Ideal and anti-ideal decision making units: A fuzzy DEA approach

Adel Hatami-Marbini*

M.S., Dep. of Industrial Engineering, Islamic Azad University, South Tehran Branch, Tehran, Iran

Saber Saati

Assistant Professor, Dep. of Mathematics, Islamic Azad University, North Tehran Branch, Tehran, Iran

Ahmad Makui

Assistant Professor, Dep. of Industrial Engineering, Iran University of Science and Technology, Tehran, Iran

Abstract

In this paper, by introducing two virtual decision-making units (DMUs) called ideal DMU (IDMU) and anti-ideal DMU (ADMU) with fuzzy inputs-outputs, the efficiency evaluation of DMUs are done by fuzzy data envelopment analysis (FDEA). Therefore, we evaluate DMUs from the perspective of the best and worst possible relative efficiency. For each DMU two efficiencies are calculated while inputs and outputs are fuzzy. These two distinctive efficiencies are combined with the closeness coefficient (CC) index. The CC index is then used for an overall ranking of all DMUs. Finally, we compare the result of proposed fuzzy DEA model with León *et al.'s* (2003) results by representing a numerical example.

Keywords: DEA; Fuzzy mathematical programming; Ideal and anti-ideal decision making unit; Ranking

1. Introduction

Data envelopment analysis (DEA) is a mathematical optimization technique that measures the relative efficiency of decision making units (DMUs) with multiple input–output. Charnes *et al.* (1978) first proposed DEA as an evaluation tool to measure and compare a DMU's relative efficiency. Their model which is commonly referred to as CCR model, assumed constant returns to scale. It was developed for variable returns to scale by Banker *et al.* (1984). That is commonly referred to as BCC model (Cooper *et al.,* 2000).

Evaluating the performance of activities or organizations by conventional DEA models requires crisp input/output data. However, in real-world problems inputs and outputs are often imprecise. The DEA models with fuzzy data can more realistically represent real-world problems than the traditional DEA models. Fuzzy set theory allows linguistic data to be used straightly within the DEA models. Fuzzy DEA (FDEA) models take the form of fuzzy linear programming models.

We can find several fuzzy approaches to the assessment of efficiency in the DEA literature. Sengupta (1992) proposed a fuzzy mathematical programming approach in which fuzziness was incorporated into DEA model by defining tolerance levels on both objective function and constraint violations. Triantis and Girod (1998) suggested a mathematical programming approach through transforming fuzzy input-output data into crisp data using membership function values. Efficiency scores were computed for different values of membership functions and then averaged. Kao and Liu (2000; 2003) introduced a technique which transforms a fuzzy DEA model to a family of crisp DEA models by applying the α-level (also called α-cuts) approach. Despotis and Smirlis (2002) and Entani *et al.* (2002) proposed a DEA model with an interval efficiency consisting of efficiencies obtained from the pessimistic and the optimistic viewpoints. Their models deal with fuzzy data. Lertworasirikul *et al.* (2003) proposed a possibility approach in which

---

* *Corresponding author. E-mail: adel_hatami@yahoo.com*

fuzzy DEA model was transformed into possibility DEA model by using possibility measures on fuzzy events. Hougaard's (2005) approach allows the decision makers to use scores of technical efficiency in combination with other sources of information as expert opinions for instance. The α-cut approach was also adopted by Saati *et al.* (2002), who defined the fuzzy CCR model as a possibilistic-programming problem and transformed it into an interval programming using the concept of α-cuts. Guo and Tanaka (2001) proposed a fuzzy CCR model in which fuzzy constraints including fuzzy equalities and fuzzy inequalities were all converted into crisp constraints by predefining a possibility level and using the comparison rule for fuzzy numbers. Recently, Hatami-Marbini *et al.* (2009) extended the idea of ranking fuzzy numbers to the fuzzy DEA model, and their method was applied to evaluation bank branches. Hatami-Marbini and Saati (2009) also developed a new method to deal with RTS of efficient DMUs in fuzzy BCC when u_0 is a fuzzy variable. Here, we are particularly interested in the approach by León *et al.* (2003), which uses the possibilistic programming. In the present paper we also utilize possibilistic programming techniques to approach the problem of the measurement of efficiency. Some interests are obtained with respect to both computational and interpretative aspects.

In this paper, CCR model is extended to be a fuzzy DEA model for evaluating DMUs from the perspective of the best and worst possible relative efficiency with the given fuzzy input-output data. Following this approach, the results obtained by two different efficiencies are combined with the closeness coefficient (CC) index. The CC index is then used for an overall ranking for all the DMUs.

The paper is organized as follows. In Section 2, first, we review some basic concepts about fuzzy numbers, then, we introduce virtual ideal DMU (IDMU) and anti-ideal DMU (ADMU) in DEA model. Section 3 presents proposed approach for solving fuzzy DEA models. Subsequently, we consider a numerical example provided by León *et al.* (2003) to validate our method in Section 4. The paper is concluded in Section 5.

