J. Ind. Eng. Int., 6 (11), 17-22, Spring 2010
ISSN: 1735-5702

Fuzzy completion time for alternative stochastic networks

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Received: 28 February 2008; Revised: 26 April 2008; Accepted: 2 September 2008

Abstract: In this paper a network comprising alternative branching nodes with probabilistic outcomes is considered. In other words, network nodes are probabilistic with exclusive-or receiver and exclusive-or emitter. First, an analytical approach is proposed to simplify the structure of network. Then, it is assumed that the duration of activities is positive trapezoidal fuzzy number (TFN). This paper combines the randomness and fuzziness and shows that the fuzzy completion time of alternative stochastic network is a fuzzy-valued random variable. Then, the probability function of network fuzzy completion time and its expected value is defined. Finally, the applications and computations are illustrated in a numerical example.

Keywords: *Fuzzy completion time; Fuzzy-valued random variables; Probabilistic nodes; Stochastic networks; Trapezoidal fuzzy numbers*

1. Introduction

In many real-world projects, the realization of activities and their durations are non-deterministic. A non-deterministic character may be stochastic or fuzzy.

Previous researches supposed that the realization of activities and their durations are stochastic (Pritsker and Happ, 1966; Pritsker and Whitehouse, 1966). This is the reason of GERT-type networks creation. On the other hand, completion of projects on time has a significant effect on their costs, revenue and usefulness. So, acquisition of network completion time will be valuable. Some new analytical methods for determining the completion time of GERT-type networks have been proposed by Shibanov (2003) and Hashemin and Fatemi Ghomi (2005). The main problem of these methods is its high complexity of relations and computations.

Recent works define the fuzzy characters for project networks because the fuzzy models are closer to reality and simpler to use (Lootsma, 1989). Many of new works are related to the fuzzy PERT-type networks (McCahon and Lee, 1988; Shipley, 1997; Chanas and Zielinski, 2001; Kuchta, 2001; Lin and Yao, 2003; Chen and Hung, 2007) and just few of them are related to the GERT-type networks. Based on fuzzy GERT-type networks, Gavareshki (2004) proposes a new applicable technique for research project scheduling. In this project, nodes are fuzzy and output activities from nodes of network belong to a fuzzy set. This method computes the network completion time as a fuzzy number. Liu *et al.*, (2004) believe that the traditional GERT networks cannot reflect the characteristics of real-world network problems and uses the triangular fuzzy numbers to formulate the fuzzy GERT model. Many of real projects complete through the realization of one and only one path of various possible network paths, and the aim of the present paper is studying this case. Here, these networks called alternative stochastic networks. This paper combines the randomness and fuzziness in the above-mentioned networks. Fuzziness and randomness are two basic types of uncertainty. In many cases, fuzziness and randomness simultaneously appear in a system. Fuzzy random variable (fuzzy-valued random variable) and random fuzzy variable are instances of hybrid variable (Liu, 2008). A fuzzy random variable is a random element that takes fuzzy variable values (Kwakernaak, 1978,1979). In addition, a random fuzzy variable is a fuzzy element that takes random variable values (Liu, 2009).

In this paper, it is supposed that nodes of considered network are probabilistic with exclusive-or receiver and exclusive-or emitter. Although many types of fuzzy sets have been used to describe uncertainties, triangular and trapezoidal fuzzy sets (fuzzy numbers) are applied to describe uncertain activity duration in the research (Zhang, 2005). So, activity durations are considered positive fuzzy number as trapezoidal. First, a new analytical approach is proposed to simplify the structure of network. This approach transforms the network to simpler equivalent network. It is shown that the network completion time of these networks is a fuzzy-valued random variable. Then, the probability function of fuzzy completion time of network has been determined. Finally, the expected value of fuzzy completion time of network is defined as a fuzzy number. The paper has the following structure: Section 2 introduces the definitions, operations and assumptions. Analytical approach is

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developed in Section 3. Section 4 describes the fuzzy completion time of network as a fuzzy-valued random variable. Section 5 introduces the applications and gives a numerical example to demonstrate how the proposed method works. Section 6 is devoted to conclusions and recommendations.

2. Definitions, operations and assumptions

2.1. Definitions

In this section, some basic notions of fuzzy theory that have been defined by Zimermann (1996) are introduced.

