	**>.25**	**>.25**	**>.25**	**>.25**	**>.25**	>.25	**>.25**
4			<.001	<.001	<.001	<.001	<.001	<.001	**>.25**	**>.25**	**>.25**	**>.25**	**>.25**
5			<.001	<.001	.021	<.001	<.001	<.001	.023	.043	**.059**	**>.25**	**>.25**

Table 4. Minimum sample size for G-statistic.

P=2	-
P=3	$n \geq 9$
P=4	$n \geq 11$
P=5	$n \geq 13$

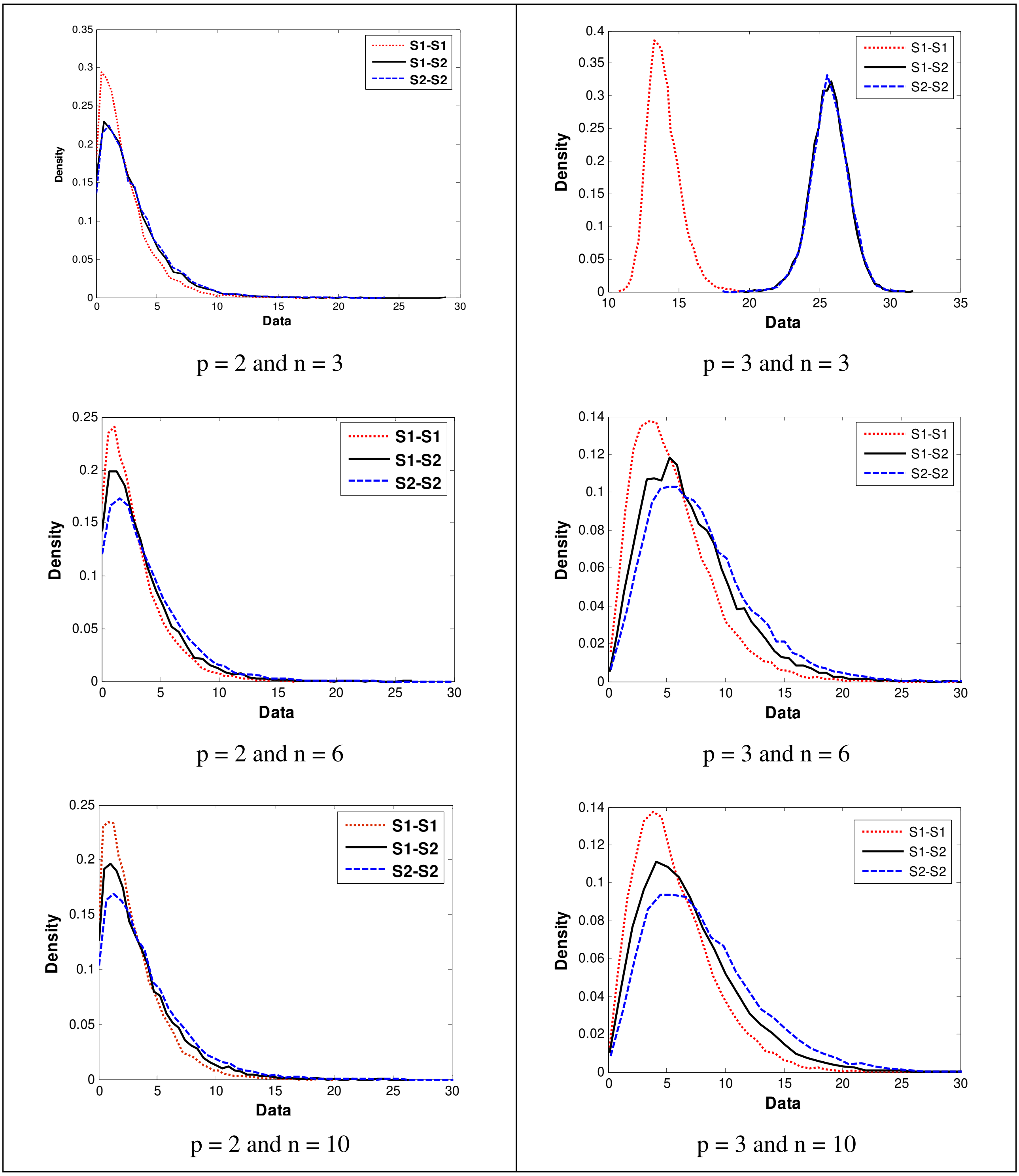

Figure 1. The distributions of G statistic under the three scenarios for p=2, p=3 and different n.

