A new approach for constraining failure probability of a critical deteriorating system

H. Babaei [1]; K. Shahanaghi [2]; A. Bakhsha [3]*

M. Sc. Student, Dep. of Industrial Engineering, Iran University of Science and Technology, Tehran, Iran

Assistant Professor, Dep. of Industrial Engineering, Iran University of Science and Technology, Tehran, Iran

M. Sc. Student, Dep. of Industrial Engineering, Iran University of Science and Technology, Tehran, Iran

Abstract: In this paper, we focus on a continuously deteriorating critical equipment which its failure cannot be measured by cost criterion. For these types of systems like military systems, nuclear systems, etc it is extremely important to avoid failure during the actual operation of the system. In this paper we propose an approach which constrains failure probability to a pre-specified value. This value guarantees a chance of failure less than or equal to the pre-specified value during real operation of the system. The inspection periods and maintenance policy are found in two phases. Failure probability is limited to a pre-specified value In the first phase, and in the second phase optimum maintenance thresholds and inspection periods are obtained in such a way that minimize long-run expected costs. Due to the complexity of the model, Monte Carlo simulation is used to obtain optimum results.

1. Introduction

Reliability is one of the important issues in the assessment of industrial equipment or products. Good product design is of course essential for products with high reliability. However, no matter how good the product design is, products deteriorate over time since they are operating under certain stress or load in real environments, often involving randomness. Maintenance has, thus, been introduced as an efficient way to assure a satisfactory level of reliability during the useful life of a physical asset.

The main idea behind Condition Based Maintenance (CBM) is to provide a decision support for maintenance actions. As such, it is natural to include maintenance policies in the consideration of the machine prognostic process. The aim of CBM is to optimize the maintenance policies according to certain criteria such as risk, cost, reliability and availability. In those CBM models, which cost is taken into account as optimization criterion, the optimum policy is often obtained by minimizing the long run cost per time unit (Kumar and Westberg, 1997).

Generally CBM models fall into two categories: completely observable systems and partially observable systems. The state of the system in a completely observable system can be completely observed or identified. First of all we discuss completely observable systems. Jardine *et*

al. (2006) focus on the analytical modeling of a condition based inspection / replacement policy for a stochastically and continuously deteriorating single unit system.

They consider both the replacement threshold and the inspection schedule as decision variables for the problem. They minimize the long run expected cost per unit time by the stationary law for the system state. Amari and McLaughlin (2004) utilized a Markov chain to describe the CBM model for a deterioration system subject to periodic inspection and the optimal inspection frequency and maintenance threshold were found to maximize the system availability. Castanier *et al.* (2005) consider a two unit system which can be maintained by good as new preventive or corrective replacements. They develop a stochastic model based on the semi-regenerative properties of the maintained system state and the associated cost model is used to optimize the performance of the maintained model. Barata *et al.* (2002) use Monte-carlo simulation to model the continuously monitored deteriorating systems. They assume that after each maintenance action a random amount of improvement is made on the state of the system which is independent of current system state. Then the optimized thresholds of maintenance are found such that the total expected cost of system is minimized. Dieulle *et al.* (2003) consider a continuously deteriorating system which is inspected in random

*Corresponding Author Email: hamid.babaei@ulg.ac.be
Tel.: +98 9122684538

times. In this model, they assume that deterioration follows a gamma distribution and system fails if its condition lies upper than a pre-specified threshold. In their model two types of replacement can be done depending on the fact that system is failed or the condition of system exceeds a critical threshold.

For partially observable systems, Barbera *et al.* (1996) propose a CBM model which assumes that failure rate of the system depend on the variables of the system state and fixed inspection periods. Then the maintenance action is optimized such that the long term costs of maintenance actions and failures are minimized. Kumar and Westberg (1997) suggest an approach based on reliability that inspection periods and maintenance thresholds are estimated such that the global cost per unit time is minimized. Chen and Trivedi (2005) build the semi-Markov decision process for the maintenance policy optimization of condition based preventive maintenance problems and present the approach for joint optimization of inspection rate and maintenance policy. Wang (2002) applied a stochastic recursive control model for CBM optimization based on the assumptions that the item monitored follows a two-period failure process with the first period of a normal life and the second one of a potential failure. A stochastic recursive filtering model was used to predict the residual, and then a decision model was established to recommend the optimal maintenance actions. The optimal condition monitoring intervals were determined by a hybrid method of simulation and analytical approach. Goode and Roylance (2000) determine the length of the next condition monitoring interval for a given risk level.