Definition 1: Let R be the set of real numbers. A fuzzy set $\tilde{A}$ is a set of ordered pairs $\{(x, \mu_{\tilde{A}}(x)) \mid x \in R\}$, where $\mu_{\tilde{A}}(x): R \to [0,1]$ and is called membership function of the fuzzy set.

Definition 2: A convex fuzzy set, $\tilde{A}$, is a fuzzy set in which: $\forall\, x, y \in R, \forall\, \lambda \in [0,1]$,

$$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min[\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)]$$

Definition 3: A fuzzy set $\tilde{A}$ is called positive if its membership function is such that:

$$\mu_{\tilde{A}}(x) = 0, \forall x \leq 0 .$$

Definition 4: Trapezoidal fuzzy number (TFN) is a convex fuzzy set, which is defined as: $\tilde{A} = (x, \mu(x))$ where:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x \leq a \\ \dfrac{x-a}{b-a} & a < x \leq b \\ 1 & b < x \leq c \\ \dfrac{x-d}{c-d} & c < x \leq d \\ 0 & x < d \end{cases}$$

For convenience, TFN represented by four real parameters a,b,c,d (a≤b≤c≤d) will be denoted by a trapezoid (a,b,c,d) (Fig.1).

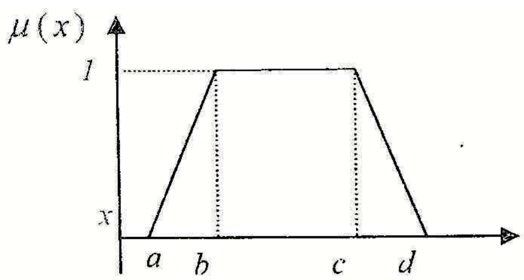

Figure1: Trapezoidal Fuzzy number (TFN).

Definition 5: A trapezoidal fuzzy number $\tilde{A}$ =(a,b,c,d) is called positive TFN if:

0≤a≤b≤c≤d.

2.2.Operation on TFNs

A number of operations can be performed on TFNs. Fuzzy addition and fuzzy scalar multiplication has been defined in (Lai and Hwang, 1992).

Fuzzy addition: Let

$$\tilde{A} = (a_1, b_1, c_1, d_1)$$

and

$$\tilde{B} = (a_2, b_2, c_2, d_2)$$

be any two TFNs, then:

$$\tilde{A} \oplus \tilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$$

Fuzzy scalar multiplication: Let

$$\tilde{A} = (a_1, b_1, c_1, d_1)$$

be any TFN and q be any real number, then:

$$q \odot \tilde{A} = (qa_1, qb_1, qc_1, qd_1)$$

2.3. Assumptions

a) Network has a single source node and it can have one or more sink node.

b) Network contains only exclusive-or, probabilistic nodes (nodes with exclusive-or receiver and exclusive-or emitter) with given probabilities.

c) Network does not contain any loop.

d) Duration of network activities is a positive TFN.

3. Analytical approach

In this section, an analytical approach is developed to simplify the structure of network. First, the notations are introduced.

N Number of activities,

M Number of sink nodes,

M_i The structural matrix of subnetwork i,

A_i The event of realization of i-th sink node,

n_i Number of paths which start from source node and terminate in i-th sink node,

A_{ij} The event of realization of i-th sink node through the realization of j-th path leading to that node,

T_{ij} Fuzzy completion time of network by realization of i-th sink node through the realization of j-th path leading to that node,

P_{ij} Realization probability of j-th path which terminates in i-th sink node,

p_i Realization probability of i-th sink node

$\overline{P}_k$ Accomplishment probability of k-th activity, given that start node of this activity has realized,

T_i Fuzzy completion time of subnetwork i (It is a fuzzy-valued random variable),

S_{ij} Activity set of j-th path which terminates in i-th sink node,

t_k Duration time of k-th activity (It is a positive TFN),

T Fuzzy completion time of network (It is a fuzzy-valued random variable).

A network can be decomposed to M subnetworks in a way that subnetwork i, i=1,2,…,M comprises all paths which terminates in i-th sink node. Structure of subnetwork i is shown by M_i, i=1,2,…,M.

