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A fuzzy random multi-objective approach for portfolio selection

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Abstract: In this paper, the portfolio selection problem is considered, where fuzziness and randomness appear simultaneously in optimization process. Since return and dividend play an important role in such problems, a new model is developed in a mixed environment by incorporating fuzzy random variable as multi-objective nonlinear model. Then a novel interactive approach is proposed to determine the preferred solution. Finally a numerical example is presented to illustrate the proposed model.

Keywords: *Fuzzy random programming; Multi-objective; Nonlinear programming; Portfolio selection.*

1. Introduction

Portfolio selection is concerned with the problem of allocating one's wealth among alternative assets so that the investment goal can be satisfied. Modern portfolio analysis started from initial research work of Markowitz (1952; 1959) that was called mean-variance model, in which an investor should always stabilize between maximizing the expected return, as expected value of returns of assets, and/or minimizing the risk, as variance from the expected value, i.e., minimizing the risk for a given expected return, and/or maximizing expected return for a given risk. Portfolio theory has been greatly improved since Markowitz. Traditionally, returns of individual assets were assumed to be stochastic variables, and many researches were focused on extending Markowitz's mean-variance models (Best and Hlouskova, 2000; Merton, 1972; Voros, 1986; Yoshimoto, 1996) and on developing new mathematical approaches to solve the problem of computation (Perold, 1984; Sharp, 1963). In those works, the investor must approve that all of the required information is brought to deal with the existing problem. However, identifying all relevant information for a decision does not mean that the investor has all information; in most cases, information is imperfect. Since decisions that must be made by the investor does not contain adequate knowledge of the problem, then s/he faces events in which reasonable probability for alternative outcomes does not exist; thus decision must be made under conditions of uncertainty. Though probability theory is one of the main techniques used for analyzing uncertainty in finance, the financial market is also affected by several non-probabilistic factors such as vagueness and ambiguity. Investors are commonly provided with information which is characterized by linguistic descriptions such as high risk, low profit, high interest rate, etc. (Sheen, 2005). With the introduction of fuzzy set theory by Zadeh (1965; 1978), scholars began to perceive that they could employ fuzzy set theory to manage portfolio in another type of uncertain environment called fuzzy environment. Among all, for instance, Tanaka and Guo (1999), Tanaka *et al.* (2000), Parra *et al.* (2001) and Carlsson *et al.* (2002) replaced probability distributions of returns of assets with possibility functions in their models.

On the other hand, however, in the decision making process we may face a hybrid uncertain environment where linguistic and frequent imprecise nature coexist. For example, a farmer specializes in raising wheat, rice, corn and cotton on his 100 acres of land. At the beginning of each year, he would make a plan about how much land to devote to each crop. The yields of the crops are required to meet the personal needs, and the surplus will be sold. On the other hand, if the yields of some crop such as corn do not meet the personal needs, the farmer buy some from the market. Generally speaking, it is a hard work for the farmer to make a proper decision, because the yields of each crop depend on the changing weather conditions. Moreover, even if the farmer learns some information about weather conditions,

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he usually expresses his judgment by *good*, *average*, *bad* or the terms *around* 100/*acre*, and *about* 100/*acre* for the yields of the crops at end of the year. Therefore, an appropriate expression about the judgment of the farmer might be as follows: the yields of the crops in the coming year may have a 50% chance of being *good*, a 30% chance of being *average* and a 20% chance of being *bad*. This is an example which fuzzy random (and not random fuzzy) variable may be used in a decision making process. There are other examples provided by some researchers in the literature, e.g., Wang and Qiao (1993), Luhandjula and Gupta (1996), Qiao and Wang (1993), Luhandjula (1996). Zadeh (2005) outlined the generalized Theory of Uncertainty in view of uncertainty in a broader perspective. Xu *et al.* (2008) proposed a fuzzy random environment applied to inventory problems. Katagiri *et al.* (2008) gave an interactive multi-objective fuzzy random linear programming. Liu (2001), where the concept of the primitive chance function of a fuzzy random event was introduced, and fuzzy random chance-constrained programming and dependent-chances programming models were built based on primitive chance. For the application of portfolio problem, Elikyurt and Ozekici (2007) studied a several multi-period portfolio optimization models where the market consists of a risk free asset and several risky assets with the returns in any period are random.

