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TECHNICAL ARTICLE

Economic design of Hotelling's T^2 control chart on the presence of fixed sampling rate and exponentially assignable causes

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Abstract Control charts are extensively used in manufacturing contexts to monitor production processes. This article illustrates economical design of a variable sample size and control limit Hotelling's T^2 control chart based on a novel cost model when occurrence times of the assignable causes are exponentially distributed. The proposed nonlinear cost model is an extension of Duncan's (J Am Stat Assoc 51: 228–242, 1956) model which was employed for univariate cases. Applying genetic algorithm to find optimum parameter values and using an L_{33} orthogonal array in sensitivity analysis on the model parameters is investigated through a numerical example to illustrate the effectiveness of the proposed approach.

Keywords Economic design · Genetic algorithms · Multivariate control chart · Statistical quality control · VSSCT^2 control chart

List of symbols

T_i^2	A random variable followed Hotelling's T^2 distribution
T_1	The expected length of in-control period
T_2	The expected length of searching period due to false alarms
T_3	The expected length of out-of-control state
$T_4(=t_1)$	The time to identify and correct the assignable cause following an action signal
h	Sampling interval
$\mathrm{ARL}_{\mathrm{out}}$	The average number of samples drawn from process when it is out of control
t	The time interval between mean shift and the latest sample point before mean shift
n'	The average sample size when the process operates in out-of-control state
G	The average time from taking a sample to the time of plotting T_i^2 statistic on the chart
$\mathrm{E(U_1)}$	The average number of sample points in the safe region when the process is in out-of-control state and current sample point belongs to safe region
$E(U_2)$	The average number of sample points in the warning region when the process is in out-of-control state and current sample point belongs to warning region
$E\left(N_i^1\right)$	The average number of sample points in the warning region when the process is in out-of-control state and current sample point belongs to safe region
$E\left(N_i^2\right)$	The average number of sample points in the safe region when the process is in out-of-control state and current sample point belongs to warning region
$E(N_1)$	The average number of samples drawn from the time of process mean shift to the time that mean shift is detected given that first sample point after mean shift falls into the safe region
$E(N_2)$	The average number of samples drawn from the time of the process mean shift to the time that mean shift is detected given that first sample point after mean shift falls into the warning region
F_{p,v_j,τ_j}	A random variable followed non-central F distribution with p and v_j degrees of freedom and non-centrality parameter τ_j

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t_0	The average amount of time exhausted searching for the assignable cause when the process is in-control
$E(FA)$	The expected number of false alarms per cycle
$E(T)$	The expected length of a production cycle
C_0	The average search cost if the given signal is false
C_1	The average cost to discover the assignable cause and adjust the process to in-control state
C_2	The hourly cost when the process is operating in control state
C_3	The hourly cost when the process is operating in out-of-control state
C_4	The cost for each inspected item
$E(C)$	The expected cost during a production cycle
$E(N_{in})$	The average numbers of samples drawn from the process given that the process is in-control
ECT	The expected cost per time

Introduction

Statistical process control (SPC) is a tremendous quality assurance tool to develop the quality of manufacture and ultimately scores on end-customer satisfaction. SPC uses control charts to monitor the most important key quality characteristics (KQCs) in manufacturing (Sharma and Rao 2013).

In general, there is at least a small variation on quality characteristics of the produced items. Hence, we should control the processes to reduce the amount of nonconforming products. Control charts basically are used to monitor processes to become aware on any alteration that may affect the quality of product.

Generally, SPC control charts are used to detect changes in a process by distinguishing between assignable causes and common causes of the process variation. When a control chart signals, process engineers initiate a search to identify and eliminate the source of variation. Knowing the time at which the process began to vary, the so-called change point would help to conduct the search more efficiently in a tighter time-frame (Assareh et al. 2013; Akhavan Niaki and Khedmati 2013). Control charts also are used to detect anomalies in the processes. They are most often used to monitor production-related processes. In many business-related processes, the quality of a process or product can be characterized by a relationship between a response variable and one or more explanatory variables which is referred to as profile (Narvand et al. 2013; Soleimani et al. 2013).

In many applications, quality of process is characterized by a single random variable called quality characteristic but some cases occur that process is characterized by more than one quality characteristic. These random variables are usually correlated and jointly distributed and cannot be controlled independently using a univariate control chart. Accordingly, multivariate statistical control methods have been proposed to investigate this issue. Most of the works on control charts in multivariate case are on problem of monitoring mean vector of the process. A measure of distance that takes into account the covariance structure was proposed by Harold Hotelling (1931). It is called Hotelling's T^2 in honour of its developer. Geometrically we can view T^2 as proportional to the squared distance of a multivariate observation from the target where equidistant points form ellipsoids surrounding the target. The higher the T^2 value, the more distance the observation from the target is. In the Hotelling's T^2 control chart the mean vector and covariance matrix are unknown and must be estimated by means of the previous data where it may affect the performance of control chart. Recently, some researchers such as Tchao and Hawkins (2011), Capizzi and Masarotto (2010) and Jensen et al. (2006) proposed solutions to investigate this issue.

