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Inventory model with composed shortage and permissible delay in payment linked to order quantity

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Abstract: In today's business transactions, it is frequently observed that a customer is allowed some grace period (permissible delay in payment) before settling the account with the supplier or producer. This policy is advantageous both for the supplier and customer since supplier attracts more customers and customer does not have to pay any interest during this fixed period either. In this paper, the researchers generalize Goyal's model (1985) with permissible delays in payment depending on the ordered quantity and shortage is the combination of backlogged and lost sales. The researchers then establish a proper mathematical model, and propose an algorithm to solve model easily. Finally, a numerical example is given to illustrate the algorithm and the theoretical results.

Keywords: *Inventory; Backlogged; Lost sale; Permissible delay in payment*

1. Introduction

The classic inventory economic order quantity (EOQ) model is based on the assumption that the supplier is paid for items immediately after they are received. However, in practice the supplier may provide the retailers many incentives such as a cash discount to motivate a faster payment and stimulate sales, or a permissible delay in payments to attract new customers and increase sales. Customers meet several suppliers too and they pay attention to quality, price and lead time of item and supplier's policies that one of the supplier's policies is grace period. The customer does not have to pay any interest during this fixed period but if the payment is delayed beyond the grace period, he has to pay a penalty.

This policy comes out to be very advantageous for the customer as he may delay the payment till the end of the permissible delay period. During the period he may sell the goods, accumulate revenues on the sales and earn interests on that revenue. Thus, it makes an economic opportunity for the customer to delay the payment of the replenishment account up to the last day of the settlement period allowed by the supplier or the producer and to earn more profit.

Goyal (1985) first developed an economic order quantity (EOQ) under the conditions of permissible delay in payments. Aggarwal and Jaggi (1995) extended Goyal's model and considered the inventory model with an exponential deterioration rate under the condition of permissible delay in payments. Shinn (1996) established permissible delay in payment model with discount, and Wang (1997) solved this model for perishable products. Jamal *et al.* (1997) then further generalized the model to allow shortages. Hwang and Shinn (1997) developed the model for determining the retailers with optimal price and lot-size simultaneously when the supplier permits delay in payments for the order of a product whose demand rate is a function of constant price elasticity. Shah (1998) considered probabilistic inventory model for deteriorating items under conditions of permissible delay in payments. Jamal *et al.* (2000) formulated a model where the retailer can pay the wholesaler either at the end of the credit period or later, incurring interest charges on the unpaid balances for the overdue periods. They developed retailer policy for the optimal cycle and payment time for a retailer in a deteriorating-item inventory scenario, in which a wholesaler allows a specified credit period for payment without penalty. Teng (2002) assumed that the selling price is not equal to the purchasing price to modify Goyal's model. The important finding from Tengs study is that it makes an economic sense for a well-established retailer to order small lot sizes and so take more frequently the benefits of the permissible delay in payments. Chung and Huang (2003) have extended Goyal's model to consider the case that the units are replenished at a finite rate under permissible delay in payments and developed an efficient solution-finding procedure to determine the retailer optimal

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ordering policy. They have extended one-level trade credit into two-level trade credit to develop the retailer replenishment model from the supply chain point of view. They assumed that not only the supplier offers the retailer trade credit but also the retailer offers the trade credit to his/her customer. This viewpoint reflected more real-life situations in the supply chain model. Arcelus et al. (2003) modeled the retailer profit-maximizing retail promotion strategy, when confronted with a vendor trade promotion offer of credit and/or price discount on the purchase of regular or perishable merchandise. Chang formulated an EOQ model depending on the ordering quantity. Next, Chung and Liao (2004) have extended it. They dealt with the problem of determining the economic order quantity for exponentially deteriorating items under permissible delay in payments depending on the ordering quantity and developed an efficient solution-finding procedure to determine the retailer optimal ordering policy. Huang (2005) analyzed an EOQ model if the trailer ordered a sufficient quantity too. Otherwise, permissible delay in payments would not be permitted. Chen and Ouyang (2006) considered a fuzzy inventory model for deteriorating items. Lately, Kumaran (2007) considered fuzzy model with allowed shortage and completely backlogged. Liao discussed model under deferrable delivery conditions without shortage and finally, Sankar and Chaudhuri (2007) extended it.