In the above-cited works, expected return and risk are two main factors which investors consider. It is often found in portfolio selection that not all the relevant information for an investment decision can be captured in terms of explicit return and risk. By considering additional and/or alternative decision criteria, a portfolio that is dominated with respect to the expected return and risk may make up for the deficit in these two criteria by a very good performance on one or several other criteria and thus be non-dominated in a multi-criteria setting. As a result, portfolio selection models that consider more criteria than the standard expected return and variance objectives of the Markowitz model have become popular. Parra *et al.* (2001) proposed a model that considers three criteria, return, risk and liquidity. Ehrgott *et al.* (2004) took into account five criteria (short and long-term return, dividend, ranking and risk) and used an MCDM approach to solve the portfolio selection problem. Fang *et al.* (2006) proposed a portfolio rebalancing model with transaction costs based on fuzzy decision theory considering three criteria (return, risk and liquidity). Gupta *et al.* (2008) used short term return, long term return, dividend, risk and liquidity through an application of fuzzy mathematical programming. The paper at hand has two important applied and theoretical contributions. First, it presents a practical, but tractable, optimization model for portfolio selection problem and it is considers two objectives; returns and dividend. And second, it introduces a novel solution procedure for finding an efficient solution to a fuzzy random multi-objective nonlinear program.

The remainder of this paper is organized as follows: In Section 2 we define our notation, state our assumptions and propose a fuzzy random multi-objective for portfolio selection problem. In Section 3 we convert the model into its crisp equivalent and present an interactive algorithm to derive a satisfying solution for an investor. In Section 4 a numerical example of the proposed model and algorithm is illustrated. Finally, conclusion remarks and further research directions are the subject of Section 5.

2. Problem formulation

In this section, we formulate portfolio selection problem as an optimization problem with multiple objectives. We assume that an investor allocates his/her wealth among n assets offering fuzzy random rates of return and annual dividend. We define a fuzzy random variable at first and then introduce some notations as follows:

Definition 1. (Liu and Liu, 2003) Let $(\Omega, A, \Pr)$ be a probability space. A fuzzy random variable is a function $\xi : \Omega \to F$ such that for any Borel set B or R, $\xi^*(B)(\omega) = Pos\{\xi(\omega) \in B\}$ is a measurable function of ω.

$\tilde{\bar{R}}_i$	Rate of return associated with i^{th} asset.
$\tilde{\bar{D}}_i$	Annual dividend on the i^{th} asset.
c_i	Fixed sum for the service associated with i^{th} asset.
C	Capital invested.
∂	Preset tolerable level.
u_i	Maximum proportion invested in i^{th} asset.
x_i	Proportion of total funds invested in the i^{th} asset.

y_i Binary variable indicating whether the i^{th} asset is contained in the portfolio or not.

In the proposed possibilistic multi-objective asset portfolio selection problem, we consider the following objectives and constraints:

2.1. Objectives

Rate of return. The expected rate of return is the most practical objective which is usually used in portfolio optimization models. However, the investor may feel that maximizing return of a portfolio itself better match one's intuition than maximizing expected value of the portfolio. For n assets portfolio, the return of the portfolio is expressed as,

$$\text{Max} \sum_i \tilde{\tilde{R}}_i x_i \tag{1}$$

Annual dividend. This objective function represents the relative annual dividend of the portfolio. It is calculated as the weighted sum of the relative annual dividends D_i of the individual assets in the portfolio, where the total dividend of an asset is set in relation to its highest sales price during the last twelve months.

Alternatively, the lowest price of an asset during the last twelve months or its current value could be used as a reference value. We decide to use the highest price of an asset here since this approach, in general, underestimate the relative annual dividend and is therefore a cautious value. Hence,

$$\text{Max} \sum_i \tilde{\tilde{D}}_i x_i \tag{2}$$

2.2. Constraints

Since portfolio return is fuzzy random variable, the goal function cannot give a deterministic number. Then it is natural for the investor to set it the goal that at a given confidence level which is considered as the safety margin, the maximal return must be achieved.

Since a portfolio with a relatively high variance can also be relatively safe if its expected value is sufficiently high (Li and Huang, 1996), the investor can change the constraint to the requirement that the result of expected value of the portfolio divided by variance should be equal to or greater than a preset level. To express the goal and constraints in mathematical expression, we have the following relation:

$$\frac{E(\sum_i \tilde{\tilde{R}}_i x_i)}{V(\sum_i \tilde{\tilde{R}}_i x_i)} \geq \partial \tag{3}$$

where $E(.)$ and $V(.)$ denote expected value and variance operators respectively.