As discussed earlier, in CBM the maintenance policies are optimized according to certain criteria such as risk, cost, reliability and availability. Risk is defined as the combination of probability and consequence. Usually, consequence can be measured by cost. In this case, risk criterion is equivalent to the cost criterion. However, there are some cases, e.g., critical equipments in a power plant, in which consequence cannot be estimated by cost. For some systems, such as aircrafts, submarines, military systems, and nuclear systems, it is extremely important to avoid failure during actual operation because it can be dangerous or disastrous (Wang, 2002). Therefore, failure probability is more important than cost in such systems. In these scenarios, probability or reliability criterion would be more appropriate. In this paper we consider a two unit series system suffering from continuous deterioration. The

inspection periods and maintenance thresholds are determined to constrain failure probability to a pre-specified value and total maintenance costs per time unit is minimized. The novelty of the present work stems from the fact that system failure due to random shocks is constrained to a pre-specified value for our supposed critical system.

2. Notations and problem formulation

First of all we declare the notations which are used in this research.

X_t — State of the system at time *t*.

Y_t — The amount of deterioration occurred in period *t*.

ξ_1 — Preventive maintenance threshold.

ξ_2 — Preventive replacement threshold.

C_{pm} — The cost incurred by preventive maintenance action.

C_{pr} — The cost incurred by preventive replacement action which is strictly bigger than C_{pm}.

C_∞ — The long-run cost of system.

$\pi(x)$ — Stationary law of the deterioration process.

$\lambda(X_t)$ — Failure rate of the system.

T — The time between two successive inspection periods.

$f(x)$ — Probability density function of deterioration occurring during one period.

$f^{(T)}(x)$ — *T* th convolution of $f(x)$.

p — Maximum allowed Failure probability determined by decision maker group.

C_d — System down time cost per time unit.

$C(t)$ — Cumulative cost per time unit till time *t*.

The measurement is taken on continuous scale. Also we assume that system is inspected at equidistant times and time to failure follows an

exponential distribution and failure rate is a linear increasing function of system condition.

For each change of time Δt, the random deteriorations $X_{t+\Delta t} - X_t$ is assumed to be independent and have the same probability density function. Natural candidates for the associated probability density function can be obtained in the class of infinitely divisible distributions, e.g. gamma distributions (Feller, 1971). The exponential distribution is a special case of gamma distributions. This distribution is easier to further investigate the evaluation of the maintenance policy.

At the end of each period a decision is made to initiate either a preventive maintenance or preventive replacement action according to each unit condition. The preventive maintenance action is initiated when the state of the unit exceeds a threshold ξ_1 and the preventive replacement action is initiated when the state of the unit exceeds a threshold ξ_2 where $\xi_2 > \xi_1$.

We assume time to failure follows a non-homogeneous Poisson process, and the failure rate is an increasing function, $\lambda(X_t)$, of the variable X_t. We assume a linear relationship between $\lambda(X_t)$ and X_t. The reliability of the system, $R(t)$, is the probability that the system will not fail by the end of time period t. The state of the system at the end of period t is $X_{t+1} = X_t + Y_t$, So the reliability of the system at the end of period t is given by:

$$R(t) = e^{-\lambda(X_t + Y_t)T} \tag{1}$$

where T is the time between two successive inspection periods. Equation (1) is identical to the probability that time between failures is greater than T.

We define conditional failure probability of the system by:

$$P_f(X_{t+1}) = 1 - R(t) = 1 - e^{-\lambda(X_t + Y_t)T} \tag{2}$$

As discussed in Section 1, we also assume that the failure of our critical equipment cannot be measured by cost. So in order to avoid failure during operation of the system, the decision maker group constrains failure probability to a maximum allowed value p. Because in optimum policy the failure probability must not exceed p, first of all in phase 1 we obtain an interval for the thresholds and the length of inspection periods which

constrains failure probability to value less than p. Then in the next phase the optimum thresholds and inspection periods are determined by minimizing the cumulative cost per time unit.