2. Problem statement

The researchers intend to generalize Goyal's model under conditions of permissible delay in payment depends on the ordered quantity and shortage is combination of backlogged and lost sales. The policy of permissible delay in payment is advantageous both for the supplier and customer since the supplier attracts more customers and the customer does not have to pay any interest during this fixed period either. To develop this model, the following assumptions are used in this paper:

1. The inventory system involves only one item.
2. Replenishment occurs instantaneously on ordering (the leading time is negligible).
3. Demand rate is known and deterministic.
4. Shortages are allowed, including backlogged and lost sale.
5. Planning horizon is infinite.
6. The price of item is fixed in relation to the amount of order (no discount for order volume).
7. Ordered item is received fully.
8. The length of the permissible delay period (M) for repaying the supplier is given by:

$$M = \begin{cases} M_1 & if \quad 0 < Q_0 < q_0 - 1 \\ M_2 & if \quad Q_0 < q_0 \end{cases}$$

where Q_0 is the ordered quantity and q_0 a specified value of Q_0, and $M_2 > M_1$.

9. No payment to the supplier is outstanding at the time of placing an order.

3. Modelling

To develop and solve the proposed model, the following notations and mathematical formulation are used in this paper:

3.1. Notations

$I(t)$	inventory level at time t
$\bar{I}$	average inventory level
$\overline{B}$	average shortage level
D	demand rate per year
A	ordering constant cost per order
p	purchasing cost per unit
H	holding cost per unit
b	backlogged shortage cost
s	lost sale shortage cost
B	the maximum of shortage
I_e	interest rate
I_r	delayed payment penalty rate $(I_r \geq I_e)$
α	the fraction of backlogged $(0 < \alpha \leq)$
$\bar{y}$	average inventory in interval (M, T_1)
Q	order quantity in completely backlogged
Q_0	order quantity per order
M	period of permissible delay in settling accounts $(0 < M < T)$
T	length of the replenishment cycle
N	number of the replenishment cycle

T_1 time when inventory level comes down to zero $(0 < T_1 < T)$

T^* optimal length of the replenishment cycle

T_1^* optimal time when inventory level comes down to zero

TC total cost

$$TC = \begin{cases} TC_1(T_1, T) \ \ for \ \ T_1 \geq M \\ \\ TC_2(T_1, T) \ \ for \ \ T_1 \geq M \end{cases}$$

3.2. Graphs

In figure 1, solid vertical line shows completely backlogged (α =1), in other words, shortage is compensable absolutely. In figure 2, dotted vertical line shows completely lost sales (α =0), in other words, shortage is not compensable; consequently, in figure 5, partial of shortage is compensable. The solid line is Q_0 (order quantity).

Since the planning horizon is infinite, the researchers have studied the model over a reorder interval, say *(0,T)*. Two situations may arise, that are described pictorially in figures 3 and 4. In figure 3, inventory level is finished after the period of permissible delay thus is met for delayed payment penalty $(M < T_1)$ and in contrast, in figure 4, customer does not pay any penalty.

According to the figures, these formulas were obtained.

$$T = \frac{Q}{D}\ ,\ T_1 = \frac{Q-B}{D}, T - T_1 = \frac{B}{D}$$

$$D = \frac{y}{T_1 - M}, \quad \bar{y} = \left(\frac{y+0}{2}(T_1 - M)\right)\frac{1}{T}$$

$$= \frac{(T_1 - M)^2 . D^2}{2Q}$$

$\bar{y}$ is average inventory in interval (M, T_1).

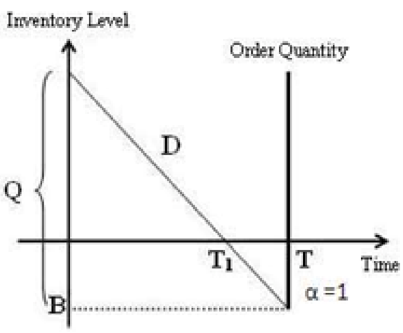

Figure 1: Completely backlogged.

3.3. Mathematical formulation

The researchers have proved mathematical formulation in two modes (case I, II) separately.

