Original Research

Modeling the Effect of Advertising and Subsidy Transfer in a Three-Level Channel Using Game Theory

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Abstract
Most cooperative advertising works assume that subsidy is usually given directly to the retailer. This work addresses a three-level supply chain setting where advertising subsidy is transferable from the manufacturer to the retailer through the distributor, together with a situation where both the retailer and the distributor directly engage in advertising. The parties in the channel are considered to play an infinite horizon Stackelberg differential game with the manufacturer as the leader, and the distributor and the retailer as the first and the last followers respectively. The work studies the effect of subsidy on the players' profits in a four-channel setting where both the manufacturer and the distributor do not subsidise advertising; the manufacturer's provided retail subsidy is not transferred to the retailer; the distributor provides retail subsidy without the manufacturer's involvement; and the distributor transfers the manufacturer's provided subsidy to the retailer. It determines the players' optimal advertising efforts, subsidy policies, and profits for the four channel settings. It also obtains the players' appropriate subsidy limits. The work shows that the channel profits are best with the distributor's participation, followed by transfer of subsidy setting, and worst with non-provision of subsidy by both manufacturer and distributor.

Keywords - Cooperative advertising; differential game; subsidy rate; Stackelberg game; supply chain

INTRODUCTION

Cooperative advertising strategy is an arrangement in which the manufacturer who leads a supply channel pays for a fraction or all the costs incurred by the follower (retailer) in the course of advertising the product of the manufacturer. We note that a lot of cooperative advertising works are game theoretic models. For instance, using a Nash and Stackelberg game [1] compared how cooperative advertising and prices affect traditional channel and online channel. Also, considering interest in price coordination, advertising effort and lot-sizing [2] used non-cooperative Nash game and a Stackelberg game to model a dual supply channel involving a manufacturer, a retailer and the consumers. Models that employ Stackelberg strategy and consider channel integration are widely used not only in cooperative advertising but other areas. For instance [3] considered vendorbuyer inventory in an imperfect manufacturing process, developed an integrated channel problem and a Stackelberg model. Another imperfect system which can run out of control leading to the production of defective items can be found in [4]. Also, based on preference for corporate social responsibility and government subsidies, [5] used game theory to model and analyze decisions under distinct power settings and subsidy settings. Using differential game [6] examined the possibility of intercepting a superior missile by a formation of pursuers with lower capabilities. [7] examined groundwater extraction using a discrete-time game on an infinite-horizon. They considered a situation where the degree of interactions between rainfall and water users are fixed, and Stackelberg game in which the extraction agency takes a decision on the cost to announce. Their simulation showed that optimal policies are possible through threshold policies. To maximize countries' stream of revenues [8] used stochastic differential game to consider the noncooperative Nash, cooperative and Stackelberg equilibria through feedback strategy. They work showed that the game outcome is dependent on the parameter and equilibrium of choice. Much recently, [9] used a dynamic duopoly to address the question of whether an alteration of the number of objectives a firm seeks to achieve influences their strategy. They considered a dynamic duopoly in which each of the firms aims at two objectives and analyzed open-loop Nash equilibria and cooperative equilibria.

In the classical cooperative advertising literature only manufacturer(s)-retailer(s) channels are considered. But, a close examination suggests that models on such channel settings do not portray reality, because a lot of manufacturers do not sell their products directly to the retailers. Most times a distributor links these two traditional channel members. Thus there is the need to study a channel with these three parties. The incorporation of the distributor as the middle man in cooperative advertising was first modeled by [10]. The origin of cooperative advertising mathematical models can be traced to [11]. He used a static setting to model cooperative advertising as price discount from a manufacturer to a retailer. Taking a clue from this work, a number of static models on cooperative advertising evolved [12-18]. These models are easily tractable leading to closed-form solutions. However, a major drawback is that they can only support a single period analysis. Although this helps with obtaining details in analyzing the interactions among a number of factors that are used in these models, dynamic models seem to be more realistic, especially on an infinite horizon.

Dynamic cooperative advertising models employ differential game theory to study the interactions between factors of interest in supply chains over time. Such works usually employ goodwill functions as can be seen in [19], some of which are based on Nerlove-Arrow model [20]. Using Sethi's sales-advertising model [21], [22] considered dynamic stochastic cooperative advertising. He *et al.* [23] extended this model to retail competition where a manufacturer supports his retailer who is competing with an independent retailer. This was further extended by [24] to consider dynamic cooperative advertising in a retail duopoly using a duopolistic extension of Sethi's sales-advertising model. Chutani and Sethi [25] improved on the model in [24] by incorporating many retailers. Ezimadu and Nwozo [26] extended [22] by incorporating national advertising into the model in a stochastic setting. Further, in an extension of [24], [27] incorporated national effort into cooperative advertising model with retail competition. In another extension of [24], [28] considered a dynamic cooperative advertising setting with independent manufacturers and retailers. They considered a situation where each manufacturer sells his product through all the retailers, and provides them with subsidy, and each retailer sells all the manufacturers' products.