2. Preliminary definitions

In this section we are simply recalling how to perform the basic operations of arithmetic of fuzzy numbers and introducing IDMU and ADMU in DEA model (Zimmermann, 2005; Klir and Yuan, 1995; Wang and Luo, 2006). They are expressed as follows.

Definition 1. Fuzzy number $\tilde{M}$ is said to be a *LR*-fuzzy number, $\tilde{M} = (p^L, q^R, \alpha^L, \beta^R)_{L,R}$ if and only if its membership function has the following form:

$$\mu_{\tilde{M}}(x) = \begin{cases} L(\dfrac{p^L - x}{\alpha^L}) & x \leq p^L \\ 1 & p^L \leq x \leq q^R \\ R(\dfrac{x - q^R}{\alpha^R}) & x \geq q^R \end{cases} \quad (1)$$

where *L* and *R* are reference functions, i.e., *L; R*: $[0, +\infty[\to [0,1]$ are strictly decreasing in $supp(\tilde{M}) = \{x : \mu_{\tilde{M}}(x) > 0\}$ and upper semi-continuous functions such that $L(0) = R(0) = 1$. If $supp(\tilde{M})$ is a bounded set, *L* and *R* are defined on $[0,1]$ and satisfy $L(1) = R(1) = 0$.

In fuzzy linear programming (FLP), the min T-norm is the most applied to evaluate a linear combination of fuzzy quantities. In particular, for a given set of *LR*-fuzzy numbers $\tilde{d}_j = (m_j^L, n_j^R, \alpha_j^L, \beta_j^R)_{L,R}$, j=1,…,n and some nonnegative x_j, we have that:

$$\sum_{j=1}^{n} \tilde{d}_j x_j = [\sum_{j=1}^{n} m_j^L x_j, \sum_{j=1}^{n} n_j^R x_j, \sum_{j=1}^{n} \alpha_j^L x_j, \sum_{j=1}^{n} \beta_j^R x_j]_{LR} \quad (2)$$

where *L* and *R* are the common left and right reference functions, and $\sum_{j=1}^{n} \tilde{d}_j x_j$ denotes the combination $\tilde{d}_1 x_1 \oplus \tilde{d}_2 x_2 \oplus ... \oplus \tilde{d}_n x_n$. Literature review reveals that multitudes of fuzzy number ranking methods exist (Wang and Kerre, 2001(a); Wang and Kerre, 2001(b); Bortolan and Degani, 1985). The ranking process depends heavily on the environment or the framework of a problem. Here, we consider an approach of fuzzy number ranking that introduce as the following two definitions:

Definition 2. Let $\tilde{M}$ and $\tilde{N}$ be two fuzzy numbers. Then, membership function $\mu_{\tilde{M} \vee \tilde{N}}(x)$ is defined as:

$$\mu_{\tilde{M} \vee \tilde{N}}(x) = \sup_{x = s \vee t} \{\mu_{\tilde{M}}(s) \wedge \mu_{\tilde{N}}(t)\} \quad (3)$$

Based on "fuzzy max" operator, Dubois and Prade (1980) recount the Definition 3.

Definition 3. Considering two fuzzy numbers $\tilde{M}$ and $\tilde{N}$,

$$\tilde{M} \succsim \tilde{N} \Leftrightarrow \tilde{M} \vee \tilde{N} = \tilde{M} \tag{4}$$

Tanaka *et al.* (1984), Leon *et al.* (2003) and Ramík and Římánek (1985) have formulated FLP problems by using this order. In fact, Ramík and Římánek (1985) provided an operative characterization of (3) in terms of the α-cut.

Theorem. Let $\tilde{M}$ and $\tilde{N}$ be two fuzzy numbers. Then, $\tilde{M} \vee \tilde{N} = \tilde{M}$ if and only if, $\forall \alpha \in [0,1]$ the two statements below hold (Ramík and Římánek, 1985):

$$\inf\{s : \mu_{\tilde{M}}(s) \geq \alpha\} \geq \inf\{t : \mu_{\tilde{N}}(t) \geq \alpha\},$$

$$\sup\{s : \mu_{\tilde{M}}(s) \geq \alpha\} \geq \sup\{t : \mu_{\tilde{N}}(t) \geq \alpha\} \tag{5}$$

In particular, if two *LR*-fuzzy numbers $\tilde{M} = (p^L, q^R, \alpha^L, \beta^R)_{L',R'}$ and $\tilde{N} = (b^L, h^R, \xi^L, \mu^R)_{L,R}$ have supported to (5) and both $L = L'$ and $R = R'$, then:

$$p^L \geq b^L, \qquad p^L - \alpha^L \geq b^L - \xi^L,$$

$$q^R \geq h^R, \qquad q^R + \beta^R \geq h^R + \mu^R \tag{6}$$

Definition 4. An IDMU is a virtual DMU that uses the least inputs to generate the most outputs. While an ADMU is a virtual DMU which consumes the most inputs but produces the least outputs.