When a control chart is used to monitor a process, three design parameters that should be selected are the sample size, the sampling interval, and the action limit(s). Duncan (1956) offered an economic model incorporated the most important relevant cost items associated with sampling and control charts. Through minimization of the proposed cost model, the optimum economical design parameters of control chart were presented.

In the literature, statistical control chart design may be applied to increase the power of any control chart such as T^2. Aparisi (1996) followed this idea through adaptive sample size and sampling interval in the multivariate case and proposed three types of modified charts with variable sample size (VSS), variable sampling interval (VSI), and variable sample size and sampling interval (VSSI) features, respectively (see Aparisi 1996; Aparisi and Haro 2001, 2003), given that the mean vector and variance–covariance matrix were known. Chen and Hsieh (2007) indicated that traditional T^2 chart gives a better performance if both sample size and control limits are variable (VSSC), and the waiting time between successive samples are fixed.

In the case of economic design of control charts, Chen (2009, 2007) used a Markov chain approach to design VSI T^2 and VSSI T^2 control charts. He showed that both of them can be more efficient than FSR (Fixed Sampling Rate) control scheme in terms of the loss. Chou et al. (2006) developed a cost function for variable sampling intervals T^2 control charts and obtained optimum design parameters using genetic algorithms. Costa and Rahim (2001) used the Markov chain approach to reach an economic design of $\bar{X}$ charts with variable parameters. De Magalhaes et al. (2001) developed a cost model for economic design of $\bar{X}$ chart with all design parameters varying in an adaptive way. They check whether the economic model for a Vp $\bar{X}$ chart reduces the quality cost of a process. Bai and Lee (1998) presented an economic design of the VSI $\bar{X}$ control charts and showed that the VSI scheme can be more efficient than the FSI scheme in terms of the expected cost per time. They applied a two-stage optimization approach to find the optimal sampling-and-charting parameters of their cost model.

In this paper, we propose a novel economic design of T^2 control chart based on the extension of Duncan's (1956) cost model. By using a genetic algorithm, the optimal design parameters of the relevant cost model besides the sensitivity analysis is proposed through an illustrative example.

The rest of the paper is organized as follows: in "Variable sample size and control limits T^2 control chart" we briefly review the VSSC T^2 control chart. We then present a method and describe the proposed formulation for the of cost model for multivariate situations in "Proposed cost model". "Illustrated example" systematically guides readers to implement the proposed procedure via a numerical example. Finally, we close with a conclusion.

Variable sample size and control limits T^2 control chart

Let $\bar{X}_1, \bar{X}_2, \bar{X}_3, \ldots$ be $p \times 1$ random vectors, each representing sample mean vector of related quality characteristics assumed jointly distributed as p-variate normal with mean vector μ_0 and variance–covariance matrix $\sum_0$ When ith sample of size n is taken at every sampling point, we calculate the following statistic:

$$\chi_i^2 = n \cdot (\bar{X}_i - \mu_0)' \cdot \Sigma_0^{-1} \cdot (\bar{X}_i - \mu_0) \quad i = 1, 2, 3 \ldots \tag{1}$$

and compare it with upper control limit (or action limit) denotes by UCL_{χ^2} which can be specified by the $(1-\alpha)$ percentile point of a χ^2 distribution with p degree of freedom $\left(\chi^2_{p,\alpha}\right)$. However, in most cases the values of μ_0 and $\sum_0$ are unknown and are estimated by sample mean vector $(\bar{\bar{X}})$, and sample variance–covariance matrix (S) of m initial random samples prior to on-line process monitoring and T^2 statistic is defined by

$$T_i^2 = n \cdot \left(\bar{X}_i - \bar{\bar{X}}\right)' \cdot S^{-1} \cdot \left(\bar{X}_i - \bar{\bar{X}}\right); \; i = 1, 2, \ldots m \tag{2}$$

that is the approximate statistic for Hotelling's multivariate chart. In this case, action limit used to monitor future random vectors is given by Alt (1984) as

Table 1 Three regions in VSSC T^2 chart

Interval	Region
Safe region	$[0, w_j]$
Warning region	$(w_j, k_j]$
Action region	$(k_j, +\infty)$

$$k = C(m, n, p) \cdot F_{p,v,\alpha}. \tag{3}$$

where $F_{p,v,\alpha}$ is the $(1-\alpha)$ percentile point of F distribution with p and v degrees of freedom. $C(m, n, p)$ and v are calculated by

$$C(m,n,p) = \begin{cases} \frac{p \cdot (m+1) \cdot (n-1)}{mn - m - p + 1}; & n > 1 \\ \frac{p \cdot (m+1) \cdot (m-1)}{m^2 - mp}; & n = 1 \end{cases}, \quad v = \begin{cases} mn - m - p + 1; & n > 1 \\ m - p; & n = 1 \end{cases} \tag{4}$$