Recently, [29] considered the effectiveness of revenue-sharing contracting scheme and cost-sharing contracting scheme in a dual channel. They developed two cooperative advertising models: a Stackelberg game model in which a manufacturer determines the sharing formula, and a Nash bargaining game in which a manufacturer and a retailer together decide the sharing formula. Using game theory [30] examined a remanufacturer and a retailer's decisions on a non-cooperative green advertising and cooperative green advertising program for distinct competitive scenarios in a low carbon supply chain. Considering an online setting [31] studied a cooperative advertising situation where the manufacturer sells a product through a traditionally direct supply channel and a situation through an online platform. They examined how the manufacturer's strategy and the choice contract of a platform distribution are related. Taking the effect of price and advertising into consideration in an online setting, [32] used Stackelberg game to consider a two-period online cooperative advertising supply channel having a manufacturer who sells his product through an online retail platform. They examined how payoff can be influenced by adopting a one-way subsidy strategy, a two-way subsidy strategy and a revenue sharing strategy.

Ezimadu [10] extended cooperative advertising from its classical manufacturer-retailer model setting to a manufacturer-distributor-retailer setting by incorporating the distributor. This is the first dynamic manufacturer-distributor-retailer cooperative advertising model. He modeled a supply chain in which the manufacturer is the Stackelberg leader, the distributor is the first follower, and the retailer is the last follower. In his model, the manufacturer and the distributor indirectly engage in advertising by subsidizing the retailer's advertising effort. Ezimadu [18] considered a static form of this model using a four-game scenario. In a quest to study the feasibility of incorporating the distributor into cooperative advertising, [17] modeled a setting where the manufacturer directly provides advertising subsidy for the retailer, and all the players deploy their prices as decision variables. Ezimadu [33] modified [10] to consider a subsidy transfer setting from the manufacturer to the retailer with the help of the distributor. In another extension, [34] modeled a setting in which the manufacturer can bypass the distributor to provide the retailer with subsidy. This work is motivated by the scarcity of three-level cooperative advertising models that simultaneously consider direct involvement of two followers' advertising efforts in a supply channel, and the transfer of advertising subsidy from the channel leader to the last follower through the first follower. These market factors can significantly determine product demand and profit for the channel.

In this work we propose a manufacturer-distributor-retailer cooperative advertising game model in which the retailer engages in local advertising, and for the first time the distributor engages in regional advertising and also provides the retailer with advertising subsidy, while the manufacturer subsidizes channel advertising by providing the distributor with advertising subsidy in hope that this (subsidy) will be transferred to the retailer. Such an aggressive advertising and subsidy approach can be employed for a product brand which is newly introduced into an already saturated market. It can also be deployed if the product brand is in competition with a superior brand which is threatening its survival in the market. This work uses an infinite horizon Stackelberg differential game to consider an extension of [33] in which the distributor supports the retailer's local advertising, directly engages in advertising, and can receive advertising subsidy support from the manufacturer. Thus we consider four channel-types which include a situation in which:

- both the manufacturer and the distributor do not subsidize advertising,
- there is partial channel subsidy provision from the manufacturer to the distributor,
- there is partial channel subsidy provision from the distributor to the retailer,
- there is total channel subsidy from the manufacturer and the distributor.

Partial channel subsidy as used above means a scenario where only the manufacturer or only the distributor is engaged in the provision of advertising subsidy as the case may be. Total subsidy means a scenario where the manufacturer and the distributor provide subsidy. From these four channel structures we will obtain the players' optimal advertising policies, the optimal subsidy policies, the market awareness share and the profits. Based on these we will consider the effect of subsidy on the market share and the profits. We will also consider the relationship between the subsidies and the profits to provide information on each player's response to the manufacturer and the distributor's subsidies. Further, we will consider the players' reactions to each subsidy commitment.

MODEL FORMULATION

I. The Players' Advertising Expenditure

To achieve a high product awareness, the retailer engages in local advertising using the effort $a_R(t)$ at time t. The distributor engages in regional advertising. He decides on the advertising effort $a_D(t)$ at time t, and in addition, he subsidises the retailer's local advertising effort with the subsidy rate α_D. This subsidy provision is premised on the retailer's closeness to the end-users of the product, and his understanding of the local terrain and ability to use local media to the advantage of the entire supply chain. Although the manufacturer is not directly engaged in advertising, he provides advertising subsidy to the distributor, and expects him to in turn extend the same gesture (subsidy) to the retailer who is the actual source of the channel revenue. His subsidy rate is α_M.

Works on cooperative advertising usually assume the advertising cost function to be quadratic in the players' advertising effort as a result of increasing marginal advertising cost [22-24, 34-38]. Thus the advertising expenditure of the retailer, the distributor and the manufacturer are $(1-\alpha_D)a_R(t)^2$, $(1-\alpha_M)a_D(t)^2+\alpha_D a_R(t)^2$ and $\alpha_M a_D(t)^2$ respectively.