Total investment. It requires that be at most equals to capital invested,

$$\sum_i c_i y_i \leq C \tag{4}$$

Capital budget constraint on the assets denotes the proportion of total amount of capital invested in a single asset,

$$\sum_i x_i = 1 \tag{5}$$

Asset satisfaction. This assures that each asset which holds a value, will be selected in the portfolio,

$$x_i \leq y_i \qquad i = 1, 2, ..., n \tag{6}$$

Maximal fraction. Proportion of the capital that can be invested in a single asset,

$$x_i \leq u_i \qquad i = 1, 2, ..., n \tag{7}$$

No short selling of assets,

$$x_i \geq 0, \qquad i = 1, 2, ..., n \tag{8}$$

Asset selected,

$$y_i \in \{0,1\}, \qquad i = 1, 2, ..., n \tag{9}$$

2.3. Model relaxation

To formulate the portfolio models, it is necessary to know the probability distribution of the portfolio return. At least, we need to know the mean vector and the covariance matrix of the return vector. However, in order to determine the covariance matrix of risky assets in a fuzzy random economic environment, it needs to estimate the joint possibility distribution of pair assets and that is nearly impossible. For this reason we provide some theorems to avoid using covariance matrix.

Theorem 1. (Liu, 2007) Let f be a convex function on $[a,b]$, and ξ a fuzzy variable that takes values in $[a,b]$. Then,

$$E[f(\xi)] \le \frac{b-E(\xi)}{b-a} f(a) + \frac{E(\xi)-a}{b-a} f(b).$$

Theorem 2. (Liu, 2007) Let ξ be a fuzzy variable that takes values in $[a,b]$. Then,

$$V[\xi] \le [E(\xi)-a][b-E(\xi)].$$

Theorem 3. Under the same assumptions as in Theorem 2, for fuzzy random variable $\sum_i \tilde{R}_i x_i$,

we have:

$$V(\sum_i \tilde{R}_i x_i) \le \left[E(\sum_i \tilde{R}_i x_i) - \sum_i \alpha_i x_i\right] \left[\sum_i \beta_i x_i - E(\sum_i \tilde{R}_i x_i)\right]$$

Proof. By definition 1 we know that fuzzy random variable $\tilde{\tilde{R}}_i$ where $\tilde{\tilde{R}}_i \in [\alpha_i, \beta_i]$ has a fuzzy nature at first therefore from Theorems 1 and 2 it is clear that the variance of $\sum_i \tilde{R}_i x_i$ is convex function in $[\sum_i \alpha_i x_i, \sum_i \beta_i x_i]$,and by $\xi = \sum_i \tilde{\tilde{R}}_i x_i$

we have:

$$V(\sum_i \tilde{\tilde{R}}_i x_i) \le \left[E(\sum_i \tilde{\tilde{R}}_i x_i) - \sum_i \alpha_i x_i\right] \left[\sum_i \beta_i x_i - E(\sum_i \tilde{\tilde{R}}_i x_i)\right]$$

This completes the proof. ■

From (3) and Theorem 3 it can be seen that:

$$V(\sum_i \tilde{\tilde{R}}_i x_i) \le \min\Big\{ \tfrac{1}{\partial} E(\textstyle\sum_i \tilde{\tilde{R}}_i x_i), \left[E(\sum_i \tilde{\tilde{R}}_i x_i) - \sum_i \alpha_i x_i\right] \left[\sum_i \beta_i x_i - E(\sum_i \tilde{\tilde{R}}_i x_i)\right] \Big\}$$

Now we introduce an additional decision variable V_{max} such that $V \le V_{max}$, It is clear that the following constraint is feasible by all feasible solution in the above constraint:

$$V_{\max} = \min\Big\{ \tfrac{1}{\partial} E(\textstyle\sum_i \tilde{\tilde{R}}_i x_i), \left[E(\sum_i \tilde{\tilde{R}}_i x_i) - \sum_i \alpha_i x_i\right] \left[\sum_i \beta_i x_i - E(\sum_i \tilde{\tilde{R}}_i x_i)\right] \Big\}$$