M_i is a matrix with n_i rows and N columns. Elements of matrix M_i are shown by m^i_{jk}, j=1,2,…,n_i , k= 1,2,…,N are defined as follows:

$$m^i_{jk} = \begin{cases} 1 & if\ activity\ k \in S_{ij} \\ 0 & Otherwise \end{cases}$$

Definition 6: Subnetwork $\bar{i}$ is Equivalent with subnetwork i if and only if $M_i = M_{\bar{i}}$ and $T_i = T_{\bar{i}}$.

For obtaining the equivalent subnetwork $\bar{i}$ notice that $A_i = \bigcup_{j=1}^{n_i} A_{ij}$ and $\forall j_1 \neq j_2 \ \ A_{ij_1} \cap A_{ij_2} = \varnothing$. Then, $p(A_i) = \sum_{j=1}^{n_i} p(A_{ij})$ and consequently $P_i = \sum_{j=1}^{n_i} P_{ij}$ (because $p(A_i) = p_i$, $p(A_{ij}) = p_{ij}$) .

Also it is evident that $p_{ij} = \prod_{k \in S_{ij}} \overline{p}_k$. Therefore, subnetwork $\bar{i}$ comprises n_i parallel alternative paths that terminate in i-th sink node. Any of these paths are related with one of the rows of M_i (we select $M_{\bar{i}} = M_i$). Realization probability of these paths is equal with P_{ij} . Length (Time) of these paths is a fuzzy number as follows:

$$T(A_{ij}) = \sum_{k \in S_{ij}} \oplus t_k$$

where $\sum \oplus$ shows the fuzzy summation.

Although some of the activities of these paths are common but $T_i = T_{\bar{i}}$, because when the subnetwork $\bar{i}$ is realized, one and only one of these paths is realized. Therefore, subnetwork $\bar{i}$ is equivalent with subnetwork i. Then, the equivalent network has ($\sum_{i=1}^{M}\sum_{j=1}^{n_i} 1$) alternative parallel paths.

4. Fuzzy completion time of network as a fuzzy-valued random variable

Kwakernaak (1978) introduced the notion fuzzy random variable in the following way. A fuzzy random variable T defined on a probability space $(\Omega, \Sigma, \mathrm{P})$ is characterized by a map $T : \Omega \to S$ such that $\omega \xrightarrow{T} T_\omega$ where S is a collection of all piecewise continuous functions $R \to [0,1]$. Each element of S is a membership function of fuzzy number. Completion time of network (T) is a fuzzy-valued random variable. Fuzzy-valued random variable is a random variable with fuzzy values as fuzzy numbers (in this paper TFN). Realization of an alternative stochastic network can be considered as a random experiment. Sample space of this experiment is:

$$\Omega = \bigcup_{\substack{i=1,\dots,M \\ j=1,\dots,n_i}} A_{ij}$$

Network completion time (T) is a fuzzy-valued random variable and defined on Ω as a function with $T(A_{ij}) = \sum_{k \in S_{ij}} \oplus t_k = T_{ij}$ rule. The probability function of T is as follows:

$$p(T = T_{ij}) = p_{ij} \qquad i = 1,\ldots,M \quad , \quad j = 1,\ldots,n_i$$

The expected value of network completion time can be defined as:

$$E(T) = \sum_{i=1}^{M} \oplus \sum_{j=1}^{n_i} \oplus P_{ij} \odot T(A_{ij})$$

or

$$E(T) = \sum_{i=1}^{M} \oplus \sum_{j=1}^{n_i} \oplus P_{ij} \odot \sum_{k \in S_{ij}} \oplus t_k \qquad (1)$$

5. Applications and numerical example

Many of real-world projects complete through the realization of one and only one of the paths of project network. These networks are called alternative stochastic networks. The network of a research project may be an alternative stochastic network. Also, the production line of a product may be presented as an alternative stochastic network. Some of repairing systems can be illustrated as an alternative stochastic network. In all above cases, activity durations can be defined as fuzzy numbers. So, computing the probability function of network completion time (It is a fuzzy-valued random variable) and its expected value will be useful.