II. Market Awareness Share Dynamics

We will use Sethi's sales-advertising model to model the awareness share dynamics. Many forms of this model have been studied, empirically validated and applied [22, 23, 34, 38-42]. The model is expressed as

$$x'(t)=\beta a_R(t)\sqrt{1-x(t)}-\delta x(t),\;\; x(0)=x_0\in[0,1],$$

where $x(t)$ is the market awareness indicating the fraction of the market informed of the product; the parameter β is an indicator of how advertising affects sales; δ is the rate at which the product awareness decays or is lost. In this work we employ a modification of the form

$$x'(t)=\beta\big(a_R(t)+a_D(t)\big)\sqrt{1-x(t)}-\delta x(t),\qquad x(0)=x_0\in[0,1]. \tag{1}$$

A similar form of (1) was employed by [26] in a manufacturer-retailer setting. This was also done by [34] in a manufacturer-distributor-retailer setting.

III. The Players' Hierarchical Decision Sequence

In this work the manufacturer is taken to be the leader of the channel. As such, the decisions of the other players depend on his decision. First he unveils his subsidy level $\alpha_M(t) \in [0,1]$ to the distributor. In reaction, the distributor decides on his advertising effort $a_D\big(x(t)\big|\alpha_M(t)\big)$ and subsidy rate $\alpha_D\big(t\big|\alpha_M(t)\big) \in [0,1]$ for retail advertising. Finally, based on these moves, the retailer decides on his local advertising effort $a_R\Big(x(t)\Big|a_D\big(x(t)\big), \alpha_D(t)\Big)$ by solving the control problem

$$f_R(x) = \max_{a_R\left(x(t)\middle|a_D(x(t)),\,\alpha_D(t)\right)\geq 0} \int_0^\infty e^{-\rho t}\left[m_R x(t) - \Big(1-\alpha_D\big(t\big|\alpha_M(t)\big)\Big)\, a_R\Big(x(t)\Big|a_D\big(x(t)\big), \alpha_D(t)\Big)^2\right] dt \quad (2)$$

subject to (1), where f_R is the retailer's profit function; ρ is the discount rate, and m_R is the retailer's price margin.

The distributor incorporates the retailer's anticipated response into his optimal control problem to obtain his advertising effort $a_D\big(x(t)\big|\alpha_M(t)\big)$ and subsidy rate $\alpha_D\big(t\big|\alpha_M(t)\big) \in [0,1]$. Thus, with f_D as the distributor's profit function and m_D as his price margin the distributor's control problem is given by

$$f_D(x) = \max_{\substack{a_D\left(x(t)\middle|\alpha_M(t)\right)\geq 0\\ 0\leq \alpha_D\left(t\middle|\alpha_M(t)\right)\leq 1}} \int_0^\infty e^{-\rho t}\left[m_D x(t) - \big(1-\alpha_M(t)\big)a_D\big(x(t)\big|\alpha_M(t)\big)^2 \right.$$
$$\left. - \alpha_D(t|\alpha_M)\, a_R\Big(x(t)\Big|a_D\big(x(t)\big), \alpha_D(t)\Big)^2\right] dt \quad (3)$$

$$x'(t) = \beta\Big(a_R\Big(x(t)\Big|a_D\big(x(t)\big), \alpha_D(t)\Big) + a_D\big(x(t)|\alpha_M\big)\Big)\sqrt{1-x(t)} - \delta x(t), \qquad x(0) = x_0 \in [0,1]. \quad (4)$$

Finally, the manufacturer incorporates the distributor's anticipated response to solve for α_M using the optimal control problem

$$f_M(x) = \max_{0\leq \alpha_M(t)\leq 1} \int_0^\infty e^{-\rho t}\left[m_M x(t) - \alpha_M(t)\, a_D\big(x(t)\big|\alpha_M(t)\big)^2\right] dt, \quad (5)$$

$$x'(t) = \beta\Big(a_R\Big(x(t)\Big|a_D\big(x(t)\big), \alpha_D(t)\Big) + a_D\big(x(t)|\alpha_M\big)\Big)\sqrt{1-x(t)} - \delta x(t), \qquad x(0) = x_0 \in [0,1], \quad (6)$$

where f_M is the manufacturer's profit function and m_M his price margin. We note that (1)-(2), (3)-(4) and (5)-(6) are infinite horizon differential games. To simplify our discussion we shall remove the arguments where there is no ambiguity.

THE ADVERTISING STRATEGIES AND PROFITS

We will determine the Stackelberg equilibrium using backward induction, which is achievable by solving the retailer's control problem first. This will be followed by the distributor's problem, and then the manufacturer's problem.