Now we use the surrogate constraints (10) to (14) instead of (3) as follows:

$$E(\sum_i \tilde{\tilde{R}}_i x_i) - \partial V_{\max} \ge 0 \tag{10}$$

$$V_{\max} \le \tfrac{1}{\partial} E(\textstyle\sum_i \tilde{\tilde{R}}_i x_i) + M\lambda \tag{11}$$

$$V_{\max} \le \left[E(\sum_i \tilde{\tilde{R}}_i x_i) - \sum_i \alpha_i x_i\right] \left[\sum_i \beta_i x_i - E(\sum_i \tilde{\tilde{R}}_i x_i)\right] + M(1-\lambda) \tag{12}$$

$$V_{\max} \le \left[E(\sum_i \tilde{\tilde{R}}_i x_i) - \sum_i \alpha_i x_i\right] \left[\sum_i \beta_i x_i - E(\sum_i \tilde{\tilde{R}}_i x_i)\right] + M(1-\lambda) \tag{13}$$

$$V_{\max} = \lambda \left[E(\sum_i \tilde{\tilde{R}}_i x_i) - \sum_i \alpha_i x_i\right]. \left[\sum_i \beta_i x_i - E(\sum_i \tilde{\tilde{R}}_i x_i)\right] + (1-\lambda)\, \tfrac{1}{\partial} E(\textstyle\sum_i \tilde{\tilde{R}}_i x_i) \tag{14}$$

$$\lambda \in \{0,1\}$$

where M is a big positive number.

From the discussions above, by integrating of (1), (2), (4) to (9) and (10) to (14), we can formulate the fuzzy random multi-objective nonlinear programming model as follows:

Model 1

$$\text{Max} \sum_i \tilde{\tilde{R}}_i x_i$$

$$\text{Max} \sum_i \tilde{\tilde{D}}_i x_i$$

Subject to:

$$E(\sum_i \tilde{\tilde{R}}_i x_i) - \partial V_{\max} \geq 0$$

$$V_{\max} \leq \frac{1}{\partial} E(\sum_i \tilde{\tilde{R}}_i x_i) + M\lambda$$

$$V_{\max} \leq \left[E(\sum_i \tilde{\tilde{R}}_i x_i) - \sum_i \alpha_i x_i \right] \left[\sum_i \beta_i x_i - E(\sum_i \tilde{\tilde{R}}_i x_i) \right] + M(1-\lambda)$$

$$V_{\max} = \lambda \left[E(\sum_i \tilde{\tilde{R}}_i x_i) - \sum_i \alpha_i x_i \right] \left[\sum_i \beta_i x_i - E(\sum_i \tilde{\tilde{R}}_i x_i) \right] + (1-\lambda) \frac{1}{\partial} E(\sum_i \tilde{\tilde{R}}_i x_i)$$

$$\sum_i c_i y_i \leq C$$

$$\sum_i x_i = 1$$

$$x_i \leq y_i \qquad i = 1, 2, ..., n$$

$$x_i \leq u_i \qquad i = 1, 2, ..., n$$

$$x_i \geq 0 \qquad i = 1, 2, ..., n$$

$$y_i \in \{0,1\} \qquad i = 1, 2, ..., n$$

$$\lambda \in \{0,1\}$$

where x is a feasible solution vector from feasible solution space (X) in Model 1. (i.e. $x \in X$).

Lemma 1. (Liu and Liu, 2003) assume that ξ is a fuzzy random variable, for any realization $\omega \in \Omega$, $E(\xi) = E[E[\xi(\omega)]]$.

By Lemma 1;

$$E(\sum_i \tilde{\tilde{R}}_i x_i) = E[0.25 \times \sum_i (\alpha_i + 2\gamma_i(\omega) + \beta_i) x_i]$$

where $\gamma_i(\omega)$ is the mid point of fuzzy random number $\tilde{\tilde{R}}_i$ and x is a feasible solution vector from feasible solution space (X) in Model 1. (i.e. $x \in X$).

3. Methodology

Generally, in order to solve the model above, we have to transform these fuzzy random variables into deterministic parameters. To solve this problem, we apply a *two-phase* approach. In the first phase, the original problem is converted into an equivalent auxiliary crisp multi-objective nonlinear model. Then, in the second phase, an interactive fuzzy random programming approach is proposed for finding a preferred solution through an interaction between the investor and model analyzer.

3.1. Prob.-Pos. approach

One way of solving a fuzzy random multi-objective programming model is to convert the constraints of problem into their corresponding crisp equivalents.