If no failure occurs during period t, the condition of system at the beginning of period $t+1$ is:

$$X_{t+1} = X_t + Y_t \tag{3}$$

where Y_t is the deterioration occurred on system in period t.

A stochastic regeneration process is characterized by accumulation of a stochastic input process and an output mechanism that removes all the present quantity whenever it exceeds a critical level. As discussed in Section 2, after replacement of the unit, it is in the good as new initial state and its future evolution does not depend any more on the past. These replacement times are regeneration points for the system. Ross (1983) shows for a regeneration process, as time increases the distribution of X_t converges to the steady state distribution. The assumption of restoring the system state to X_0 when unit state reaches ξ_2 tells us that after each preventive replacement action the system state is independent of what has happened before.

Let $\pi(x)$ denote the stationary law of the deterioration process and x and y be the system deterioration levels observed at the end of two successive maintenance operations. The possible scenarios are the followings:

Scenario 1. $x < \xi_1$. System is left as it is and the probability density function of amount of occurring deterioration is $f^{(T)}(y - x)$ where T is the length of inspection period.

Scenario 2. $\xi_1 < x < \xi_2$. The system is preventively maintained to its initial state and the probability density function of amount of occurring deterioration will be $f^{(T)}(y)$.

Scenario 3. $\xi_2 < x$. The system is replaced with a new one and the probability density function of amount of occurring deterioration is $f^{(T)}(y)$.

By integration on the whole state space, the description of the different maintenance actions can lead to the following expression of the stationary probability density for the deterioration process at inspection times.

$$\pi(y) = \underbrace{\left(\int_0^{\xi_1} \pi(x) f^{(T)}(y-x)dx \right)}_{\text{Scenario 1}} +$$

$$\underbrace{\int_{\xi_1}^{\xi_2} \pi(x)dx f^{(T)}(y)}_{\text{Scenario 2}} + \qquad (4)$$

$$\underbrace{\int_{\xi_2}^{\infty} \pi(x)dx f^{(T)}(y)}_{\text{Scenario 3}}$$

The goal is to determine T, ξ_1 and ξ_2 in a way that failure probability at steady state remains less than or equal to p and maintenance costs are as low as possible.

The evaluation of the probability density function $\pi(y)$ is rather tricky and requires to solve a one sided integral equation. Hence due to complexity of Equation (4) we approximate stationary probability density function via simulation in the next section.

3. Simulation modeling and two phase solution procedure for the proposed approach

In order to simulate continuous deterioration process we discretize state space. We assume that between exponentially distributed times there exists a small amount of deterioration Δ. Therefore we have $X_{t+1} = X_t + N\Delta$ where N follows Poisson distribution.

3.1. Approximation of steady state distribution through simulation

In order to approximate stationary probability density function for a given maintenance thresholds and inspection period, the deterioration process is simulated for 10,000,000 periods. In the beginning of each period, if the system condition appears to equal $X_0 + n\Delta$, then +1 is added to the frequency of observing condition variable in state $X_0 + n\Delta$. Since we discretized state space, the number of different states at inspection times cannot be countless. This allows us to calculate observing frequency for a finite number of states. Finally the frequency of each state is divided by 10,000,000 to obtain a probability distribution (see Appendix). Figure 1 shows stationary probability density function of system state for different thresholds and $T = 10$ obtained by Monte Carlo simulation. Each subfigure

corresponds to a different maintenance threshold which are $(\xi_1, \xi_2) = (1, 2), (2, 3), (3, 4)$ and $(5, 8)$.

Expected failure probability and long run cost are obtained through approximated probability density function in the next two subsections.

3.2. First phase optimization

Let triple (ξ_1, ξ_2, T) denote a combination of thresholds and inspection period for our critical deteriorating system. First of all we split feasible space into different thresholds and inspection periods and select several points (ξ_1, ξ_2, T) from the feasible space for simulation.