I. The Retailer's Advertising Strategy and Profit

From (1) and (2), we have the Hamilton-Jacobi-Bellman (HJB) equation

$$\rho f_R(x) = \max_{a_R(x|a_D,\,\alpha_D,\,\alpha_M)\geq 0} \Big\{ m_R x - \big(1-\alpha_D(t|\alpha_M)\big)a_R(x|a_D,\alpha_D)^2$$
$$+ f_{Rx}\big[\beta\big(a_R(x|a_D,\alpha_D) + a_D(x|\alpha_M)\big)\sqrt{1-x} - \delta x\big]\Big\}, \quad (7)$$

where f_{Rx} is the first derivative of f_R with respect to x. Performing the maximization in (7) we have

$$-2\big(1-\alpha_D(t|\alpha_M)\big)a_R(x|a_D,\alpha_D) + \beta f_{Rx}\sqrt{1-x} = 0. \Longrightarrow \quad a_R(x|a_D,\alpha_D) = \frac{\beta f_{Rx}\sqrt{1-x}}{2\big(1-\alpha_D(t|\alpha_M)\big)}. \quad (8)$$

Putting (8) into (7) we have

$$\rho f_R(x) = m_R x + \frac{\beta^2 (f_{Rx})^2 (1-x)}{4\big(1-\alpha_D(t|\alpha_M)\big)} + \beta f_{Rx} a_D(t|\alpha_M)\sqrt{1-x} - \delta x. \qquad (9)$$

Equation (8) shows the retailer's willingness to increase effort as his profit rate, advertising effectiveness and the subsidy to him increase. However, the subsidy should not be total (one hundred percent), else it would imply that the retailer would engage in unbounded advertising effort which makes no sense since it is unrealistic and practically impossible. Also the rational and benefit of the distributor totally subsidizing the effort is brought to question considering the certainty of the retail expenditure $(1-\alpha_D)a_R^2$ amounting to zero because this will adversely affect the distributor and the entire channel. On the other hand, the retailer has the leverage of reducing advertising commitment as the product awareness gets large. This is understandable since it is natural to reduce effort towards patronage when the market is saturated with a product's good-will information.

Now, the second term in (9) means that the retailer's profit increases with price margin, advertising effectiveness, awareness, distributor's subsidy rate and effort, while it is negatively affected by the discount and the decay rates. These can be respectively salvaged if the retailer is foresighted and adopts advertising method(s) that will leave lasting effect on the mind of the end-user.

II. The Distributor's Strategies and Profit

From (3) and (4) we have the HJB equation

$$\rho f_D(x) = \max_{\substack{a_D(x|\alpha_M)\ge 0 \\ 0\le \alpha_D(t|\alpha_M)\le 1}} \{m_D x - (1-\alpha_M)a_D(x|\alpha_M)^2 - \alpha_D(t|\alpha_M)a_R(x|a_D,\alpha_D)^2$$
$$+ f_{Dx}\big[\beta\big(a_R(x|a_D,\alpha_D) + a_D(x|\alpha_M)\big)\sqrt{1-x} - \delta x\big]\}. \qquad (10)$$

where f_{Dx} is the first derivative of f_D with respect to x. Considering $a_D(x|\alpha_M)$ in performing the given maximization on the right-hand side of (10), the first order condition for an interior solution is

$$-2a_D(x|\alpha_M)(1-\alpha_M) + \beta f_{Dx}\sqrt{1-x} = 0, \Longrightarrow \quad a_D(x|\alpha_M) = \frac{\beta f_{Dx}\sqrt{1-x}}{2(1-\alpha_M)}. \qquad (11)$$

Using (8) and (11) in (10) we have

$$\rho f_D(x) = \max_{0\le \alpha_D(t|\alpha_M)\le 1} \left\{ m_D x + \frac{\beta^2 (f_{Dx})^2 (1-x)}{4(1-\alpha_M)} - \frac{\alpha_D(t|\alpha_M)\beta^2 (f_{Rx})^2 (1-x)}{4\big(1-\alpha_D(t|\alpha_M)\big)^2} + \frac{\beta^2 f_{Rx} f_{Dx}(1-x)}{2\big(1-\alpha_D(t|\alpha_M)\big)} - \delta f_{Dx} x \right\}. \qquad (12)$$