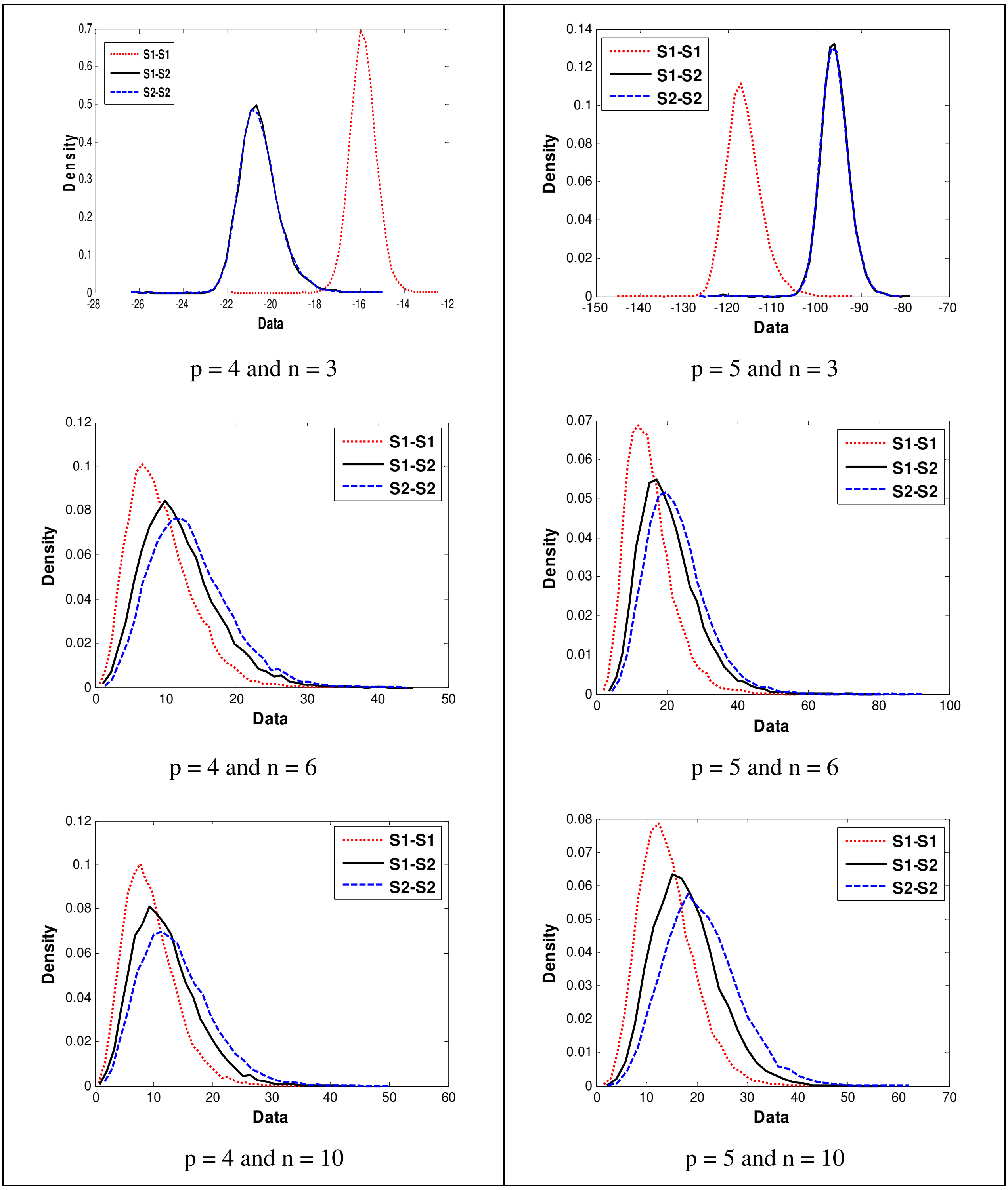

Figure 2. The distributions of G statistic under the three scenarios for p=4, p=5 and different n.

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