Traditional Hotelling's T^2 chart operates with a fixed sample of size n_0 drawn every h_0 hours from process, and T^2 statistic is plotted on a control chart with $k_0 = C(m, n_0, p) \cdot F_{p,v,\alpha}$ as the action limit. The VSSC T^2 chart is a modification of traditional T^2 chart. Let (n_1, w_1, k_1) be minimum sample size, largest warning and action limits, and (n_2, w_2, k_2) be maximum sample size, smallest warning and action limits, respectively, such that $n_1 < n_0 < n_2$ while keeping sampling interval fixed at h. The warning (w_j) and action $(k_j = C(m, n_j, p) \cdot F_{p,v,\alpha})$ limits divide T^2 chart to three regions as shown in Table 1:

The decision to switch between maximum and minimum sample size depends on position of the prior sample point on the control chart and summarizes as following function:

$$(n(i), w(i), k(i)) = \begin{cases} (n_1, w_1, k_1) & \text{if} \quad 0 \le T_{i-1}^2 \le w_{i-1} \\ (n_2, w_2, k_2) & \text{if} \quad w_{i-1} < T_{i-1}^2 \le k_{i-1} \end{cases} \tag{5}$$

During the in-control period, it is assumed that the size of samples is chosen at random between two values when the process starts or after a false alarm. Small size is selected with probability of p_0, whereas large sample size is selected with probability of $1 - p_0$, where p_0 is the conditional probability of a sample point falling in the safe region, given that it did not fall in the action region and is calculated as follows:

$$p_0 = \Pr\left(T_i^2 < w_1 | T_i^2 < k_1\right) = \Pr\left(T_1^2 < w_2 | T_i^2 < k_2\right) \tag{6}$$

$$\begin{aligned} 1 - p_0 &= \Pr\left(w_1 < T_i^2 < k_1 | T_i^2 < k_1\right) \\ &= \Pr\left(w_2 < T_i^2 < k_2 | T_i^2 < k_2\right) \end{aligned} \tag{7}$$

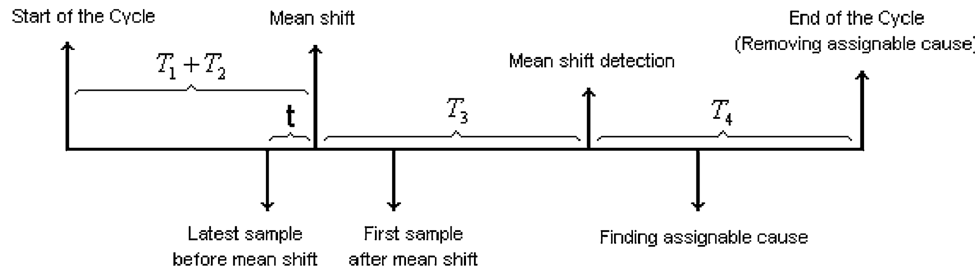

Fig. 1 Production cycle considered in the cost model

Proposed cost model

Cost model is an extension of Duncan (1956) model which was employed in a univariate case. First, we make a number of assumptions as follows:

- The mean vector and variance–covariance matrix of process are unknown.
- At beginning, the process is in-control but after a random time it will be disturbed by an assignable cause that causes a fixed shift in the process mean vector.
- The process after the shift remains out of control until the assignable cause is eliminated (if possible).
- When the T_i^2 value falls outside the action limit, the process is stopped and then a search is started to find the assignable cause and adjust the process.
- The interval between starting the process and occurring of an assignable cause follows an exponential distribution with λ as its parameter.

In the economic design of VSSC T^2 control chart we tend to find the optimal design parameters that minimize the expected cost per time. Figure 1 depicts the production cycle, which is divided into four time intervals of in-control period, out-of-control period, searching period due to false alarm, and the time period for identifying and correcting the assignable cause. Individuals are now illustrated before they are grouped together.

(T_1) The expected length of in-control period is $\frac{1}{\lambda}$.

(T_3) The expected length of out-of-control state represents the average time needed for the control chart to produce a signal after the process mean shift. T_3 is given by

$$T_3 = h \cdot \mathrm{ARL}_{\mathrm{out}} - t + n' \cdot G \tag{8}$$

where G is the average time from taking a sample to the time of plotting T_i^2 statistic on the chart, and n' is the average sample size when the process operates in out-of-control state, and $\mathrm{ARL}_{\mathrm{out}}$ is the average number of samples drawn from process when it is out of control. n' and $\mathrm{ARL}_{\mathrm{out}}$ are given by

$$n' = p_0 \cdot \frac{n_1 \cdot E(U_1) + n_2 \cdot E(N_i^1)}{E(N_1)} + (1 - p_0) \cdot \frac{n_1 \cdot E(N_i^2) + n_2 \cdot E(U_2)}{E(N_2)} \tag{9}$$

$$\mathrm{ARL}_{\mathrm{out}} = p_0 \cdot E(N_1) + (1 - p_0) \cdot E(N_2) \tag{10}$$

where as indicated by Chen (2007a, b), $E(U_1)$ is the average number of sample points in the safe region when the process is in out-of-control state and current sample point belongs to safe region. Then,

$$E(U_1) = \frac{1 - p_{22}}{1 + p_{11} \cdot p_{22} - p_{11} - p_{22} - p_{12} \cdot p_{21}} \tag{11}$$