Definition 2. Consider the following multi-objective programming problem with fuzzy random coefficients (Lia *et al.*, 2006):

$$\text{Max } f_1(x, \xi)$$
$$\vdots$$
$$\text{Max } f_m(x, \xi)$$

Subject to:

$$g_r(x, \xi) \leq 0, \qquad r = 1, 2, ..., P$$

By the definition of primitive possibility, the fuzzy random multi-objective programming model is proposed as:

$$\text{Max } f_1$$
$$\vdots$$
$$\text{Max } f_m$$

Subject to:

$$\Pr\{\omega | Pos\{f(x, \xi) \geq f_j\} \geq \delta_j\} \geq \rho_j, \; j = 1, ..., m$$

$$\Pr\{\omega | Pos\{g(x, \xi) \leq 0\} \geq \delta_r\} \geq \rho_r, \; r = 1, ..., p$$

Based on Definition 2 and Model 1 we can transform uncertain objectives into deterministic ones.

Model 2

$$\text{Max } f_1 \qquad (15)$$

$$\text{Max } f_2 \qquad (16)$$

Subject to:

$$\Pr\{\omega \,|\, Pos\{\sum_i \tilde{\bar{R}}_i(\omega)x_i \geq f_1\} \geq \delta_1\} \geq \rho_1 \qquad (17)$$

$$\Pr\{\omega \,|\, Pos\{\sum_i \tilde{\bar{D}}_i(\omega)x_i \geq f_2\} \geq \delta_2\} \geq \rho_2 \qquad (18)$$

$x \in X$

To solve this problem we consider a special case and present the result.

Lemma 2. (Sakawa, 1993) Let $\bar{m}$ and $\bar{n}$ are two independently fuzzy numbers with continuous membership functions. For given confidence level $\eta \in [0,1]$,

$$Pos\{\bar{m} \geq \bar{n}\} \geq \eta \;\Leftrightarrow m_\eta^R \geq n_\eta^L,$$

where m_η^R and n_η^L are the right and left side extreme points of the η–level sets $[m_\eta^L, m_\eta^R]$ and $[n_\eta^L, n_\eta^R]$ of $\bar{m}$ and $\bar{n}$, respectively, and $Pos\{\bar{m} \geq \bar{n}\}$ means the degree of possibility that $\bar{m}$ is greater than or equal to $\bar{n}$.

Theorem 4. Assume that the fuzzy random variable $\tilde{\bar{R}}_i$ is characterized by the fuzzy triangular number $(\alpha_i, \gamma_i(\omega), \beta_i)$, and then we have:

$$\Pr\{\omega \,|\, Pos\{\sum_i \tilde{\bar{R}}_i(\omega)x_i \geq f_1\} \geq \delta_1\} \geq \rho_1$$

if and only if:

$$f_1 \leq \sum_i (1-\delta_1)\beta_i x_i + \delta_1 \sum_i d_i x_i + \\ + \delta_1 \Phi^{-1}(1-\rho_1)\sqrt{\sum_i \sigma_i^2 x_i^2}$$

Proof. We denote that:

$$\sum_i \tilde{\bar{R}}_i x_i = (\sum_i \alpha_i x_i, \sum_i \gamma_i(\omega)x_i, \sum_i \beta_i x_i),$$

where $\gamma_i(\omega) \sim N(d_i, \sigma_i^2)$,

then $\sum_i \gamma_i(\omega)x_i \sim N(\sum_i d_i x_i, \sum_i \sigma_i^2 x_i^2)$ hence

$$\sum_i \tilde{\bar{R}}_i(\omega)x_i = (\sum_i \alpha_i x_i, \sum_i \gamma_i(\omega)x_i, \sum_i \beta_i x_i).$$

By lemma 2 we have:

$$Pos\{\sum_i \tilde{\bar{R}}_i(\omega)x_i \geq f_1\} \geq \delta_1\} \geq \rho_1$$

$$\Leftrightarrow \sum_i \beta_i x_i - \delta_1 \sum_i (\beta_i - \gamma_i(\omega))x_i \geq f_1$$

Then we have:

$$\Pr\{\omega \,|\, Pos\{\sum_i \tilde{\bar{R}}_i(\omega)x_i \geq f_1\} \geq \delta_1\} \geq \rho_1$$

$$\Leftrightarrow \Pr\{\omega \,|\, \sum_i \beta_i x_i - \delta_1 \sum_i (\beta_i - \gamma_i(\omega))x_i \geq f_1\}$$

$$\Leftrightarrow \Pr\{\omega \,|\, \sum_i \gamma_i(\omega)x_i \geq \\ {}^1\!/_{\delta_1}(f_1 - \sum_i (1-\delta_1)\beta_i x_i)\} \geq \rho_1$$