Now, performing the given maximization in (12) we have

$$-\frac{\beta^2 (f_{Rx})^2 (1-x)}{4}\left[\frac{\big(1-\alpha_D(t|\alpha_M)\big)^2}{\big(1-\alpha_D(t|\alpha_M)\big)^4} + \frac{2\alpha_D\big(1-\alpha_D(t|\alpha_M)\big)}{\big(1-\alpha_D(t|\alpha_M)\big)^4}\right] + \frac{\beta^2 f_{Rx} f_{Dx}(1-x)}{2\big(1-\alpha_D(t|\alpha_M)\big)^2} = 0,$$

$$\Longrightarrow \quad \alpha_D(t|\alpha_M) = \frac{2f_{Dx} - f_{Rx}}{2f_{Dx} + f_{Rx}} > 0 \ \text{ if } \ 2f_{Dx} > f_{Rx} \qquad (13)$$

which is positive if $2f_{Dx} > f_{Rx}$, or zero otherwise

Just as stated above, (11) shows that the distributor's commitment to advertising increases with his profit rate, advertising effectiveness, awareness, and subsidy from the manufacturer. Equation (12) shows that the distributor's profit depends on the distributor and the manufacturer's subsidy rates. For large α_D, that is, as $\alpha_D \longrightarrow 1$, the term $(1-\alpha_D)^2$ approaches 0 faster than $1-\alpha_D$ so that the term $\frac{\alpha_D \beta^2 (f_{Rx})^2 (1-x)}{4(1-\alpha_D)^2}$ tends to be larger than $\frac{\beta^2 f_{Rx} f_{Dx}(1-x)}{2(1-\alpha_D)}$, and so grows out of bound faster. This implies that it is not in the distributor's best interest to provide large subsidy let alone to totally support the retailer. Obviously, the reverse is the case for very low subsidy. That is, as α_D approaches 0, the term $\frac{\alpha_D \beta^2 (f_{Rx})^2 (1-x)}{4(1-\alpha_D)^2}$ approaches a very small value, and eventually becomes 0 (at $\alpha_D = 0$), while $\frac{\beta^2 f_{Rx} f_{Dx}(1-x)}{2(1-\alpha_D)}$ tends to $\frac{\beta^2 f_{Rx} f_{Dx}(1-x)}{2} \neq 0$. Thus, the distributor may have to opt for optimal subsidy. In short, (13) shows that he should only provide subsidy if the rate of increase of his profit is twice greater than that of the retailer.

III. The Manufacturer's Subsidy Rate and Profit

From (5) and (6) we have the HJB equation

$$\rho f_M(x) = \max_{0 \leq \alpha_M \leq 1} \left\{ m_M x - \frac{\alpha_M \beta^2 (f_{Dx})^2 (1-x)}{4(1-\alpha_M)^2} + \frac{\beta^2 f_{Rx} f_{Mx}(1-x)}{2\big(1-\alpha_D(t|\alpha_M)\big)} + \frac{\beta^2 f_{Mx} f_{Dx}(1-x)}{2(1-\alpha_M)} - \delta f_{Mx} x \right\}, \quad (14)$$

where f_{Mx} is the first derivative of f_M with respect to x. Performing the given maximization on the right-hand side of (14), leads to the first order condition for interior solution

$$-\frac{\beta^2 (f_{Dx})^2 (1-x)}{4} \left[\frac{(1-\alpha_M)^2 + 2\alpha_M (1-\alpha_M)}{(1-\alpha_M)^4} \right] + \frac{\beta^2 f_{Mx} f_{Dx} (1-x)}{2(1-\alpha_M)^2} = 0.$$

$$\Rightarrow \quad \alpha_M = \frac{2 f_{Mx} - f_{Dx}}{2 f_{Mx} + f_{Dx}} \quad \text{if } 2 f_{Mx} > f_{Dx}. \qquad (15)$$

From (14) we notice that large manufacturer's subsidy adversely affects his profit just like the case of the distributor discussed earlier; and his provision of subsidy should be based on (15) that is, his profit rate of increase should be twice greater than that of the retailer. Further, just as was noted earlier, we observe in the last two subsections that while the profits increase with the players' price margins, they are negatively affected by the discount and the decay rates. These can be salvaged as stated earlier.

STACKELBERG EQUILIBRIUM FOR THE PLAYERS' PROFITS AND RATES OF INCREASE FOR UNSUBSIDIZED ADVERTISING

This section deals with a channel setting in which both the manufacturer and the distributor do not subsidize advertising. Thus, $\alpha_D(t|\alpha_M) = \alpha_M = 0$. This channel setting serves as reference point in considering the other settings. Putting (11) in (9) we have

$$\rho f_R(x) = m_R x + \frac{\beta^2 (f_{Rx})^2 (1-x)}{4\big(1-\alpha_D(t|\alpha_M)\big)} + \frac{\beta^2 f_{Rx} f_{Dx}(1-x)}{2(1-\alpha_M)} - \delta f_{Rx} x. \qquad (16)$$

Since there is no provision of subsidy (16) becomes

$$f_R(x) = \frac{\beta^2 f_{Rx}^2 + 2\beta^2 f_{Rx} f_{Dx}}{4\rho} + \left[\frac{4 m_R - \beta^2 f_{Rx}^2 - 2\beta^2 f_{Rx} f_{Dx} - 4\delta f_{Rx}}{4\rho} \right] x. \quad (17)$$