Then probability density function of the deterioration at steady state is used to calculate failure probability (see Equation (2)) of the system at infinity.

Those triples for which failure probability does not exceeds p are selected for the optimization in the next period.

In order to obtain more precise solutions, this search can be performed finer. Accumulation of deterioration during operation period results in higher failure rate at the end of the period. Therefore during the operation of the system, the failure probability is also at most p. From Equation (2) we can infer that failure probability for the simulated system at infinity must be calculated from Equation (5).

$$P_f = \int_{x=0}^{x=\infty} \left(1 - e^{\lambda(x)T} \right) \times \pi(x)dx \qquad (5)$$

In order to calculate the failure probability we fix maintenance policy (including maintenance thresholds and inspection periods), and simulate the deterioration process to obtain corresponding probability density function at steady state, then failure probability is calculated from Equation (5).

The maintenance thresholds and inspection periods for which the failure probability is less than or equal to p are selected for the next phase optimization. In the next stage the optimum value for ξ_1, ξ_2 and T are obtained by minimizing long run expected maintenance costs.

3.3. Second phase optimization

In this phase of optimization procedure we select triple (ξ_1, ξ_2, T) among triples provided by
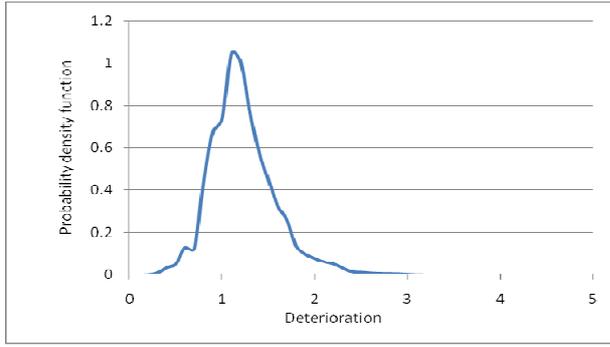
Figure 1-a: The Probability density functions for the deterioration process at steady state, $\xi_1 = 1, \ \xi_2 = 2$.
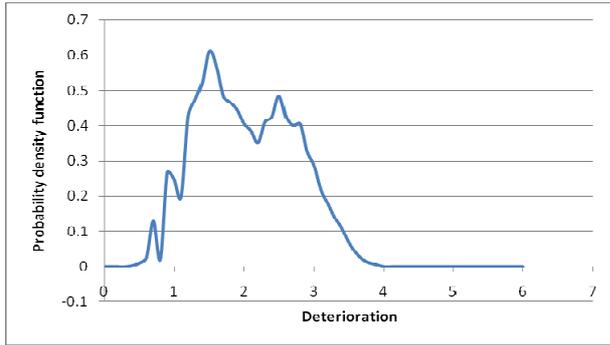
Figure 1-b: The Probability density functions for the deterioration process at steady state, $\xi_1 = 2, \ \xi_2 = 3$.
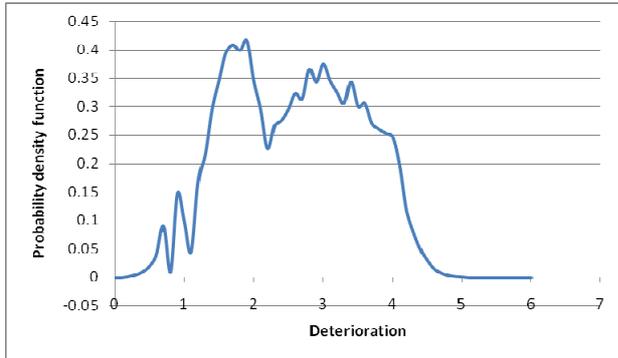
Figure 1-c: The Probability density functions for the deterioration process at steady state, $\xi_1 = 3, \ \xi_2 = 4$.
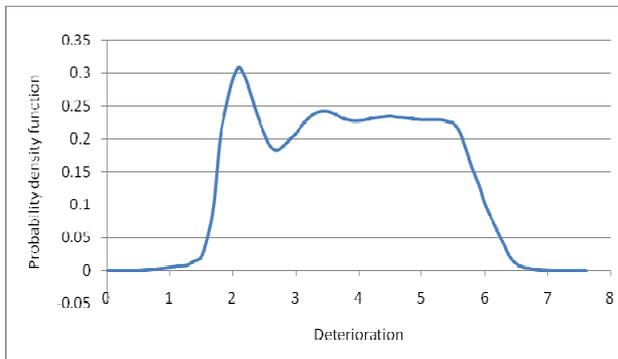
Figure 1-d: The Probability density functions for the deterioration process at steady state, $\xi_1 = 5, \ \xi_2 = 8$.