Let x_{ij} $(i = 1,\ldots,m, j = 1,\ldots,n)$ and y_{rj} $(r = 1,\ldots,s, j = 1,\ldots,n)$ are the *i*th input and the *r*th output of DMU_j, respectively. An ideal IDMU and ADMU in DEA are defined as follows (exact data) (Wang and Luo, 2006):

$$IDMU(x_i^{\min}, y_r^{\max})$$

$$ADMU(x_i^{\max}, y_r^{\min}) \tag{7}$$

where:

$$x_i^{\min} = \min_j\{x_{ij}\}, \quad x_i^{\max} = \max_j\{x_{ij}\}, \quad \forall i$$

$$y_r^{\min} = \min_j\{y_{rj}\}, \quad y_r^{\max} = \max_j\{y_{rj}\}, \quad \forall r$$

On the other hand, Let $\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^R, \alpha_{ij}^L, \alpha_{ij}^R)$ and $\tilde{y}_{rj} = (y_{rj}^L, y_{rj}^R, \beta_{rj}^L, \beta_{rj}^R)$ are the *i*th fuzzy input and the *r*th fuzzy output of DMU_j, respectively. Hence, an IDMU and ADMU in DEA are obtained as follows:

$$\tilde{x}_i^{\min} = (x_i^{L\min}, x_i^{R\min}, \alpha_i^{L\max}, \alpha_i^{R\min}), \quad \forall i$$

$$\tilde{x}_i^{\max} = (x_i^{L\max}, x_i^{R\max}, \alpha_i^{L\min}, \alpha_i^{R\max}), \quad \forall i$$

$$\tilde{y}_r^{\min} = (y_r^{L\min}, y_r^{R\min}, \beta_r^{L\max}, \beta_r^{R\min}), \quad \forall r$$

$$\tilde{y}_r^{\max} = (y_r^{L\max}, y_r^{R\max}, \beta_r^{L\min}, \beta_r^{R\max}), \quad \forall r \tag{8}$$

where

$$x_i^{L\min} = \min\{x_{ij}^L\}, \qquad x_i^{R\min} = \min\{x_{ij}^R\},$$

$$\alpha_i^{L\max} = \max\{\alpha_{ij}^L\}, \qquad \alpha_i^{R\min} = \min\{\alpha_{ij}^R\},$$

$$x_i^{L\max} = \max\{x_{ij}^L\}, \qquad x_i^{R\max} = \max\{x_{ij}^R\},$$

$$\alpha_i^{L\min} = \min\{\alpha_{ij}^L\}, \qquad \alpha_i^{R\max} = \max\{\alpha_{ij}^R\},$$

$$y_r^{L\min} = \min\{y_{rj}^L\}, \qquad y_r^{R\min} = \min\{y_{rj}^R\},$$

$$\beta_r^{L\max} = \max\{\beta_{rj}^L\}, \qquad \beta_r^{R\min} = \min\{\beta_{rj}^R\},$$

$$y_r^{L\max} = \max\{y_{rj}^L\}, \qquad y_r^{R\max} = \max\{y_{rj}^R\},$$

$$\beta_r^{L\min} = \min\{\beta_{rj}^L\}, \qquad \beta_r^{R\max} = \max\{\beta_{rj}^R\},$$

Note that a virtual IDMU and virtual ADMU may not exist in practical production activity.

3. Fuzzy DEA models

Suppose that we have *n* DMUs. Each DMU_j $(j = 1,\ldots,n)$ produces *s* different outputs, y_{rj} $(r = 1,\ldots,s)$, using *m* different inputs, x_{ij} $(i = 1,\ldots,m)$. DEA models are first developed for crisp data and then extended to fuzzy data. Note

that fuzzy data can expressed as *LR*-fuzzy number $\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^R, \alpha_{ij}^L, \beta_{ij}^R)_{L_{ij}, R_{ij}}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ and $\tilde{y}_{rj} = (y_{rj}^L, y_{rj}^R, \alpha_{rj}^L, \beta_{rj}^R)_{L'_{rj}, R'_{rj}}$, $r = 1, \ldots, s$, $j = 1, \ldots, n$ as the trapezoidal fuzzy numbers where first, second, third and forth components display the left, right, left spread and right spread of the related numbers, respectively. Meanwhile for simplifying, data can be described by means of *LR*-fuzzy numbers of the same type as follows:

$$L_{i1} = \ldots = L_{in} = L_i, \qquad i = 1, \ldots, m$$

$$L'_{r1} = \ldots = L'_m = L'_r, \qquad r = 1, \ldots, s$$

$$R_{i1} = \ldots = R_{in} = R_i, \qquad i = 1, \ldots, m$$

$$R'_{r1} = \ldots = R'_m = R'_r, \qquad r = 1, \ldots, s \qquad (9)$$

Let the input-oriented CCR model is used with fuzzy data. Then, we are developed FDEA models with a virtual IDMU and ADMU to get the best and the worst possible relative efficiencies, respectively.

However, the IDMU is a virtual unit, its production behavior should be come the goal of each DMU. According to Wang and Luo (2006), the efficiency of the IDMU can be defined as:

$$\text{Max} \quad \theta_{ID} = \frac{\sum_{r=1}^{s} u_r y_r^{\max}}{\sum_{i=1}^{m} v_i x_i^{\min}}$$

S.t.