$$\Leftrightarrow \Pr\left\{\omega \,\middle|\, \frac{\sum_i \gamma_i(\omega)x_i - \sum_i d_i x_i}{\sqrt{\sum_i \sigma_i^2 x_i^2}} \geq \frac{{}^1\!/_{\delta_1}(f_1 - \sum_i (1-\delta_1)\beta_i x_i) - \sum_i d_i x_i}{\sqrt{\sum_i \sigma_i^2 x_i^2}}\right\} \geq \rho_1$$

$$\Leftrightarrow \Phi\left(\frac{{}^1\!/_{\delta_1}(f_1 - \sum_i (1-\delta_1)\beta_i x_i) - \sum_i d_i x_1}{\sqrt{\sum_i \sigma_i^2 x_i^2}}\right) \leq 1-\rho_1$$

$$\Leftrightarrow f_1 \leq \sum_i (1-\delta_1)\beta_i x_i + \delta_1 \sum_i d_i x_i + \\ \delta_1 \Phi^{-1}(1-\rho_1)\sqrt{\sum_i \sigma_i^2 x_i^2}$$

This completes the proof. ■

Similarly, second objective can be converted into its crisp equivalent.

Theorem 5. Assume that the fuzzy random variable $\tilde{\tilde{D}}_i$ is characterized by the fuzzy triangular number $(\alpha_i', \gamma_i'(\omega), \beta_i')$, and then we have:

$$\Pr\{\omega \mid Pos\{\sum_i \tilde{\tilde{D}}_i(\omega)x_i \geq f_2\} \geq \delta_2\} \geq \rho_2$$

if and only if:

$$f_2 \leq \sum_i (1-\delta_2)\beta_i' x_i + \delta_2 \sum_i d_i' x_i + \delta_2 \Phi^{-1}(1-\rho_2)\sqrt{\sum_i \sigma_i'^2 x_i^2}$$

Proof. We denote that:

$$\sum_i \tilde{\tilde{D}}_i x_i = (\sum_i \alpha_i' x_i, \sum_i \gamma_i'(\omega)x_i, \sum_i \beta_i' x_i),$$

where $\gamma_i'(\omega) \sim N(d_i', \sigma_i'^2)$,
then $\sum_i \gamma_i'(\omega)x_i \sim N(\sum_i d_i' x_i, \sum_i \sigma_i'^2 x_i^2)$ hence

$$\sum_i \tilde{\tilde{D}}_i(\omega)x_i = (\sum_i \alpha_i' x_i, \sum_i \gamma_i'(\omega)x_i, \sum_i \beta_i' x_i).$$

By lemma 2 we have:

$$Pos\{\sum_i \tilde{\tilde{D}}_i(\omega)x_i \geq f_2\} \geq \delta_2\} \geq \rho_2$$

$$\Leftrightarrow \sum_i \beta_i' x_i - \delta_2 \sum_i (\beta_i' - \gamma_i'(\omega))x_i \geq f_2,$$

Then we have:

$$\Pr\{\omega \mid Pos\{\sum_i \tilde{\tilde{D}}_i(\omega)x_i \geq f_2\} \geq \delta_2\} \geq \rho_2$$

$$\Leftrightarrow \Pr\{\omega \mid \sum_i \beta_i' x_i - \delta_2 \sum_i (\beta_i' - \gamma_i'(\omega))x_i \geq f_2\}$$

$$\Leftrightarrow \Pr\{\omega \mid \sum_i \gamma_i'(\omega)x_i \geq \frac{1}{\delta_2}(f_2 - \sum_i (1-\delta_2)\beta_i' x_i)\} \geq \rho_2$$

$$\Leftrightarrow \Pr\left\{\omega \left| \frac{\sum_i \gamma_i'(\omega)x_i - \sum_i d_i' x_i}{\sqrt{\sum_i \sigma_i'^2 x_i^2}} \geq \frac{\frac{1}{\delta_2}(f_2 - \sum_i (1-\delta_2)\beta_i' x_i) - \sum_i d_i' x_i}{\sqrt{\sum_i \sigma_i'^2 x_i^2}} \right.\right\} \geq \rho_2$$

$$\Leftrightarrow \Phi\left(\frac{\frac{1}{\delta_2}(f_2 - \sum_i (1-\delta_2)\beta_i' x_i) - \sum_i d_i' x_i}{\sqrt{\sum_i \sigma_i'^2 x_i^2}}\right) \leq 1-\rho_2$$

$$\Leftrightarrow f_2 \leq \sum_i (1-\delta_2)\beta_i' x_i + \delta_2 \sum_i d_i' x_i + \delta_2 \Phi^{-1}(1-\rho_2)\sqrt{\sum_i \sigma_i'^2 x_i^2}$$