$E(U_2)$ is the average number of sample points in the warning region when the process is in out-of-control state and current sample point belongs to warning region. Then,

$$E(U_2) = \frac{1 - p_{11}}{1 + p_{11} \cdot p_{22} - p_{11} - p_{22} - p_{12} \cdot p_{21}} \tag{12}$$

$E(N_i^1)$ is the average number of sample points in the warning region when the process is in out-of-control state and current sample point belongs to safe region. Then,

$$E(N_i^1) = 1 + \frac{p_{12}}{1 - p_{22}} \tag{13}$$

$E(N_i^2)$ is the average number of sample points in the safe region when the process is in out-of-control state and current sample point belongs to warning region. Then,

$$E(N_i^2) = 1 + \frac{p_{21}}{1 - p_{11}} \tag{14}$$

$E(N_1)$ is the average number of samples drawn from the time of process mean shift to the time that mean shift is detected given that first sample point after mean shift falls into the safe region. Then,

$$E(N_1) = \frac{1 - p_{22} + p_{12}}{1 + p_{11} \cdot p_{22} - p_{11} - p_{22} - p_{12} \cdot p_{21}} \tag{15}$$

$E(N_2)$ is the average number of samples drawn from the time of the process mean shift to the time that mean shift is detected given that first sample point after mean shift falls into the warning region. Then,

Table 2 Cost and process parameters for numerical example

C_0	C_1	C_2	C_3	C_4	m	λ	δ	t_0	t_1	G
\$10	\$30	\$100	\$0.5	\$0.1	50	0.01	2	0.1 h	0.3 h	0.1 h

$$E(N_2) = \frac{1 - p_{11} + p_{21}}{1 + p_{11} \cdot p_{22} - p_{11} - p_{22} - p_{12} \cdot p_{21}} \quad (16)$$

where

$$p_{11} = \Pr\{T_i^2 < w_1 | T_i^2 \sim C(m, n_1, p) \cdot F_{p,v_1,\tau_1}\}$$

$$p_{12} = \Pr\{w_1 < T_i^2 < k_1 | T_i^2 \sim C(m, n_1, p) \cdot F_{p,v_1,\tau_1}\}$$

$$p_{21} = \Pr\{T_i^2 < w_2 | T_i^2 \sim C(m, n_2, p) \cdot F_{p,v_2,\tau_2}\}$$

$$p_{22} = \Pr\{w_2 < T_i^2 < k_2 | T_i^2 \sim C(m, n_2, p) \cdot F_{p,v_2,\tau_2}\}$$

where $C(m, n_j, p)$ and v_j for $j = 1, 2$ is calculated by Eq. (4), and F_{p,v_j,τ_j} for $j = 1, 2$ is a random variable followed non-central F distribution with p and v_j degrees of freedom and non-centrality parameter τ_j defined by $\tau_j = n_j \cdot (\mu_1 - \mu_0)' \sum^{-1} (\mu_1 - \mu_0)$. If we define $\delta = \sqrt{(\mu_1 - \mu_0)' \sum^{-1} (\mu_1 - \mu_0)}$, then $\tau_j = n_j \cdot \delta^2$, where δ is the Mahalanobis distance that is a measure of change in process mean vector.

(T_2) Let t_0 denote the average amount of time exhausted searching for the assignable cause when the process is in-control, and $E(FA)$ denote the expected number of false alarms per cycle, which is given by

$$E(\mathrm{FA}) = \frac{\left[\frac{1}{h\lambda}\right]}{\mathrm{ARL}_{in}} = \frac{\left[\frac{1}{h\lambda}\right]}{\frac{1}{\alpha}} = \alpha \cdot \left[\frac{1}{h\lambda}\right]; \quad (17)$$

then, the expected length of searching period due to false alarms can be expressed by $T_2 = t_0 \cdot E(\mathrm{FA})$.

(T_4) The time to identify and correct the assignable cause following an action signal is a constant t_1.

Aggregating the foregoing four time intervals, the expected length of a production cycle would be expressed by

$$E(T) = \frac{1}{\lambda} + t_0 \cdot \alpha \cdot \left[\frac{1}{h\lambda}\right] + h \cdot \mathrm{ARL}_{\mathrm{out}} - t + n' \cdot G + t_1 \quad (18)$$

If one defines C_0, the average search cost if the given signal is false; C_1, the average cost to discover the assignable cause and adjust the process to in-control state; C_2, the hourly cost when the process is operating in control state; C_3, the hourly cost when the process is operating in out-of-control state; C_4, the cost for each inspected item; then the expected cost during a production cycle is given by