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1 \qquad \forall j$$

$$u_r, v_i \ge 0 \qquad \forall r, i \qquad (10)$$

where u_r $(r = 1, \ldots, s)$ and v_i $(i = 1, \ldots, m)$ being the weight on *r*th output and *i*th input, respectively. Model (10) can be solved through a linear form as shown below by performing the Charnes–Cooper transformation (1962):

$$\text{Max} \quad \theta_{ID} = \sum_{r=1}^{s} u_r y_r^{\max}$$

S.t.

$$\sum_{i=1}^{m} v_i x_i^{\min} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \qquad \forall j$$

$$u_r, v_i \ge 0 \qquad \forall r, i \qquad (11)$$

The dual program of (11) is obtained as:

$$\text{Min} \quad \theta_{ID}$$

S.t.

$$\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_{ID} x_i^{\min} \qquad \forall i$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_r^{\max} \qquad \forall r$$

$$\lambda_j \ge 0 \qquad \forall j \qquad (12)$$

When input-output data are *LR*-fuzzy number, (12) can be expressed as the following fuzzy LP problem:

$$\text{Min} \quad \theta_{ID}$$

S.t.

$$\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \lesssim \theta_{ID} \tilde{x}_i^{\min} \qquad \forall i$$

$$\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \gtrsim \tilde{y}_r^{\max} \qquad \forall r$$

$$\lambda_j \ge 0 \qquad \forall j \qquad (13)$$

where,

$$\tilde{x}_i^{\min} = (x_i^{L\min}, x_i^{R\min}, \alpha_i^{L\max}, \alpha_i^{R\min})$$

$$\tilde{y}_r^{\max} = (y_r^{L\max}, y_r^{R\max}, \beta_r^{L\min}, \beta_r^{R\max})$$

$$\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^R, \alpha_{ij}^L, \alpha_{ij}^R)$$

$$\tilde{y}_{rj} = (y_{rj}^{L}, y_{rj}^{R}, \beta_{rj}^{L}, \beta_{rj}^{R})$$

Using inequality (6) and linear combinations as (2), (13) can be transformed as the following equivalent LP model:

$$\text{Min} \quad \theta_{ID}$$

S.t.

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{L} \leq \theta_{ID} x_i^{L\,\min} \qquad \forall i$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{R} \leq \theta_{ID} x_i^{R\,\min} \qquad \forall i$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{L} - \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{L} \leq \theta_{ID} x_i^{L\,\min}$$

$$-\theta_{ID} \alpha_i^{L\,\max} \qquad \forall i$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{R} + \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{R} \leq \theta_{ID} x_i^{R\,\min}$$

$$+\theta_{ID} \alpha_i^{R\,\min} \qquad \forall i$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{L} \geq y_r^{L\,\max} \qquad \forall r$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{R} \geq y_r^{R\,\max} \qquad \forall r$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{L} - \sum_{j=1}^{n} \lambda_j \beta_{rj}^{L} \geq y_r^{L\,\max}$$

$$-\beta_r^{L\,\min} \qquad \forall r$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{R} + \sum_{j=1}^{n} \lambda_j \beta_{rj}^{R} \geq y_r^{R\,\max}$$

$$+\beta_r^{R\,\max} \qquad \forall r$$

$$\lambda_j \geq 0 \qquad \forall j \quad (14)$$

Therefore, the optimal value of (14) provides an evaluation of the efficiency of IDMU in which all the possible values of the different variables for all the DMUs at all the possibility levels are considered. Model (15) is utilized to determine the best relative efficiency of DMU_p using the efficiency of IDMU (with exact data) (Wang and Luo, 2006):

$$\text{Max} \quad \theta_p = \frac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip}}$$

S.t.

$$\frac{\sum_{r=1}^{s} u_r y_r^{\max}}{\sum_{i=1}^{m} v_i x_i^{\min}} \geq \theta_{ID}^{*}$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1 \qquad \forall j$$

$$u_r, v_i \geq 0 \qquad \forall r, i \qquad (15)$$

Model (15) can be expressed as the following linear programming problem:

$$\text{Max} \quad \theta_p = \sum_{r=1}^{s} u_r y_{rp}$$

S.t.

$$\sum_{i=1}^{m} v_i x_{ip} = 1$$

$$\sum_{r=1}^{s} u_r y_r^{\max} - \sum_{i=1}^{m} v_i \theta_{ID}^{*} x_i^{\min} \geq 0$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \qquad \forall j$$

$$u_r, v_i \geq 0 \qquad \forall r, i \qquad (16)$$

The dual program of (16) is as follows:

$$\text{Min} \quad \theta_p$$

S.t.

$$\theta_{ID}^* x_i^{\min} \lambda^* + \sum_{j=1}^n \lambda_j x_{ij} \le \theta_p x_{ip} \qquad \forall i$$

$$y_r^{\max} \lambda^* + \sum_{j=1}^n \lambda_j y_{rj} \ge y_{rp} \qquad \forall r$$

$$\lambda^* \le 0$$

$$\lambda_j \ge 0 \qquad \forall j \qquad (17)$$

Now, since input-output data are *LR*-fuzzy number and θ_{ID}^* is taken from (14), the best relative efficiency of DMU_p can be determined through the following model:

$\mathrm{Min} \ \theta_p$

S.t.

$$\theta_{ID}^* \tilde{x}_i^{\min} \lambda^* + \sum_{j=1}^n \lambda_j \tilde{x}_{ij} \lesssim \theta_p \tilde{x}_{ip} \qquad \forall i$$

$$\tilde{y}_r^{\max} \lambda^* + \sum_{j=1}^n \lambda_j \tilde{y}_{rj} \gtrsim \tilde{y}_{rp} \qquad \forall r$$

$$\lambda^* \le 0$$

$$\lambda_j \ge 0 \qquad \forall j \qquad (18)$$

Similarly, suppose that we have:

$$\tilde{y}_{rj} = (y_{rj}^L, y_{rj}^R, \beta_{rj}^L, \beta_{rj}^R)$$

$$\tilde{y}_r^{\max} = (y_r^{L\max}, y_r^{R\max}, \beta_r^{L\min}, \beta_r^{R\max})$$

$$\tilde{x}_i^{\min} = (x_i^{L\min}, x_i^{R\min}, \alpha_i^{L\max}, \alpha_i^{R\min})$$

$$\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^R, \alpha_{ij}^L, \alpha_{ij}^R)$$

Therefore, (18) can be transformed as the following equivalent LP model:

$\mathrm{Min} \ \theta_p$

S.t.

$$\theta_{ID}^* x_i^{L\min} \lambda^* + \sum_{j=1}^n \lambda_j x_{ij}^L \le \theta_p x_{ip}^L \qquad \forall i$$

$$\theta_{ID}^* x_i^{R\min} \lambda^* + \sum_{j=1}^n \lambda_j x_{ij}^R \le \theta_p x_{ip}^R \qquad \forall i$$

$$\theta_{ID}^* x_i^{L\min} \lambda^* + \sum_{j=1}^n \lambda_j x_{ij}^L - \theta_{ID}^* \alpha_i^{L\max} \lambda^*$$
$$- \sum_{j=1}^n \lambda_j \alpha_{ij}^L \le \theta_{ip} x_{ip}^L - \theta_{ip} \alpha_{ip}^L \quad \forall i$$

$$\theta_{ID}^* x_i^{R\min} \lambda^* + \sum_{j=1}^n \lambda_j x_{ij}^R + \theta_{ID}^* \alpha_i^{R\min} \lambda^*$$
$$+ \sum_{j=1}^n \lambda_j \alpha_{ij}^R \le \theta_{ip} x_{ip}^R + \theta_{ip} \alpha_{ip}^R \quad \forall i$$

$$y_r^{L\max} \lambda^* + \sum_{j=1}^n \lambda_j y_{rj}^L \ge y_{rp}^L \qquad \forall r$$

$$y_r^{R\max} \lambda^* + \sum_{j=1}^n \lambda_j y_{rj}^R \ge y_{rp}^R \qquad \forall r$$

$$y_i^{L\max} \lambda^* + \sum_{j=1}^n \lambda_j y_{rj}^L - \beta_i^{L\min} \lambda^*$$
$$- \sum_{j=1}^n \lambda_j \beta_{rj}^L \ge y_{rp}^L - \beta_{rp}^L \qquad \forall r$$

$$y_r^{R\max} \lambda^* + \sum_{j=1}^n \lambda_j y_{rj}^R + \beta_r^{R\max} \lambda^*$$
$$+ \sum_{j=1}^n \lambda_j \beta_{rj}^R \ge y_{rp}^R + \beta_{rp}^R \qquad \forall r$$

$$\lambda^* \le 0 \qquad\qquad (19)$$

$$\lambda_j \ge 0 \qquad \forall j$$

When input-output data are LR-fuzzy number, the efficiency of the ADMU can be expressed as the following fuzzy LP problem:

Max φ_{AD}

S.t.

$$\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \gtrsim \varphi_{AD} \tilde{x}_i^{\max} \qquad \forall i$$

$$\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \lesssim \tilde{y}_{rp} \qquad \forall r$$

$$\lambda_j \geq 0 \qquad \forall j \qquad (20)$$

Similarly, (20) can be obtained by the following LP:

Max φ_{AD}

S.t.

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{L} \geq \varphi_{AD} x_i^{L\max} \qquad \forall i$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{R} \geq \varphi_{AD} x_i^{R\max} \qquad \forall i$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{L} - \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{L} \geq \varphi_{AD} x_i^{L\max}$$

$$-\varphi_{AD} \alpha_i^{L\min} \qquad \forall i$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{R} + \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{R} \geq \varphi_{AD} x_i^{R\max}$$

$$+\varphi_{AD} \alpha_i^{R\max} \qquad \forall i$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{L} \leq y_r^{L\min} \qquad \forall r$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{R} \leq y_r^{R\min} \qquad \forall r$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{L} - \sum_{j=1}^{n} \lambda_j \beta_{rj}^{L} \leq y_r^{L\min} - \beta_r^{L\max} \qquad \forall r$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{R} + \sum_{j=1}^{n} \lambda_j \beta_{rj}^{R} \leq y_r^{R\min} + \beta_r^{R\min} \qquad \forall r$$