the first phase which minimizes system costs. Total cost of system is calculated as follows:

$$E[C_\infty] = C_{insp}P_{insp} + C_{pm}P_{pm} + \\ C_{pr}P_{pr} + C_d P_{insp} \qquad (6)$$

where P_{insp}, P_{pm}, P_{pr} are respectively the probability at steady state to have an inspection, preventive maintenance action and preventive replacement action. Computation of these probabilities requires the knowledge of probability density function of system deterioration at infinity. The probabilities P_{insp}, P_{pm}, P_{pr} are calculated from Equation (7):

$$P_{insp} = \int_{x=\xi_1}^{\infty} \pi(x)dx$$
$$P_{pm} = \int_{x=\xi_1}^{x=\xi_2} \pi(x)dx \qquad (7)$$
$$P_{pr} = \int_{x=\xi_2}^{x=\infty} \pi(x)dx$$

Hence the long run expected cost of system is given by:

$$E[\xi_1, \xi_2, T] = C_{insp}\int_{x=\xi_1}^{\infty} \pi(x)dx + \\ C_{pm}\int_{x=\xi_1}^{x=\xi_2} \pi(x)dx + \\ C_{pr}\int_{x=\xi_2}^{x=\infty} \pi(x)dx + \\ C_d\int_{x=\xi_1}^{\infty} \pi(x)dx \qquad (8)$$

where ξ_1, ξ_2 and T are provided by the first phase as discussed in subsection 3.2.

4. Numerical results

We simulated the deterioration process for 10,000,000 periods and $\Delta = 0.1$ by a C++ program to obtain stationary distribution function. Failure probability and maintenance costs are calculated for

$$\mu = 1, \ \lambda = 0.3, C_{pm} = 1, \ C_{pr} = 4, \ C_d = 16, \ p = 0.2$$

by excel. The C++ program is presented in the appendix. The stationary probability density function was estimated for thresholds:

(ξ_1,ξ_2)=(0.5,1), (0.5,2), (1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,8), (2,8), (3,8), (4,8), (5,8), (6,8) (9)

and for inspection periods *T*=3,5,7,8,10 which are illustrated orderly in Figure 2. The horizontal axis represents maintenance thresholds as shown in (9). The six graphs in Figure 2 corresponds to the 6 inspection periods 3,5,7,8,10. The lowest curve corresponds *T*=3 and others respectively correspond *T*=5,7,8,10.

As Figure 2 shows for certain pair of maintenance thresholds, the larger the inspection periods, the greater the failure probability; because larger inspection periods allow deterioration to accumulate without any improvement on the system state and consequently this leads to a larger failure rate. This is also true for maintenance thresholds. If inspection periods are fixed, then larger maintenance thresholds will postpone recovery of system. This makes deterioration accumulate and resulting in larger failure rate. Selected maintenance policies for optimization of next phase are as follows:

(0.5,1,3), …, (0.5,1,5), (0.5,2,5), (1,2,5), (1,3,5), (2,3,5), (1,4,5), (2,4,5), (0.5,1,7), (0.5,2,7), (1,2,7), (1,3,7), (0.5,1,8), (0.5,2,8).

Here each triple represents (ξ_1, ξ_1, *T*). Figure 3 shows long-run expected cost for the feasible space provided in the first phase of the optimization procedure.

Every point on each graph has failure probability at most *p* = 0.2.