Table 3 Level plan for the three control parameters in the GA

	PS	CP	MR	GN
Level 1	20	0.1	0.05	30
Level 2	25	0.3	0.07	40
Level 3	30	0.5	0.10	50

$$E(C) = C_0 \cdot E(\mathrm{FA}) + C_1 + C_2 \cdot \frac{1}{\lambda} + C_3 \cdot (h \cdot \mathrm{ARL}_{\mathrm{out}} - t + n' \cdot G) + C_4 \cdot E(N) \quad (19)$$

where $E(N)$ is the average number of inspected items and is calculated by

$$E(N) = n \cdot E(N_{\mathrm{in}}) + n' \cdot \mathrm{ARL}_{\mathrm{out}} \quad (20)$$

where given that the process is in-control, $E(N_{\mathrm{in}})$ is the average numbers of samples drawn from the process, and n is the average sample size. They are given by

$$E(N_{in}) = \left[\frac{1/\lambda}{h}\right] = \left[\frac{1}{h\lambda}\right] \quad (21)$$

$$n = n_1 \cdot p_0 + n_2 \cdot (1 - p_0) \quad (22)$$

Finally, the expected cost per time ECT is given by

$$\mathrm{ECT} = \frac{E(C)}{E(T)} \quad (23)$$

Illustrated example

The usefulness and effectiveness of the proposed procedure beside the optimal approximation and sensitivity analysis on main parameters is demonstrated using a numerical example which acts as a modification of Lin et al. (2009). Suppose that a production process is monitored by the VSSC T^2 control chart. The cost and process parameters are as shown in Table 2,

The cost model given in Eq. (23) has some specification abbreviated as follows:

- It is a nonlinear model and a function of mixed continuous-discrete decision variable
- Mathematically, model space is a discrete and non-convex.

Hence, using nonlinear programming techniques for optimizing this model is time consuming and inefficient. Hence, we decided to use the Genetic Algorithms (GA)

Table 4 Experimental layout of L9 array and the experimental results

Trial	PS	CP	MR	GN	y_1	y_2	y_3	SN
1	1	1	1	1	136.9070	141.6008	136.9152	−42.8285
2	1	2	2	2	133.0914	135.1014	135.8494	−42.5864
3	1	3	3	3	135.6946	131.6555	135.4612	−42.5604
4	2	1	2	3	132.0809	135.3874	132.7288	−42.5036
5	2	2	3	1	137.1651	131.7730	134.5210	−42.5747
6	2	3	1	2	134.0452	135.6475	138.0805	−42.6666
7	3	1	3	2	136.0331	134.1237	135.2717	−42.6160
8	3	2	1	3	132.9111	133.6228	134.7701	−42.5272
9	3	3	2	1	134.2257	132.4884	133.5089	−42.5037

Table 5 Response table of S/N's for the three control parameters in the GA

Level	PS	CP	MR	GN
1	−127.9754	−127.9481	−128.0223	−127.9070
2	−127.7449	**−127.6883**	**−127.5937**	−127.8690
3	**−127.6469**	−127.7308	−127.7512	**−127.5912**

introduced by Holland (1975) with a mathematical software package (MATLAB 7.1) to obtain the optimal values of $n_1, n_2, h, w_1, w_2, k_1, k_2$ that minimize the expected cost per time. Some advantages of GA are as follows:

- GA uses the fitness function and the stochastic concepts (not deterministic rule) to search for optimal solution. Therefore, the GA can be applied for many kinds of optimization problems.
- Mutation and crossover techniques in the GA avoid trapping in the local optimum.
- The GA is able to search for many possible solutions at the same time.
- We applied the solution procedure used in Lin et al. (2009) to our example as follows:

Step 1. *Initialization* Generating 30 initial solutions randomly, which satisfy the following constraints:

$$n_1 < n_2, \quad w_1 < w_2, \quad k_j = C(m, n_j, p) \cdot F_{p,v_j,\alpha}, \quad 0 < h < 20$$

Step 2. *Evaluation* Calculating the value of the cost function in Eq. (23) to evaluate each solution.

Step 3. *Selection* Replacing the solution with highest cost by the solution with lowest cost.

Step 4. *Crossover* Selecting a pairs of solutions in step 3 randomly to use them as the parents for crossover operations. In this example, we apply the arithmetical crossover method with crossover probability 0.3 as follows:

Offspring 1 = 0.3 Parents 1 + 0.7 Parent 2; Offspring 2 = 0.7 Parents 1 + 0.3 Parents 2

Where offsprings are new chromosomes. At the end of GA steps, determination of crossover probability is described in detailed.

Step 5. *Mutation* Here, we use non-uniform method to carry out the mutation operation with the rate of 0.07. Thus, we can randomly select 7 % of chromosomes to mutate some parameters (or genes). At the end of GA steps, determination of mutation rate is described in detailed.

Step 6. Repeat Step 2–5 until the stopping criteria is found. In this example, we use "50 generations" as our stopping criteria.