Obviously f_R is linear in x with slope

$$f_{Rx} = \frac{4 m_R - \beta^2 f_{Rx}^2 - 2\beta^2 f_{Rx} f_{Dx} - 4\delta f_{Rx}}{4\rho},$$

$$\Rightarrow \quad \beta^2 f_{Rx}^2 + \big(2\beta^2 f_{Dx} + 4(\delta+\rho)\big) f_{Rx} - 4 m_R = 0,$$

which is quadratic in f_{Rx}. Thus, we have that

$$f_{Rx(\alpha_D,\, \alpha_M = 0)} = \frac{-\big(2\beta^2 f_{Dx} + 4(\delta+\rho)\big)}{2\beta^2} \pm \frac{\sqrt{\big(2\beta^2 f_{Dx} + 4(\delta+\rho)\big)^2 + 16\beta^2 m_R}}{2\beta^2}$$

Similarly (12) becomes

$$f_D(x) = \frac{\beta^2 f_{Dx}^2 + 2\beta^2 f_{Rx} f_{Dx}}{4\rho} + \left[\frac{4 m_D - \beta^2 f_{Dx}^2 - 2\beta^2 f_{Rx} f_{Dx} - 4\delta f_{Dx}}{4\rho} \right] x \quad (18)$$

$$\Longrightarrow \quad f_{Dx}=\frac{4m_D-\beta^2 f_{Dx}^2-2\beta^2 f_{Rx}f_{Dx}-4\delta f_{Dx}}{4\rho}$$

which is quadratic in f_{Dx}. Thus

$$f_{Dx(\alpha_D,\,\alpha_M=0)}=\frac{-\left(2\beta^2 f_{Rx}+4(\delta+\rho)\right)}{2\beta^2}\pm\frac{\sqrt{\left(2\beta^2 f_{Rx}+4(\delta+\rho)\right)^2+16\beta^2 m_D}}{2\beta^2}.$$

Also (14) becomes

$$f_M(x)=\frac{\beta^2 f_{Rx}f_{Mx}+\beta^2 f_{Dx}f_{Mx}}{2\rho}+\left[\frac{2m_M-\beta^2 f_{Rx}f_{Mx}-\beta^2 f_{Dx}f_{Mx}-2\delta f_{Mx}}{2\rho}\right]x \qquad (19)$$

$$\Longrightarrow \quad f_{Mx}=\frac{2m_M-\beta^2 f_{Rx}f_{Mx}-\beta^2 f_{Dx}f_{Mx}-2\delta f_{Mx}}{2\rho}.$$

This is linear in f_{Mx}. Thus we have

$$f_{Mx(\alpha_D,\,\alpha_M=0)}=\frac{2m_M}{\beta^2(f_{Rx}+f_{Dx})+2(\delta+\rho)}.$$

Hence:

Proposition 1 Suppose that neither the manufacturer nor the distributor subsidises any of the advertising efforts then, the retailer and distributor's advertising efforts are

$$a_R(x)=\frac{\beta f_{Rx(\alpha_D,\,\alpha_M=0)}\sqrt{1-x(t)}}{2}$$

and

$$a_D(x)=\frac{\beta f_{Dx(\alpha_D,\,\alpha_M=0)}\sqrt{1-x(t)}}{2}$$

respectively; while the retailer, the distributor and the manufacturer's payoffs are given by (17), (18) and (19) respectively.

STACKELBERG EQUILIBRIUM FOR THE PLAYERS' PROFIT AND RATES OF INCREASE FOR PARTIAL CHANNEL SUBSIDY BY THE MANUFACTURER

This section deals with a channel setting in which only the manufacturer provides subsidy. That is, while the distributor receives subsidy from the manufacturer, he does not in turn transfer or provide subsidy to the retailer. This can be seen as a case of non-transfer of subsidy to the retailer. As such, $\alpha_D(t|\alpha_M)=0$, and $\alpha_M>0$ as obtained in (15), so that (16) becomes

$$f_R(x)=\frac{\beta^2 f_{Rx}^2+2\beta^2 f_{Rx}f_{Dx}}{4\rho}+\left[\frac{4m_R-\beta^2 f_{Rx}^2-2\beta^2 f_{Rx}f_{Dx}-4\delta f_{Rx}}{4\rho}\right]x \quad (20)$$

$$\Longrightarrow \quad f_{Rx}=\frac{4m_R-\beta^2 f_{Rx}^2-2\beta^2 f_{Rx}f_{Dx}-4\delta f_{Rx}}{4\delta}.$$

Thus

$$f_{Rx(\alpha_D=0,\,\alpha_M>0)}=\frac{-\left(2\beta^2 f_{Dx}+4(\delta+\rho)\right)}{2\beta^2} \qquad \# \pm\frac{\sqrt{\left(2\beta^2 f_{Dx}+4(\delta+\rho)\right)^2+16\beta^2 m_R}}{2\beta^2}$$