This completes the proof.■

From Theorems 4 and 5, we have that Model 2 is equivalent to the following multi-objective programming problem:

Max f_1

Max f_2

Subject to:

$$f_1 \leq \sum_i (1-\delta_1)\beta_i x_i + \delta_1 \sum_i d_i x_i + \delta_1 \Phi^{-1}(1-\rho_1)\sqrt{\sum_i \sigma_i^2 x_i^2}$$

$$f_2 \leq \sum_i (1-\delta_2)\beta_i' x_i + \delta_2 \sum_i d_i' x_i + \delta_2 \Phi^{-1}(1-\rho_2)\sqrt{\sum_i \sigma_i'^2 x_i^2}$$

$x \in X$

or equivalently:

Model 3

$$\text{Max } Z_1 = \sum_i (1-\delta_1)\beta_i x_i + \delta_1 \sum_i d_i x_i + \delta_1 \Phi^{-1}(1-\rho_1)\sqrt{\sum_i \sigma_i^2 x_i^2} \tag{19}$$

$$\text{Max } Z_2 = \sum_i (1-\delta_2)\beta_i' x_i + \delta_2 \sum_i d_i' x_i + \delta_2 \Phi^{-1}(1-\rho_2)\sqrt{\sum_i \sigma_i'^2 x_i^2} \tag{20}$$

Subject to:

$x \in X$

4. Proposed interactive fuzzy random programming approach

Now we are ready to construct an interactive algorithm for multi-objective nonlinear programming with fuzzy random coefficients Model 1 in order to derive a satisfying solution. It is noteworthy that we inspired the 6th step from Torabi and Hassini (2008). The proposed interactive solution procedure to solve the original the model is as follows:

1. Elicit the appropriate triangular fuzzy distributions from the investor for the parameters.

2. Ask the investor to set the aspiration levels ρ_1, ρ_2 and δ_1, δ_2.

3. Convert the original fuzzy random multi-objective problem into the crisp one regarding to Model 3.

4. Calculate the individual $Z_1^+ = \max_{x \in X} Z_1$, $Z_1^- = \min_{x \in X} Z_1$, the individual $Z_2^+ = \max_{x \in X} Z_2$ and $Z_2^- = \min_{x \in X} Z_2$ under the given constraints.

5. Specify a linear membership function for each objective function as follows for $j = 1,2$:

$$\mu_j(x) = \begin{cases} 1; & Z_j > Z_j^+ \\ \dfrac{Z_j - Z_j^-}{Z_j^+ - Z_j^-}; & Z_j^- \le Z_j \le Z_j^+ \\ 0; & Z_j < Z_j^- \end{cases} \quad (21)$$

6. Convert the multi-objective model into an equivalent single-objective using the following given crisp formulation:

Model 4

$$\mathrm{Max}\,\pi(x) = \tau\pi_0 + (1-\tau)\sum_j w_j \mu_j(x)$$

Subject to:

$$\pi_0 \le \mu_j(x), \qquad j = 1,2 \quad (22)$$

$$x \in X$$

$$\pi_0, \tau \in [0,1]$$

where $\pi_0 = \min_j \{\mu_j(x)\}$ also, w_j and τ indicate relative importance of the j^{th} objective function and the coefficient of compensation, respectively.

7. If the investor is satisfied with the current solution of (22), then stop. The current ideal solution is a satisfying solution for the investor. Otherwise, ask him/her to update the current controllable parameters such as ρ, δ and τ, and return to Step 3.

5. Numerical example

In this section the computational results of a real case study in IML Co. are given to illustrate details of the proposed model and algorithm. This company is a leading consultant in financing and stock market.

At the time of this study, the aim was to construct a mathematical model to determine the optimal investment on different assets. Since each investor should consult with experts of the company, s/he is given some information about different parameters of assets such as rate of return (R_i), dividend (D_i) and unit purchase cost (c_i), so the proposed model is applied to solve this problem.

As a result, Tables 1, 2 and 3 provide an insight into the data characteristics of the model. IML Co. has chosen 10 top ranking assets from 3856 existing assets based on the related coefficients of rate of return (R_i), dividend (D_i) and unit purchase cost (c_i). All information has gathered from Iranian Stock Exchange. The corresponding computational results are summarized in Tables 4 and 5.

Table 1: The source of data set.