$$\lambda_j \geq 0 \qquad \forall j \qquad (21)$$

Note that, in (21), we have:

$$\tilde{y}_{rj} = (y_{rj}^{L}, y_{rj}^{R}, \beta_{rj}^{L}, \beta_{rj}^{R}),$$

$$\tilde{x}_i^{\max} = (x_i^{L\max}, x_i^{R\max}, \alpha_i^{L\min}, \alpha_i^{R\max})$$

and

$$\tilde{x}_{ij} = (x_{ij}^{L}, x_{ij}^{R}, \alpha_{ij}^{L}, \alpha_{ij}^{R})$$

$$(i = 1, 2, ..., m \ , r = 1, 2, ..., s \ , j = 1, 2, ..., n).$$

Let φ_{AD}^* be the worst efficiency of the ADMU with fuzzy data, its efficiency is clearly worse than other units. Model (22) is utilized to determine the worse relative efficiency of DMU_p using the efficiency of ADMU (with exact data) (Wang and Luo, 2006):

$$\text{M in} \ \varphi_p = \frac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip}}$$

S.t.

$$\frac{\sum_{r=1}^{s} u_r y_r^{\min}}{\sum_{i=1}^{m} v_i x_i^{\max}} \leq \varphi_{AD}^*$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \geq 1 \qquad \forall j$$

$$u_r, v_i \geq 0 \qquad \forall r, i \qquad (22)$$

The fractional programming problem (22) can be expressed as the following linear programming:

$$\text{M in} \ \varphi_p = \sum_{r=1}^{s} u_r y_{rp}$$

S.t.

$$\sum_{i=1}^{m} v_i x_{ip} = 1$$

$$\sum_{r=1}^{s} u_r y_r^{\min} - \sum_{i=1}^{m} v_i \varphi_{AD}^* x_i^{\max} \leq 0$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \geq 0 \qquad \forall j$$

$$u_r, v_i \geq 0 \qquad \forall r, i \qquad (23)$$

The dual of the above model can be formulated as follows:

Max φ_p

S.t.

$$\varphi_{AD}^* x_i^{\max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij} \geq \varphi_p x_{ip} \qquad \forall i$$

$$y_r^{\min} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{rj} \leq y_{rp} \qquad \forall r$$

$$\lambda^* \leq 0$$

$$\lambda_j \geq 0 \qquad \forall j \qquad (24)$$

Now, since input-output data are *LR*-fuzzy number and φ_{ID}^* is taken from (21), the worst possible relative efficiency of DMU_p can be determined through the following model:

Max φ_p

S.t.

$$\varphi_{AD}^* \tilde{x}_i^{\max} \lambda^* + \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \gtrsim \varphi_p \tilde{x}_{ip} \qquad \forall i$$

$$\tilde{y}_r^{\min} \lambda^* + \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \lesssim \tilde{y}_{rp} \qquad \forall r$$

$$\lambda^* \leq 0$$

$$\lambda_j \geq 0 \qquad \forall j \qquad (25)$$

where

$$\tilde{y}_{rj} = (y_{rj}^L, y_{rj}^R, \beta_{rj}^L, \beta_{rj}^R)$$

$$\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^R, \alpha_{ij}^L, \alpha_{ij}^R)$$

$$\tilde{x}_i^{\max} = (x_i^{L\max}, x_i^{R\max}, \alpha_i^{L\min}, \alpha_i^{R\max})$$

$$\tilde{y}_r^{\min} = (y_r^{L\min}, y_r^{R\min}, \beta_r^{L\max}, \beta_r^{R\min})$$

Similarly, model (25) can be transformed as the following equivalent LP model:

Max φ_p

S.t.

$$\varphi_{ID}^* x_i^{L\max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij}^L \geq \varphi_p x_{ip}^L \qquad \forall i$$

$$\varphi_{ID}^* x_i^{R\max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij}^R \geq \varphi_p x_{ip}^R \qquad \forall i$$

$$\varphi_{AD}^* x_i^{L\max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij}^L - \varphi_{AD}^* \alpha_i^{L\min} \lambda^*$$
$$-\sum_{j=1}^{n} \lambda_j \alpha_{ij}^L \geq \varphi_p x_{ip}^L - \varphi_p \alpha_{ip}^L \quad \forall i$$

$$\varphi_{AD}^* x_i^{R\max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij}^R + \varphi_{AD}^* \alpha_i^{R\max} \lambda^*$$
$$+\sum_{j=1}^{n} \lambda_j \alpha_{ij}^R \geq \varphi_p x_{ip}^R + \varphi_p \alpha_{ip}^R \quad \forall i$$

$$y_r^{L\min} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{rj}^L \leq y_{rp}^L \qquad \forall r$$

$$y_r^{R\min} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{rj}^R \leq y_{rp}^R \qquad \forall r$$

$$y_r^{L\min} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{rj}^L - \beta_r^{L\max} \lambda^*$$