For implementing the GA, we need to determine its parameters: the population size (PS), the crossover Probability (CP), the mutation rate (MR), and the number of generations (GN). Here, we use the orthogonal array experiment to determine the values of these parameters. As shown in Table 3, three levels of each parameter are planned in this orthogonal array experiment. An L9

Table 6 Solutions of the cost model for different process mean shifts in VSSC scheme

δ	n_1	n_2	h	w_1	w_2	k_1	k_2	ECT	AATS	E(FA)	ARL_{out}
0.25	1	38	0.1532	4.7620	4.4635	12.3465	10.8455	16.1316	10.1591	3.2611	66.8100
0.5	1	9	0.3237	4.1545	3.9175	12.3465	10.8574	6.9146	3.0725	1.5419	9.9902
0.75	1	16	0.6411	2.7997	2.6768	12.3465	10.8999	4.6862	2.5296	0.7775	4.4455
1	1	2	0.6592	2.1328	2.0534	12.3465	10.9334	3.9928	1.5015	0.7560	2.7772
1.25	2	11	0.7493	2.3173	2.2302	12.3125	10.9460	3.5507	1.1314	0.6648	2.0093
1.5	1	11	0.8853	1.9289	1.8611	12.3465	10.9460	3.2754	1.1873	0.5623	1.8404
1.75	1	11	0.8577	1.9289	1.8611	12.3465	10.9460	3.2310	1.0437	0.5805	1.7162
2	1	11	3.2385	1.9289	1.8611	12.3465	10.9460	0.9174	1.0393	0.5425	1.6321

Table 7 Solution of the cost model for different process mean shifts in FSR scheme

δ	n	h	k	ECT	AATS	E(FA)	ARL_{out}
0.25	40	0.6461	10.8436	18.3653	8.5210	0.7714	13.6875
0.50	21	0.8519	10.8770	9.0118	3.5064	0.5844	4.6151
0.75	13	0.9534	10.9229	6.0519	2.2647	0.5219	2.8745
1.00	9	0.9254	10.9808	4.6358	1.5696	0.5378	2.1953
1.25	7	1.0263	11.0392	3.8283	1.2858	0.4847	1.7521
1.50	5	0.9818	11.1573	3.3034	1.1945	0.5068	1.7157
1.75	4	1.0051	11.2774	2.9494	1.0911	0.4950	1.5847
2	4	1.1042	11.2774	2.6609	0.8421	0.4502	1.2617

Table 8 Different levels of model and cost parameters

Model and cost parameters	Level −1	Level 0	Level 1
C_0	25	50	75
C_1	50	100	200
C_2	10	20	30
C_3	50	75	100
C_4	5	10	15
p	2	4	5
m	10	50	100
λ	0.01	0.03	0.05
δ	0.5	1.0	1.5
t_0	0.1	0.3	0.5
t_1	0.5	1.0	1.5

orthogonal array is employed and the GA parameters are then assigned to it. In the L9 orthogonal array experiment, there are nine different level combinations of the four parameters. For each trial or combination, three cost values, denoted by y_1, y_2, and y_3, are obtained from the GA and the results are recorded in Table 4.

Based on the information in Table 5, the optimal level combination of the four control parameters in the GA is that PS = 30, CP = 0.3, MR = 0.07, and GN = 50.

By running MATLAB for different values of process mean shift, we achieved the optimal approximate solution of the example as shown in Tables 6 and 7.

In order to investigate the effect of model's parameters on the final solution, sensitivity analysis is arranged using orthogonal-array experimental design and multiple linear regression analysis.

Based on the proposed model, $n_1, n_2, h, w_1, w_2, k_1, k_2$ are determined as the responses. Eleven control factors ($C_0, C_1, C_2, C_3, C_4, m, \ldots t_1$) each with three levels, shown in Table 8, are allocated sequentially to an L_{33} orthogonal array, as shown in Table 9. The experiments are conducted randomly. The experimental data were analysed by following the proposed procedure strictly. For each trial, genetic algorithm was applied to produce the best approximate solution of the economic design of VSSC T^2 chart and the results are recorded in Table 10.

Consecutively to examine the effects of parameters on the responses, regression analysis concerned by Minitab statistical package. The outputs of Minitab include the ANOVA and regression coefficients tables beside normal probability plot of residuals evaluated for models adequacy and validity which show the final set of regression lines and summary of regression models (Table 11) as follows:

Table 9 Experimental design based on the L_{33} orthogonal array

Trial	C_0 A	C_1 B	C_2 C	C_3 D	C_4 E	m F	p G	δ H	λ J	t_0 K	t_1 L
1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1
2	−1	1	1	−1	1	−1	−1	−1	1	1	−1
3	1	−1	1	−1	1	−1	1	−1	−1	−1	1
4	−1	−1	1	1	1	1	−1	1	−1	−1	−1
5	1	1	−1	−1	−1	−1	1	−1	1	1	1
6	−1	−1	1	−1	−1	1	1	1	1	−1	−1
7	0	0	0	0	0	0	0	0	0	0	0
8	1	−1	−1	−1	−1	1	−1	−1	1	1	−1
9	−1	1	−1	1	−1	1	−1	1	1	1	−1
10	1	1	1	1	−1	1	1	−1	1	−1	−1
11	1	−1	1	−1	−1	−1	1	1	−1	1	−1
12	1	−1	1	1	−1	−1	−1	−1	1	−1	1
13	−1	1	−1	−1	1	1	1	1	−1	1	−1
14	1	1	1	−1	−1	1	−1	1	−1	1	1
15	1	−1	−1	−1	1	1	−1	1	1	−1	1
16	1	−1	−1	1	1	1	1	−1	−1	1	−1
17	−1	−1	−1	1	1	−1	1	−1	1	−1	−1
18	−1	1	1	−1	−1	−1	−1	1	1	−1	1
19	−1	−1	1	−1	1	1	1	−1	1	1	1
20	−1	1	1	1	1	−1	1	1	−1	−1	1
21	1	1	−1	1	1	−1	−1	−1	−1	1	1
22	1	1	1	−1	1	1	−1	1	−1	−1	−1
23	1	−1	−1	1	−1	1	1	1	−1	−1	1
24	−1	1	1	1	−1	−1	1	−1	−1	1	−1
25	1	1	1	1	1	1	1	1	1	1	1
26	−1	−1	−1	−1	1	−1	−1	1	−1	1	1
27	−1	−1	−1	1	−1	−1	1	1	1	1	1
28	−1	−1	1	1	−1	1	−1	−1	−1	1	1
29	1	−1	1	1	1	−1	−1	1	1	1	−1
30	1	1	−1	1	−1	−1	−1	1	−1	−1	−1
31	−1	1	−1	1	1	1	−1	−1	1	−1	1
32	1	1	−1	−1	1	−1	1	1	1	−1	−1
33	−1	1	−1	−1	−1	1	1	−1	−1	−1	1