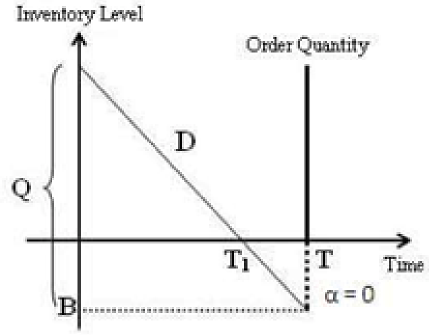

Figure 2: Completely lost sale.

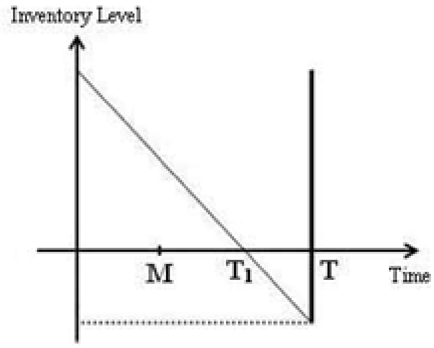

Figure 3: $M \leq T_1$.

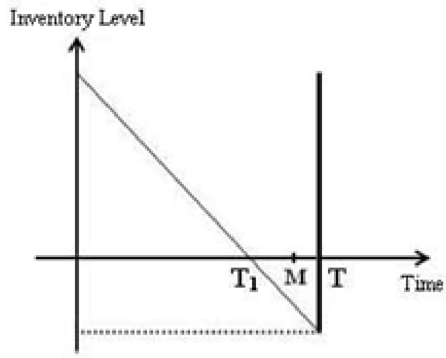

Figure 4: $M > T_1$.

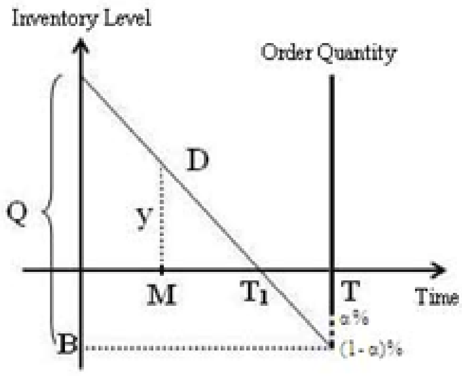

Figure 5: %α Backlogged, %(1-α) lost sale.

3.3.1. Case 1: *($M \leq T_1$)*

In this situation, the holding cost is obtained at two modes (backlogged and lost sale) separately where both are similar.

$$\bar{I} = (\bar{I}_{t1}.T_1 + \bar{I}_{t-t1} .(T - T_1))\frac{1}{T}$$

$$= \frac{1}{T}(\frac{Q-B}{2}T_1 + 0) = \frac{DT_1^2}{2T}$$

$$HC = \frac{hDT_1^2}{2T} \quad (1)$$

The ordering cost is obtained at two modes separately where both are similar.

$$TO = A.N = \frac{A}{T} \quad (2)$$

The shortage cost is obtained at backlogged mode.

$$\overline{B} = \frac{1}{T}(\overline{B}_{t1}.T_1 + \overline{B}_{t-t1}.(T - T_1))$$

$$= \frac{1}{T}(0 + \frac{B}{2}(T - T_1)) = \frac{D(T-T_1)^2}{2T} \quad (3)$$

$$SC = \frac{D.b(T - T_1)^2}{2T}$$

The shortage cost is obtained at lost sale mode.

$$SC = B.N.S = \frac{B.S}{T} = \frac{D.S(T - T_1)}{T} \quad (4)$$

The earned interest is obtained at two modes separately that both are similar.

$$IE_1 = p.I_e.\bar{I} = \frac{p.I_e.D.T_1^2}{2T} \quad (5)$$

The delayed payment penalty is obtained at two modes separately where both are similar.

$$IP = p.I_r.\bar{y} = \frac{p.I_r.(T_1 - M)^2.D^2}{2Q}$$

$$\frac{p.I_r.(T_1 - M)^2.D}{2T} \quad (6)$$

Therefore the total average cost per unit time is:

$$TC_1 = \frac{A}{T} + \frac{hDT_1^2}{2T} + \alpha(\frac{D.b(T - T_1)^2}{2T})$$

$$+ (\frac{D.S(T - T_1)}{T})(1 - \alpha)$$

$$+ \frac{p.I_r.(T_1 - M)^2.D}{2T} - \frac{p.I_e.D.T_2^2}{2T} \quad (7)$$

Optimal values of T_1 and T which minimize TC_1 are obtained by solving the equations

$$\frac{\partial TC_1(T_1,T)}{\partial T_1} = 0 \qquad \frac{\partial TC_1(T_1,T)}{\partial T} = 0$$