As Figure 3 shows the optimum maintenance policy for the second phase of the algorithm is (ξ_1,ξ_2,T)=(1,4,5). The maintenance policy (1,4,3) incurs higher maintenance costs while its thresholds are the same of (1,4,5); because frequent inspections on the system leads to more downtime costs on it.

Other maintenance policies like (1,3,5) or (0.5,2,5) which have the same inspection periods of the optimum policy (1,4,5) incur higher costs too; since they cause much maintenance costs due to their lower thresholds.
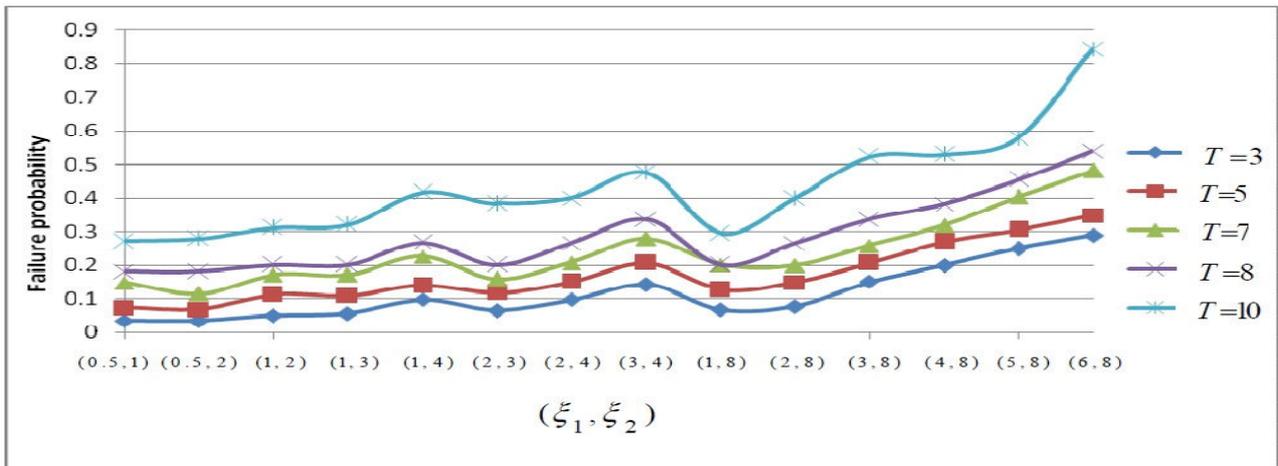
Figure 2: Expected failure probability of the critical system at the steady state.
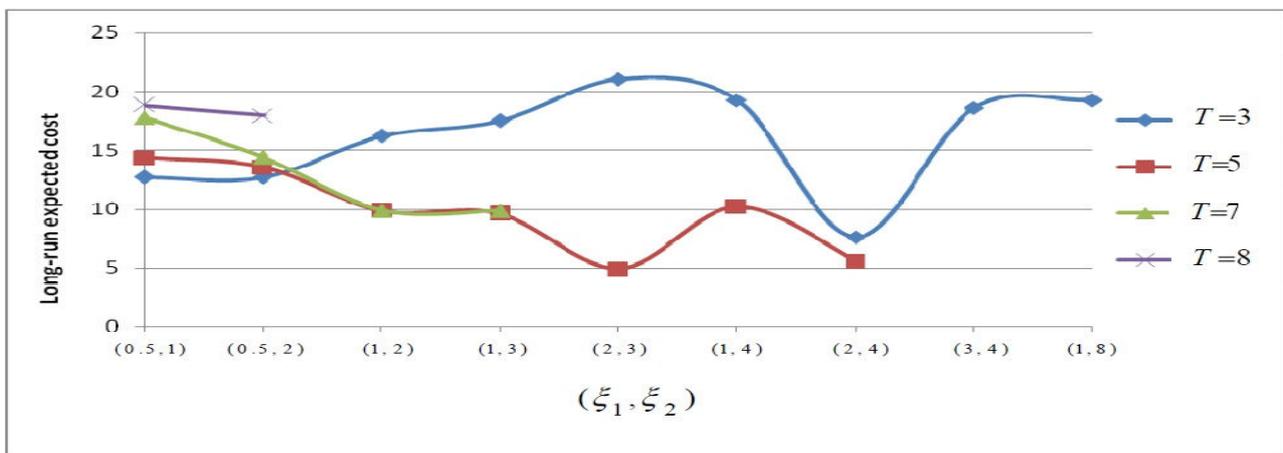
Figure 3: Expected long-run cost for the policies optimized in first phase.