Table 10 The optimal approximate solution of the proposed cost model of the VSSCT^2 control chart

Trial	n_1	n_2	h	w_1	w_2	k_1	k_2	ARL_{out}	AATS	E(FA)	ECT
1	6	36	8.6307	4.3721	2.1336	13.2782	11.869	4.6072	34.7011	0.055	28.3332
2	1	8	19.4509	7.807	1.4075	27.33	12.7816	51.6911	1004.889	0.005	50.6387
3	1	33	19.8268	56.4761	5.47	147.9021	19.0688	31.1459	616.6575	0.025	48.9769
4	10	14	16.8168	4.2554	2.1541	10.7779	10.7546	1.0493	1.7172	0.25	44.1161
5	3	24	6.8896	15.7204	4.1693	35.8304	19.331	16.2543	105.7648	0.01	48.4318
6	1	14	12.0857	0.7276	0.2112	18.9592	17.0596	1.0447	4.7116	0.005	41.4693
7	14	20	16.8939	2.4731	0.3704	15.38	15.3091	1.2945	5.4297	0.005	46.3325
8	1	35	2.8264	4.6535	3.3237	11.418	10.7225	9.9457	27.8953	0.035	44.1854
9	1	12	5.2623	0.3726	0.3475	11.418	10.7642	1.188	2.0385	0.015	40.3002
10	1	38	1.7955	9.4517	5.5023	18.9592	16.9669	10.5816	18.7498	0.055	86.2823
11	4	10	16.9209	3.8303	3.3686	27.8239	20.8952	1.4676	9.4376	0.025	34.9397
12	11	38	10.8506	0.1221	0.014	12.4273	11.8573	2.1015	13.6531	0.005	78.1498
13	2	8	16.8382	1.0869	0.8627	18.9368	17.1833	1.4079	7.8975	0.025	22.1606
14	5	9	14.5433	1.2604	0.4427	10.8732	10.7874	1.2599	5.5829	0.03	33.45
15	5	10	10.7392	1.3942	0.2272	10.8732	10.7779	1.2645	4.3189	0.005	33.4729
16	1	28	1.1855	12.9405	1.9591	18.9592	16.9855	25.6543	29.9952	0.42	58.026
17	1	33	1.7218	55.1002	2.795	147.9021	19.0688	25.5857	42.9933	0.055	96.5649
18	7	9	10.2913	3.2215	0.5105	12.9851	12.6321	1.242	3.0731	0.005	44.3691
19	1	9	17.2101	10.8951	6.3101	18.9592	17.1497	93.8402	1612.209	0.005	50.6235
20	8	9	14.3343	12.8888	4.5003	21.7024	21.2422	1.4717	7.1016	0.03	46.2782
21	3	28	6.7594	3.5652	1.0936	16.427	11.9331	5.3641	30.8897	0.07	55.9246
22	1	17	12.9653	4.2219	0.4535	11.418	10.7449	12.3608	151.0186	0.035	47.2727
23	10	12	16.8003	10.2577	6.4063	17.1236	17.0857	1.1439	3.2194	0.025	18.2835
24	15	39	8.3496	8.6465	2.7015	19.9533	18.9646	4.3019	27.7647	0.055	60.7434
25	9	10	10.1197	10.6164	2.4248	17.1497	17.1236	1.2262	2.528	0.005	65.1906
26	6	7	14.4806	1.2635	1.0096	13.2782	12.9851	1.3907	7.0222	0.03	20.8448
27	9	11	10.045	4.9621	4.9375	21.2422	20.6242	1.2213	2.3128	0.005	32.2659
28	4	34	7.2101	2.6445	0.9745	10.9309	10.7231	3.9808	22.4338	0.065	53.2787
29	6	7	6.7828	11.5497	7.0929	13.2782	12.9851	1.5254	3.9124	0.01	61.5481
30	10	14	16.8147	4.9424	2.5018	12.5177	12.2426	1.0639	1.9627	0.025	19.2499
31	3	24	6.8896	4.8472	1.2856	11.0479	10.732	9.4055	58.579	0.01	92.2198
32	5	6	11.1845	5.4761	1.3211	24.8404	23.2901	2.5056	19.208	0.005	43.9865
33	2	38	3.8552	8.3731	3.5426	18.9368	16.9669	7.1008	23.7565	0.125	31.0165