Second differential of T_1 and T are positive, so because of an evident response, it is not shown.

$$T_1^* = \frac{(1-\alpha)S + pI_rM + \alpha bT}{h + \alpha b + p(I_r - I_e)} \quad (8)$$

$$T^* = \sqrt{\frac{2A + DhT_1^2 + \alpha DbT_1^2 - 2SDT_1(1-\alpha) +}{\alpha Db} \quad \frac{pDI_r(T_1 - M)^2 - DpI_eT_1^2}{}} \quad (9)$$

3.3.2. Case 2: *($M > T_1$)*

The holding, ordering and shortage costs are like the last part. Since $M>T_1$, the buyer pays no delayed payment penalty. The earned interest is obtained from sum of average inventory in interval $(0,T_1)$ and average sales revenue in interval (T_1,M) (Money interest in interval $(0,M)$).

The earned interest is obtained at two modes separately where both are similar.

$$IE_2 = \frac{DpI_eT_1^2}{2T} + \frac{(M - T_1)pQI_e}{T} = \frac{DpI_eT_1}{2T}(M - \frac{T_1}{2}) \quad (10)$$

Therefore the total average cost per unit time is:

$$TC_2 = \frac{A}{T} + \frac{hDT_1^2}{2T} + \alpha(\frac{D.b(T - T_1)^2}{2T})$$

$$+ (\frac{D.S(T - T_1)}{T})(1 - \alpha)$$

$$\frac{p.I_e.T_1}{T}\left(M-\frac{T_1}{2}\right) \quad (12)$$

In the meantime, if $M=T_1$ hence $TC_1=TC_2$.

Optimal values of T_1 and T which minimize TC_1 are obtained by solving the equations

$$\frac{\partial TC_2(T_1,T)}{\partial T_1}=0 \qquad \frac{\partial TC_2(T_1,T)}{\partial T}=0$$

Second differential of T_1 and T are positive, so because of an evident response, it is not shown.

$$T_1^*=\frac{(1-\alpha)S+pI_eM+\alpha bT}{h+\alpha b+pI_e} \quad (12)$$

$$T^*= \sqrt{\frac{\dfrac{2A+DhT_1^2+\alpha DbT_1^2-2SDT_1}{\alpha Db}}{(1-\alpha)+2pDI_eMT_1+DpI_eT_1^2}} \quad (13)$$

3.3.3. Completely lost sale: (α =0)

Zipkin (2000) showed that if shortage was a completely lost sale, the optimal policy is to have either no stockouts or all stockouts. Solved model at mode of a completely lost sale proved if that (α = 0), we do not order; therefore, model is soved without any shortage.

Mode 1, if $M < T_1$, $B = 0$ and $T = T_1$, Hence,

$$TC = \frac{A}{T_1}+\frac{hDT_1}{2}+\frac{DpI_r(T_1-M)^2}{2T_1}-\frac{pI_eDT_1}{2} \quad (14)$$

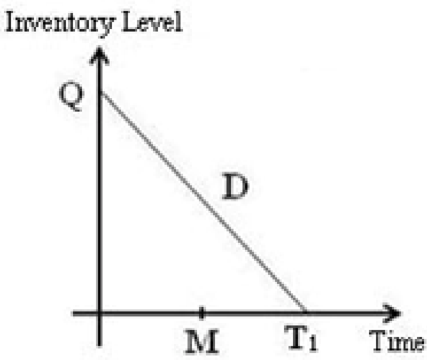

Fig. 6: Completely lost sale without any shortage.