Parameter	Corresponding value
∂	0.9
C	2700
M	1000
u_i	0.25
w_1	0.75
w_2	0.25

Table 2: Information of the top 10 returns.

Asset	Return	
1	$(-0.21,\gamma_1,2.25)$	$\gamma_1 \sim N(1,0.05)$
2	$(0.1,\gamma_2,3)$	$\gamma_2 \sim N(1,0.2)$
3	$(-0.34,\gamma_3,4.1)$	$\gamma_3 \sim N(2,0.31)$
4	$(0.15,\gamma_4,2.88)$	$\gamma_4 \sim N(1,0.28)$
5	$(-0.6,\gamma_5,4.2)$	$\gamma_5 \sim N(1.9,0.25)$
6	$(0.12,\gamma_6,3.35)$	$\gamma_6 \sim N(1.5,0.22)$
7	$(-0.21,\gamma_7,4)$	$\gamma_7 \sim N(1.8,0.25)$
8	$(0,\gamma_8,4.2)$	$\gamma_8 \sim N(1.5,0.38)$
9	$(-0.3,\gamma_9,3.4)$	$\gamma_9 \sim N(1,0.35)$
10	$(-0.22,\gamma_{10},4.31)$	$\gamma_{10} \sim N(2.7,0.15)$

Table 3: Information of the top 10 dividends.

Asset	Dividend		Cost
1	$(0,\gamma'_1,0.38)$	$\gamma'_1 \sim N(0.03,0.01)$	312
2	$(0.31,\gamma'_2,0.67)$	$\gamma'_2 \sim N(0.31,0.01)$	218
3	$(0.13,\gamma'_3,0.92)$	$\gamma'_3 \sim N(0.23,0.04)$	324
4	$(0.05,\gamma'_4,1)$	$\gamma'_4 \sim N(0.12,0.06)$	159
5	$(0.12,\gamma'_5,0.75)$	$\gamma'_5 \sim N(0.21,0.02)$	340
6	$(0.43\ \gamma'_6,1)$	$\gamma'_6 \sim N(0.51,0.01)$	192
7	$(0.51,\gamma'_7,0.92)$	$\gamma'_7 \sim N(0.6,0.05)$	331
8	$(0,\gamma'_8,1.12)$	$\gamma'_8 \sim N(0.24,0.06)$	224
9	$(0.2,\gamma'_9,1.1)$	$\gamma'_9 \sim N(0.31,0.03)$	209
10	$(0.31,\gamma'_{10},0.72)$	$\gamma'_{10} \sim N(0.35,0.01)$	312

Table 4: Results corresponding to step 4.

Z_1^+	Z_1^-	Z_2^+	Z_2^-
$\rho_1 = 0.90$		$\rho_2 = 0.85$	
3.4607	2.5393	0.71620	0.54075

Table 5: Interactive process ($\tau = 0.4$).

δ_1	δ_2	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*
0.8	0.8	0	0	0.1	0	0.057	0.043
0.8	0.85	0	0	0.1	0	0.008	0.092
0.9	0.85	0	0	0.1	0	0.037	0.1
0.9	0.9	0	0	0.1	0	0	0.1
0.95	0.9	0	0	0.1	0	0.010	0.1
0.95	0.95	0	0	0.1	0	0.1	0.1

Table 5: Interactive process ($\tau = 0.4$) (Continued).

x_7^*	x_8^*	x_9^*	x_{10}^*	$V_{\max}^*$	Z_1^*	Z_2^*	π^*
0.1	0.1	0.1	0.1	2.900	3.447	0.714	0.985
0.1	0.1	0.1	0.1	2.875	3.405	0.706	0.939
0.1	0.1	0.1	0.1	2.894	3.360	0.697	0.891
0.1	0.1	0.1	0.1	2.870	3.322	0.687	0.842
0.1	0.1	0.1	0.1	2.877	3.295	0.685	0.820
0.1	0.1	0.1	0.1	2.870	3.285	0.667	0.760

6. Conclusion

It has been observed through the paper that portfolio selection would be considered under fuzzy environment which is a suitable for modeling vagueness and uncertainty of decision making in the real world. The proposed model is a comprehensive and practical one that could be constructed easily by real data as the researchers have done. The proposed algorithm could be changed to enhance other criteria of multi-objective decision making models well. Another development would be when the assets are not independent of each other in price, rate of return, dividend and etc, where for lack of controllability there may be different categories of assets in which all experts are agreed not to invest on more than one.

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