Similarly, (12) becomes

$$f_D(x) = \frac{2\beta^2 f_{Dx} f_{Mx} + \beta^2 f_{Dx}^2 + 2\beta^2 f_{Rx} f_{Dx}}{8\rho} + \left[\frac{8m_D}{8\rho}\right]x + \left[\frac{-2\beta^2 f_{Dx} f_{Mx} - \beta^2 f_{Dx}^2 - 2\beta^2 f_{Rx} f_{Dx} - 8\delta f_{Dx}}{8\rho}\right]x, \quad (21)$$

implying that

$$f_{Dx} = \frac{8m_D - 2\beta^2 f_{Dx} f_{Mx} - \beta^2 f_{Dx}^2 - 2\beta^2 f_{Rx} f_{Dx} - 8\delta f_{Dx}}{8\rho}.$$

Thus

$$f_{Dx(\alpha_D=0,\,\alpha_M>0)} = \frac{-\left(2\beta^2(f_{Mx} + f_{Rx}) + 8(\delta+\rho)\right)}{2\beta^2} \pm \frac{\sqrt{\left(2\beta^2(f_{Mx} + f_{Rx}) + 8(\delta+\rho)\right)^2 + 32\beta^2 m_D}}{2\beta^2}.$$

Also (14) becomes

$$f_M(x) = \frac{4\beta^2 f_{Mx}^2 + \beta^2 f_{Dx}^2 + 8\beta^2 f_{Rx} f_{Mx} + 4\beta^2 f_{Mx} f_{Dx}}{16\rho} + \left[\frac{16m_M - 4\beta^2 f_{Mx}^2 - \beta^2 f_{Dx}^2 - 8\beta^2 f_{Rx} f_{Mx}}{16\rho}\right]x + \left[\frac{-4\beta^2 f_{Mx} f_{Dx} - 16\delta f_{Mx}}{16\rho}\right]x, \quad (22)$$

$$\Rightarrow \quad f_{Mx} = \frac{16m_M - 4\beta^2 f_{Mx}^2 - \beta^2 f_{Dx}^2 - 8\beta^2 f_{Rx} f_{Mx}}{16\rho} \frac{-4\beta^2 f_{Mx} f_{Dx} - 16\delta f_{Mx}}{16\rho}$$

Thus

$$f_{Mx(\alpha_D=0,\,\alpha_M>0)} = \frac{-U_1}{8\beta^2} \pm \frac{\sqrt{U_1^2 - U_2}}{8\beta^2},$$

where

$$U_1 = \left(4\beta^2(2f_{Rx} + f_{Dx}) + 16(\delta+\rho)\right)$$

$$U_2 = 16\beta^2(\beta^2 f_{Dx} - 16m_M)$$

Hence:

Proposition II Suppose that the manufacturer provides subsidy which is however not transferred to the retailer, then the advertising efforts are given by

$$a_{R(\alpha_D=0,\,\alpha_M>0)}(x) = \frac{\beta f_{Rx(\alpha_D=0,\,\alpha_M>0)}\sqrt{1-x(t)}}{2}$$

and

$$a_{D(\alpha_D=0,\,\alpha_M>0)}(x) = \frac{\beta\left(2f_{Mx(\alpha_D=0,\,\alpha_M>0)} + f_{Dx(\alpha_D=0,\,\alpha_M>0)}\right)}{4} \times \frac{\sqrt{1-x(t)}}{4}$$

and the payoffs are given by (20), (21) and (22).

STACKELBERG EQUILIBRIUM FOR THE PLAYERS' PROFITS AND RATES OF INCREASE FOR PARTIAL CHANNEL SUBSIDY BY THE DISTRIBUTOR

This section deals with a channel setting in which the distributor provides advertising subsidy for the retailer without receiving support from the manufacturer. This can be as a result of the retailer's importance as the actual source of channel members' revenue. Thus, this is a form of intervention subsidy. Since only the distributor subsidises advertising, we regard this as partial channel subsidy irrespective of the rate. Thus, $\alpha_M = 0$, while $\alpha_D(t|\alpha_M) > 0$ as given in (13), so that (16) becomes

$$\rho f_R(x) = m_R x + \frac{\beta^2 f_{Rx}^2(1-x)}{4\left(1-\frac{2f_{Dx}-f_{Rx}}{2f_{Dx}+f_{Rx}}\right)} + \frac{\beta^2 f_{Rx} f_{Dx}(1-x)}{2} - \delta f_{Rx} x, \tag{23}$$

which implies

$$f_{Rx} = \frac{8m_R - \beta^2 f_{Rx}(2f_{Dx}+f_{Rx}) - 4\beta^2 f_{Rx} f_{Dx} - 8\delta f_{Rx}}{8\rho}.$$