$$-\sum_{j=1}^{n} \lambda_j \beta_{rj}^L \leq y_{rp}^L - \beta_{rp}^L \qquad \forall r$$

$$y_r^{R\min}\lambda^* + \sum_{j=1}^{n}\lambda_j y_{rj}^R + \beta_r^{R\min}\lambda^*$$

$$+\sum_{j=1}^{n}\lambda_j \beta_{rj}^R \le y_{rp}^R + \beta_{rp}^R \qquad \forall r$$

$$\lambda^* \le 0$$

$$\lambda_j \ge 0 \qquad \forall j \qquad (26)$$

Let θ_j^* and φ_j^* be the best and the worst possible relative efficiencies of DMU_j, respectively. These two distinctive efficiency assessments may lead to completely different results. Hence, it is essential to consider them together to give an overall assessment of each DMU. Wang and Luo (2006) propose that the coefficient closeness value should be calculated in the similar manner as used in the technique for order preference by similarity to ideal solution (TOPSIS). Coefficient closeness considers the distances from the ideal (to be minimized) and for the nadir (to be maximized) simultaneously through the TOPSIS formula as follows:

$$CC_j = \frac{\varphi_j^* - \varphi_{AD}^*}{(\varphi_j^* - \varphi_{AD}^*) + (\theta_{ID}^* - \theta_j^*)} \qquad (27)$$

Obviously, $0 \le CC_j \le 1$ where $j=1,2,...,n$. The bigger difference between φ_j^* and φ_{AD}^* and the smaller difference between θ_j^* and θ_{ID}^* mean the better the performance of DMU_j. Note that overall ranking for all DMUs provides in uncertain environment based on the CC_j index.

In the next section, a numerical experiment is presented to illustrate the proposed approach. It is obvious, if inputs and outputs are assumed to be symmetrical triangular membership functions, then the two constraints associated with the main values reduce to only one, and can be eliminated since it is redundant as a consequence of the symmetry.

4. Numerical example

A simple numerical example with fuzzy single-input and single-output was introduced by Leon *et al.* (2003). We will consider this example with its data listed in Table 1 and it should be noted, each DMU consumes a symmetrical triangular fuzzy input to produce a symmetrical triangular fuzzy output. The efficiencies of DMUs with proposed method in Leon *et al.* (2003) are listed in last column of Table 1.

From Table 1 we see *A*, *B*, *C* and *G* are efficient with proposed method in Leon *et al.* (2003). Their final result for efficient units is the fuzzy set as follows:

$$\tilde{E}_f = \{(A, 1), (B, 0.3), (C, 1), (G, 0.9)\}$$

Now, we use our proposed models with virtual IDMU and ADMU for these DMUs. The IDMU and ADMU are shown in the last two rows of Table 1. The resulting efficiency scores and *CC* values are listed in Table 2.

From Table 2 that the DEA models based on IDMU evaluations DMUs *A* and *C* are efficient, while other DMUs are not efficient. On the other hand, is based on ADMU appraisals *C* is the best DMU, which is followed by DMUs *A*, *B*, *D*, *E* and *G*, respectively, and both *F* and *H* are the worst DMU. When IDMU and ADMU efficiencies obtained are aggregated with *CC* index for each DMU, a fully ranking order is achieved that are shown in the fourth and fifth columns of Table 2. The overall ranking from proposed models is nearly like to result of Leon et al. such as *A* and *C* that are important for analyst because they can be pattern for future variation in input-output data of other DMUs.

5. Conclusions

In this paper, two kinds of fuzzy DEA models based on virtual IDMU and IDMU are suggested for evaluating the efficiencies of DMUs with fuzzy input and output data. The two distinctive efficiencies are integrated using a closeness coefficient index. Using the obtained *CC* index for each DMU, the overall ranking can be provided. The proposed models for performance assessment of DMUs in fuzzy environment can be applied easily in analyzing management evaluation problem of units. It can be concluded that the proposed fuzzy DEA models extend BCC model to more general forms where crisp, fuzzy and hybrid data can be handled easily. Because uncertainty always exists in human thinking and judgment, fuzzy DEA models can play an important role for perceptual evaluation problems comprehensively existing in the real world.

Table 1. Fuzzy data of DMUs in numerical example and Leon et al. (2003) results.

DMUs	Input	Output	Loen et al. efficiency
A	(3, 2, 2)	(3, 1, 1)	1
B	(4, 0.5, 0.5)	(2.5, 1, 1)	1
C	(4.5, 1.5, 1.5)	(6, 1, 1)	1
D	(6.5, 0.5, 0.5)	(4, 1.25, 1.25)	0.75
E	(7, 2, 2)	(5, 0.5, 0.5)	0.6429
F	(8, 0.5, 0.5)	(3.5, 0.5, 0.5)	0.605
G	(10, 1, 1)	(6, 0.5, 0.5)	1
H	(6, 0.5, 0.5)	(2, 1.5, 1.5)	0.6923
IDMU	(3, 2, 0.5)	(6, 0.5, 1.5)	-
ADMU	(10, 0.5, 2)	(2, 1.5, 0.5)	-

Table 2. The efficiencies by proposed models.