$$T_1^*=\sqrt{\frac{2A+DpI_rM^2}{(h+pI_r-pI_e)D}} \quad (15)$$

$$Q_0^*=DT_1^* \quad (16)$$

Mode 2, if $M=T_1$, $B = 0$ and $T=T_1$, Hence,

$$TC=\frac{A}{M_1}+\frac{hDT}{2}-\frac{DpI_eM}{2} \quad (17)$$

$$Q_0^*=DM \quad (18)$$

4. Algorithm

Since the researchers' goal is finding the optimal solution quickly, they have proposed an algorithm based on mathematical solutions such as differentiation and boundary conditions, which helps the inventory managers to decide easily. First step is resulted from differentiation of function and next two steps are resulted out of checking boundary conditions**.**

Step 1: Find (T_1^* , T^*) by two equations – two unknown quantities in cases I and II, and compute TC, then according to M that the supplier is given, first check Q_0^* according to M is correct or not (if it was incorrect, show it by E_Q). Then check $T_1 \geq M$ and $T_1<M$ for T_1^* is correct or not (if it was incorrect, show it by E_T).

The four answers of this step are resulted from differentiation of T_1 and T.

Step 2: Solve Q_0 at the boundary conditions (BC) for largest M in case I and II, then check E_T and E_Q , and compute TC.

After optimal quantities of differentiation in Step 1, the best result may be found in boundary conditions. Therefore check Q_0.

Step 3: Solve M at the boundary conditions for largest M in case II for two modes ($T=M$ and $T_1=M$), then check E_T and E_Q . Compute TC. After two steps checked above, we check M when conforms on T or T_1 . These conditions help to minimize total cost too.

Step 4: Among correct answers, minimum TC is optimal.

Table 1: The solution of problem.

Mode	Quantity of M	Optimal of T_1	Optimal of T	TC	Error
I	0.042	0.079	0.106	4053	
I	0.083	0.098	0.122	3623	
II	0.042	0.064	0.094	4487	E_T
II	0.083	0.068	0.092	3600	E_Q
I , Q_0 in BC	0.083	0.105	0.133	3646	
II , Q_0 in BC	0.083	0.094	0.133	3965	E_T
II , $M=T_1$	0.083	0.083	0.108	3698	E_Q
II , M=T	0.083	0.062	0.083	3627	E_Q

5. Numerical example

Consider an inventory control problem with the ordering constant cost per order is $A = 250$, the holding cost per unit is $h = 20$, the backlogged shortage cost is $b = 50$, the lost sale shortage cost is $s = 60$, the demand rate per year is $D = 3000$, the interest rate is $I_e = 0.1$, the delayed payment penalty rate is $I_r = 0.15$, the purchasing cost per unit is $p = 100$ and consider a year with 360 days. Find optimal quantities of T_1^*, T^* and TC? ($\alpha = 100\ \%$)

$$M = \begin{cases} 15 \quad if \quad 0 < Q_0 \leq 399 \\ \\ 30 \quad if \quad Q_0 \geq 400 \end{cases}$$

According to Table 1, by 30 days of permissible delay ($M=0.083$), optimal quantities are $T_1^* = 0.098$ (36 days), $T^* = 0.122$ (44 days) and $TC^* = 3623$.

The results in Table 1 indicate that it is better that buyer applies period of permissible delay in settling accounts and liquidates his/her debt after six days. In this problem, the buyer has not paid any money in the beginning of period and has sold his/her all of the goods in 36 days and in addition, has earned an interest of money during that period.

6. Conclusion

In this paper, the researchers have studied an inventory problem, where the shortage was combination of backlogged and lost sales, and the permissible delay in payment depends on the order quantity. For this reason the period of permissible delay (M) will depend on the order quantity, thus solution of problem is intricate; consequently, an algorithm is also suggested to find the optimal ordering policy, which helps the inventory managers to decide easily. In recent years, grace period model has been the focus of considerable research activities because it has a very practical application. For future researches, considering this model with partial payment of cost at first under conditions of probabilistic demand is recommended or in this paper; however, the researchers have considered only one break in the delay period. A natural extension of the model would be to studying case of N breaks in permissible delay periods, i.e. to assume:

$$M = M_i, \quad if \quad q_{i-1} \leq q < q_i, \quad i=1,2,3,\ldots,N$$

Where

$$M_1 < M_2 < \ldots < M_N \text{ and } q_0 = 0 < q_1 < q_2 < \ldots < q_N$$

It would also be interesting to study the problem discussed in the paper for a deteriorating item.

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