$$n_1 = -0.885401 + 0.06375\ C_3 + 0.0277627\ m + 2.6875\delta$$

$$n_2 = 41.6714 + 0.0975\ C_3 - 0.7625\ C_4 - 18.75\ \delta - 10.3125\ t_0$$

$$h = 10.7 + 0.183\ C_2 - 0.0713\ C_3 + 4.23\ \delta - 81.5\ \lambda$$

$$w_1 = 1.69 + 0.755\ C_4 - 0.0771\ m + 3.42\ p - 8.23\ \delta$$

$$w_2 = 0.231 + 0.640\ p$$

$$k_1 = 11 + 1.60\ C_4 - 0.22\ m + 7.92\ p - 17.4\ \delta$$

$$k_2 = 7.54 + 0.0272\ m + 2.36\ p + 0.785\ \delta$$

$$\mathrm{ECT} = 9.23 + 0.506\ C_2 + 0.355\ C_3 + 0.894\ C_4 - 20.5\ \delta + 448\ \lambda$$

$$\mathrm{AATS} = 268 + 9.70\ C_2 - 4.21\ C_3 + 20.6\ C_4 - 223\ \delta$$

$$E(\mathrm{FA}) = 0.0059 + 0.000844\ C_3 + 0.000486\ m - 1.65\ \lambda$$

$$\mathrm{ARL_{out}} = 16.1 + 1.24\ C_4 - 18.3\ \delta$$

The first estimated regression line indicates the hourly cost when the process operates in out-of-control state (C_3), the number of rational subgroups (m) and the magnitude of mean shift (δ) affect the small sample size (n_1). The second estimated line shows the Minitab output for the large sample size. Seeing the regression line, the hourly cost when the process operates in out-of-control state (C_3), the cost of inspecting each item (C_4), the magnitude of mean shift (δ) and the average time wasted due to searching for

Table 11 Summary of regression models and ANOVA table

Parameter	S	F	P
n_1	3.43838	5.32959	0.004762
n_2	5.43917	30.4540	0.000
h	4.27896	5.97	0.001
w_1	10.3899	5.16	0.003
w_2	1.85314	8.61	0.006
k_1	27.3234	4.16	0.009
k_2	0.812818	230.02	0.000
AATS	276.511	4.68	0.005
E(FA)	0.0718125	4.16	0.014
ECT	9.42799	22.35	0.000
ARL_{out}	15.6108	8.01	0.002

assignable cause when the process is in control (t_0), significantly influence the value of large sample size (n_2). The sign of coefficient of C_3 is positive indicating a larger magnitude of C_3 results in a larger amount of n_2. Also, Because of the coefficients of C_4, δ and t_0 are negative, by increasing each of them the value of large sample size decreases. The estimated small action limit regression line (k_2) noted that a higher number of rational subgroups (m) will reduce the amount of k_2. On the other hand, if the number of quality characteristics (p) and the magnitude of mean shift (δ) increases, the amount of k_2 increases. Regression line of optimal value of cost function (ECT) is affected significantly by three cost parameters and two process parameters (i.e., $C_2, C_3, C_4, \lambda, \delta$). A larger shift magnitude of process mean (δ) leads a lower value of ECT. Meanwhile, increase in values of C_2, C_3, C_4 and λ results in increase in the value of ECT. Also similar analysis may be conducted for the sampling interval (h), the large warning limit (w_1), the small warning limit (w_2), the large action limit (k_1), adjusted average time to signal (AATS), the average number of false alarms (E(FA)), and the number of samples drawn when the process operates in out-of-control state (ARL_{out}), respectively.

Concluding remarks

Delivering economical design of the VSSC T^2 control chart on the presence of fixed sampling intervals and exponentially distributed assignable causes is the main contribution of the present study which provides more sensitivity in the traditional Hotelling's T^2 control chart in rapid detecting of small drifts in the process mean vector. The real assumption on the occurrence times of the assignable cause is allowed us in applying the Markov chain approach on constructing the proposed expected hourly cost model as a novel extension of the priors. The main accomplished results on the proposed model are

- Larger changes in the process mean vector cause to increase value of small action limit. Additionally, it tends to generate a lower expected cost per time and large sample size.
- The large sample size tends to be raised when the hourly cost of operating process in out-of-control state increases. Also, it decreases when cost of inspecting each item or wasted time due to each false alarm increases.
- By growth in value of the hourly cost of operating the process in control, the hourly cost of operating the process out of control or the cost for each inspected item, the expected cost per time increases.
- The small action limit will be large by adding to the number of quality characteristics or deduction in the number of rational subgroups.
- If the duration of in control period increases, the expected cost per time will decrease.

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