5. Conclusion

In this paper we considered a critical system suffering from continuous deterioration. Since failure of this system can be disastrous it had better to find a policy which constrains failure chance to a pre-specified value. The failure probability was obtained for different policies, and then optimum policy based on long-run expected cost was found.

Adopting this policy guarantees a chance of failure less than $p = 0.2$ with the most possible low cost. The complexity of the system motivated us to use simulation for optimizing maintenance policy. To find the optimum policy which fulfils two objectives failure probability and cost, a search in the feasible space of maintenance thresholds and inspection periods was employed. This procedure can be performed finer and more exhaustive especially for inspection periods, but it will be time consuming.

In order to obtain an optimum result with a high degree of confidence, probability density function of the deterioration process at infinity can be numerically computed, though its calculations is time consuming. The stationary distribution of the deterioration process can be obtained from numerical analysis methods like trapezoidal or Simpson's method.

This work can be easily extended to multi-component systems. Since downtime incurs high costs on the system if a replacement or preventive maintenance action is performed on a component then an opportunistic maintenance action (including preventive replacement or preventive maintenance action) can be performed on any components of the system depending on its condition to save in costs.

Acknowledgments

References

Amari, S. V. and McLaughlin, L., (2004), *Optimal design of a condition-based maintenance model*. Proceedings of the Annual Reliability and Maintainability Symposium, Los Angeles, CA, USA, 528 -533.

Barata, J.; Soares, C. G.; Marseguerra, M. and Zio, E., (2002), Simulation modeling of repairable multi-component deteriorating systems for 'on condition' maintenance optimization. *Reliabil-ity Engineering and System Safety,* 76(3), 255-264.

Barbera, F.; Schneider, H. and Kelle, P., (1996), A condition based maintenance model with exponential failures and fixed inspection intervals. *Journal of the Operational Research Society*, 47(8), 1037-1045.

Castanier, B.; Grall, A. and Berenguer, C., (2005), A condition based maintenance policy with non-periodic inspections for a two unit series system. *Reliability Engineering and System Safety,* 87(1), 109-120.

Chen, D. and Trivedi, K. S., (2005), Optimization for condition based maintenance with semi-Markov decision process. *Reliability Enginee-ring and System Safety*, 90(1), 25-29.

Dieulle, L.; Berenguer, C.; Grall, A. and Roussignol, M., (2003), Sequential condition-based maintenance scheduling for a deteriorating system. *European Journal of Operational Research,* 150(2), 451-461.

Feller, W., (1971), *An introduction to probability theory and its applications*. Wiley series in probability and mathematical statistics. 2nd edition. New York.

Goode, K. B.; Roylance B. J. and Moore J., (2000), Development of model to predict condition monitoring interval times. *Iron making and Steelmaking,* 27(1), 63-68.

Grall, A.; Berenguer, C. and Dieulle, L., (2002), A condition based maintenance policy for stochastically deteriorating systems. *Reliabi-lity Engineering and System Safety*, 76(2), 167-180.

Jardine, A. K. S.; Daming, L. and Dragan, B., (2006), A review on machinery diagnostics and prognostics implementing condition based maintenance. *Mechanical Systems and Signal Processing,* 20(7), 1483-1510.

Kumar, D. and Westberg, U., (1997), Mainten-ance scheduling under age replacement policy using proportional hazards model and TTT plotting. *European Journal of Operational Research,* 99(3), 507-515.

Ross, S. M., (1983), *Stochastic processes*. Wiley, New York.

Wang, W. B., (2002), *A stochastic control model for on line condition based maintenance decision support.* in: Sixth World Multi-conference on Systematics, Cybernetics and Informatics, Proceedings - Industrial Systems and Engineering, Orlando, 6, 370-374.