DMUs	IDMU efficiency	ADMU efficiency	CC	Ranking
A	1	1.6118	0.379	2
B	0.6667	1.5425	0.330	3
C	1	2.4608	0.508	1
D	0.6429	1.5189	0.323	4
E	0.6	1.2986	0.274	5
F	0.4235	1	0.191	7
G	0.6	1.2557	0.265	6
H	0.4615	1	0.193	8
IDMU	3	-		
ADMU	-	0.3922		

References

[1] Banker, R. D., Charnes, A., and Cooper, W. W., 1984, Some models for estimating technical and scale inefficiency in data envelopment analysis. *Management Science*, 30, 1078-1092.

[2] Bortolan, G. and Degani, R., 1985, A review of some methods for ranking fuzzy subsets. *Fuzzy Sets and Systems*, 15, 1-19.

[3] Charnes A. and Cooper W., 1962, Programming with linear fractional functions. *Navel Research Logistics Quarterly*, 9, 181-185.

[4] Charnes, A., Cooper, W. W. and Rhodes, E., 1978, Measuring the efficiency of decision-making units. *European Journal of Operational Research*, 2, 429-444.

[5] Cooper W. W., Seiford, L. M., and Tone, K., 2000, *Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software*. Kluwer Academic Publisher, Dordrecht.

[6] Despotis, D. K. and Smirlis, Y. G., 2002, Data envelopment analysis with imprecise data. *European Journal of Operational Research*, 140, 24-36.

[7] Dubois, D. and Prade H., 1980, Systems of linear fuzzy constraints. *Fuzzy Sets and Systems*, 3, 37-48.

[8] Entani, T., Maeda, Y. and Tanaka, H., 2002, Dual models of interval DEA and its extension to interval data. *European Journal of Operational Research*, 136, 32-45.

[9] Guo, P. and Tanaka, H., 2001, Fuzzy DEA: A perceptual evaluation method. *Fuzzy Sets and Systems,* 119, 149-160.

[10] Hatami-Marbini, A., Saati, S. and Makui, A., 2009, An application of fuzzy numbers ranking in performance analysis. *Journal of Applied Sciences,* 9(9), 1770-1775.

[11] Hatami-Marbini, A., Saati, S., 2009, Stability of RTS of efficient DMUs in DEA with fuzzy u_0 under fuzzy data. *Applied Mathematical Sciences,* 3(44), 2157-2166.

[12] Hougaard, J. L., 2005, A simple approximation of productivity scores of fuzzy production plans. *Fuzzy Sets and Systems*, 152, 455-465.

[13] Kao, C. and Liu, S. T., 2000, Fuzzy efficiency measures in data envelopment analysis. *Fuzzy Sets and Systems*, 113, 427-437.

[14] Kao, C. and Liu, S. T., 2003, A mathematical programming approach to fuzzy efficiency ranking. *International Journal of Production Economics*, 86, 145-154.

[15] Klir, G. J. and Yuan, B., 1995, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Prentice-Hall Inc., USA.

[16] León, T., Liern, V., Ruiz, J. L. and Sirvent, I., 2003, A fuzzy mathematical programming approach to the assessment of efficiency with DEA models. *Fuzzy sets and systems*, 139, 407-419.

[17] Lertworasirikul, S., Fang, S. C., Joines, J. A. and Nuttle, H. L. W., 2003, Fuzzy data envelopment analysis (DEA): A possibility approach. *Fuzzy Sets and Systems*, 139, 379-394.

[18] Ramík, J. and Římánek, J. T, 1985, Inequality relation between fuzzy numbers and its use in fuzzy optimization. *Fuzzy Sets and Systems*, 16, 123-138.

[19] Saati, M. S., Memariani, A. and Jahanshahloo, G. R., 2002, Efficiency analysis and ranking of DMUs with fuzzy data. *Fuzzy Optimization and Decision Making*, 1, 255-267.

[20] Sengupta, J. K., 1992, A fuzzy systems approach in data envelopment analysis. *Applied Mathematics and Computation*, 24, 259-266.

[21] Tanaka, H., Ichihasi, H. and Asai, K., 1984, A formulation of fuzzy linear programming problem based on comparison of fuzzy numbers. *Control and Cybernetics*, 13, 185-194.

[22] Triantis, K. and Girod, O., 1998, A mathematical programming approach for measuring technical efficiency in a fuzzy environment, *Journal of Productivity Analysis*, 10, 85-102.

[23] Wang, X. and Kerre, E. E., 2001(a), Reasonable properties for the ordering of fuzzy quantities (I). *Fuzzy Sets and Systems*, 118, 375-385.

[24] Wang, X. and Kerre, E. E., 2001(b), Reasonable properties for the ordering of fuzzy quantities (II). *Fuzzy Sets and Systems*, 118, 387-405.

[25] Wang, Y. M. and Luo, Y., 2006, DEA efficiency assessment using ideal and anti-ideal decision-making units. *Applied Mathematics and Computation*, 173, 902-915.

[26] Zimmermann, H. J., 2005, *Fuzzy Set Theory and Its Applications*. Forth edition, Springer, New York.