On a production line a part is manufactured at the beginning of the line. Before the finishing, it is inspected, with 25% of parts failing the inspection and requiring rework. Manufacturing and inspection is called activity 1. Reworking is called activity 2. 30% of the reworked parts fail in the next inspection (activity 4). Parts that fail in this inspection are scrapped (activity 6). If the part passes either of the above inspections, it is sent to the final finishing operation (activity 3,5) A final inspection (activity 7) rejects 5% of the parts; these are scrapped (activity 8) and accepts 95% of the parts (activity 9). The manufacturer intends to know what are the probabilities of having non-defective and scrapped parts and the corresponding probability distribution function of the times will take for receipt of non-defective and scrapped parts. The alternative stochastic network for the above production line is illustrated in Fig. 2. The network in Fig. 2 comprises two sink nodes. In other words, it comprises two subnetworks:

Subnetwork 1 and subnetwork 2 are shown in Fig. 3 and Fig. 4 respectively.

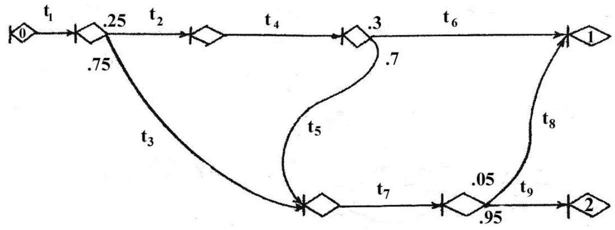

Figure 2: Network in the numerical example.

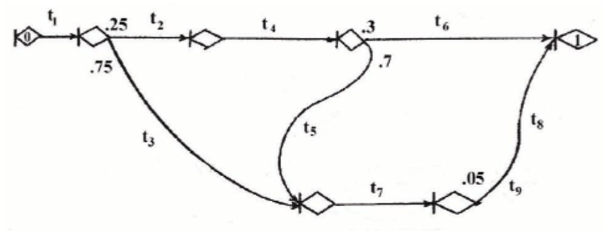

Figure 3: Subnetwork 1.

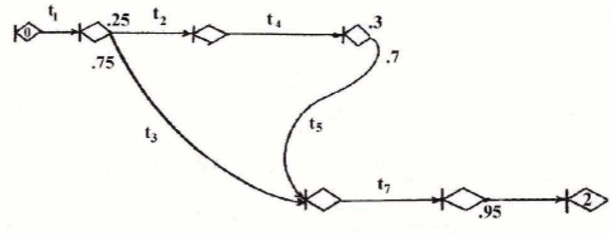

Figure 4: Subnetwork 2.

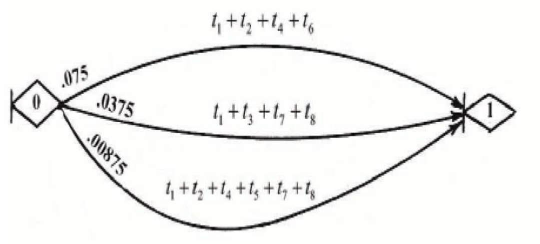

Figure 5: Subnetwork 1.

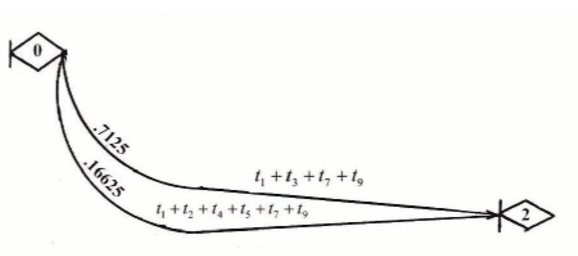

Figure 6: Subnetwork 2.

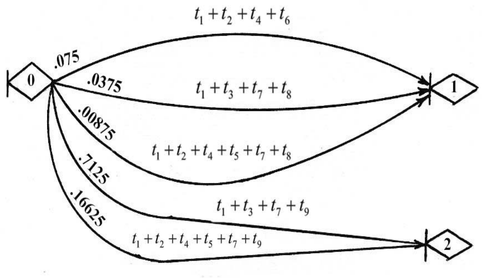

Figure 7: Equivalent network.

Structural matrixes of above subnetworks are as follows:

$$M_1=\begin{bmatrix}1&1&0&1&0&1&0&0&0\\1&0&1&0&0&0&1&1&0\\1&1&0&1&1&0&1&1&0\end{bmatrix}$$

$$M_2=\begin{bmatrix}1&0&1&0&0&0&1&0&1\\1&1&0&1&1&0&1&0&1\end{bmatrix}$$

Activity set of paths and their realization probabilities are:

$$t_1+t_3+t_7+t_9$$

$$P_{21}=\overline{P}_1\overline{P}_3\overline{P}_7\overline{P}_9=(1)\ (.75)\ (1)\ (.95)=.7125$$

$$t_1+t_2+t_4+t_5+t_7+t_9$$

$$P_{22}=\overline{P}_1\overline{P}_2\overline{P}_4\overline{P}_5\overline{P}_7\overline{P}_9=(1)(.25)(1)(.7)(1)(.95)=.16625$$

$$t_1+t_2+t_4+t_6$$

$$P_{11}=\overline{P}_1\overline{P}_2\overline{P}_4\overline{P}_6=(1)\ (.25)\ (1)\ (.3)=.075$$

$$t_1+t_3+t_7+t_8$$

$$P_{12}=\overline{P}_1\overline{P}_3\overline{P}_7\overline{P}_8=(1)\ (.75)\ (1)\ (.05)=.0375$$

$$t_1+t_2+t_4+t_5+t_7+t_8$$

$$P_{13}=\overline{P}_1\overline{P}_2\overline{P}_4\overline{P}_5\overline{P}_7\overline{P}_8=(1)(.25)(1)(.7)(1)(.05)=.00875$$

$$\mathrm{P}_1=\mathrm{P}_{11}+\mathrm{P}_{12}+\mathrm{P}_{13}=.075+.0375+.00875=.12125$$

$$\mathrm{P}_2=\mathrm{P}_{21}+\mathrm{P}_{22}=.7125+.16625=.87875$$

Therefore, the equivalent subnetworks $\overline{1}$ and $\overline{2}$ will be as shown in Fig. 5 and Fig. 6. Hence, the equivalent network is as shown in Fig. 7.

Suppose that the duration of activities are trapezoidal numbers as shown in Table 1.

Now, the probability function of completion time of network (which is a fuzzy-valued random variable) can be obtained as follows:

Table 1: Duration of activities as positive trapezoidal numbers.

k	a	b	c	d
1	1	2	4	5
2	2	3	5	6
3	1.5	2	3	3.5
4	2.5	3	4	4.5
5	0.5	1	2	2.5
6	0.5	1.5	3.5	4.5
7	1	2	3	4
8	2	3	4	5
9	1.5	2.5	4	5

$$\begin{aligned}p(T=T(A_{ij}))&=0.075 \quad i=1,j=1,T(A_{11})=(6,9.5,16.5,20)\\&=0.0375 \quad i=1,j=2,T(A_{12})=(5.5,9,14,17.5)\\&=0.00875 \quad i=1,j=3,T(A_{13})=(9,14,22,27)\\&=0.7125 \quad i=2,j=1,T(A_{21})=(5,8.5,14,17.5)\\&=0.16625 \quad i=2,j=2,T(A_{22})=(8.5,13.5,22,27)\end{aligned}$$

It is obvious that, $P(T_1=T(A_{1j})), j=1,2,3$ and $P(T_2=T(A_{2j})), j=1,2$ for sink nodes 1, 2 can be computed.

By using formula (1) the expected network completion time can be computed as trapezoidal number.

$$E(T)=\sum_{i=1}^{M}\oplus\sum_{j=1}^{n_i}\oplus P_{ij}\odot T(A_{ij})$$

$$\begin{aligned}&=0.075\odot(6,9.5,16.5,20)\\&\oplus 0.0375\odot(5.5,9,14,17.5)\\&\oplus 0.00875\odot(9,14,22,27)\\&\oplus 0.7125\odot(5,8.5,14,17.5)\\&\oplus 0.16625\odot(8.5,13.5,22,27)\\&=(5.935625,9.473125,15.5875,19.35)\end{aligned}$$

6. Conclusion

This paper has shown that a stochastic network with alternative branching node and fuzzy activity durations is a suitable tool to describe non-deterministic projects. In this paper, the network completion time, as a non-deterministic character of alternative networks, has been determined as a fuzzy-valued random variable. Then, the probability function of this fuzzy-valued random variable and its expected value has been defined. Consequently, it is shown that the expected network completion time can be obtained as a fuzzy number. In this study, nodes of network were probabilistic with exclusive-or receiver and exclusive-or emitter. Considering the other types of nodes can be good subjects for future studies. Also, it is supposed that the activity durations are positive trapezoidal fuzzy numbers. Future studies can be considered with other types of fuzzy numbers. Similar computations can be done with α -cut set of activity durations.

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