Thus

$$f_{Rx(\alpha_D>0,\,\alpha_M=0)} = \frac{-\left(6\beta^2 f_{Dx} + 8(\delta+\rho)\right)}{2\beta^2} \pm \frac{\sqrt{\left(6\beta^2 f_{Dx} + 8(\delta+\rho)\right)^2 + 32\beta^2 m_R}}{2\beta^2}.$$

Similarly (12) becomes

$$\rho f_D(x) = m_D x + \frac{\beta^2 f_{Dx}^2(1-x)}{4} - \frac{2f_{Dx}-f_{Rx}}{2f_{Dx}+f_{Rx}}\left(\frac{\beta^2 f_{Rx}^2(1-x)}{4}\right)\left(\frac{2f_{Dx}+f_{Rx}}{2f_{Rx}}\right)^2 + \frac{\beta^2 f_{Rx} f_{Dx}(1-x)}{2}\left(\frac{2f_{Dx}+f_{Rx}}{2f_{Rx}}\right) - \delta f_{Dx} x \quad (24)$$

$$= \frac{4\beta^2 f_{Dx}(3f_{Dx}+f_{Rx}) - \beta^2((2f_{Dx})^2 - f_{Rx}^2)}{16\rho} + \left[\frac{16m_D - 4\beta^2 f_{Dx}(3f_{Dx}+f_{Rx})}{16\rho}\right]x + \left[\frac{\beta^2((2f_{Dx})^2 - f_{Rx}^2) - 16\delta f_{Dx}}{16\rho}\right]x.$$

$$f_{Dx} = \frac{16m_D - 4\beta^2 f_{Dx}(3f_{Dx}+f_{Rx})}{16\rho} + \frac{\beta^2((2f_{Dx})^2 - f_{Rx}^2) - 16\delta f_{Dx}}{16\rho}$$

Thus

$$f_{Dx(\alpha_D>0,\,\alpha_M=0)} = \frac{-\left(4\beta^2 f_{Rx} + 16(\delta+\rho)\right)}{16\beta^2} \pm \frac{\sqrt{\left(4\beta^2 f_{Rx} + 16(\delta+\rho)\right)^2 - 32\beta^2(\beta^2 f_{Rx} - 16m_D)}}{16\beta^2}.$$

Also (14) becomes

$$\rho f_M(x) = m_M x + \frac{\beta^2 f_{Rx} f_{Mx}(1-x)}{2}\left(\frac{2f_{Dx}+f_{Rx}}{2f_{Rx}}\right) + \frac{\beta^2 f_{Mx} f_{Dx}(1-x)}{2} - \delta f_{Mx} x \tag{25}$$

which implies

$$f_{Mx} = \frac{4m_M - \beta^2 f_{Mx}(2f_{Dx}+f_{Rx}) - 2\beta^2 f_{Mx} f_{Dx} - 4\delta f_{Mx}}{4\rho}.$$

Thus

$$f_{Mx(\alpha_D>0,\,\alpha_M=0)} = \frac{4m_M}{\beta^2(4f_{Dx}+f_{Rx}) + 4(\delta+\rho)}.$$

Hence:

Proposition III Suppose that the distributor provides subsidy to the retailer irrespective of the manufacturer not providing subsidy, then the advertising efforts are given by

$$a_{R(\alpha_D>0,\,\alpha_M=0)}(x) = \frac{\beta\left(2f_{Dx(\alpha_D>0,\,\alpha_M=0)} + f_{Rx(\alpha_D>0,\,\alpha_M=0)}\right)}{4} \times \frac{\sqrt{1-x(t)}}{4}$$

and

$$a_{D(\alpha_D>0,\,\alpha_M=0)}(x) = \frac{\beta f_{Dx(\alpha_D>0,\,\alpha_M=0)}\sqrt{1-x(t)}}{2}$$

and the payoffs are given by (23), (24) and (25).

STACKELBERG EQUILIBRIUM FOR THE PLAYERS' PROFITS AND RATES OF INCREASE FOR TOTAL CHANNEL SUBSIDY

This section deals with a channel setting in which the manufacturer provides subsidy for the distributor's advertising, while the distributor provides subsidy for the retailer's local advertising effort. Thus, we have total channel subsidy provision in the sense that there is commitment by the two channel members to subsidize advertising. In this channel structure all hands are on deck to ensure the best possible performance as individuals and as a channel. This level of commitment may be seen in a setting in which the manufacturer trusts the first follow (the distributor), who in turn is faithful to his responsibility and commitment to the retailer. Thus, using (13) and (15) in (16) we have

$$f_R(x) = \frac{\beta^2 f_{Rx}(2f_{Dx}+f_{Rx}) + 2\beta^2 f_{Rx}(2f_{Mx}+f_{Dx})}{8\rho} + \left[\frac{8m_R - \beta^2 f_{Rx}(2f_{Dx}+f_{Rx})}{8\rho}\right]x + \left[\frac{-2\beta^2 f_{Rx}(2f_{Mx}+f_{Dx}) - 8\delta f_{Rx}}{8\rho}\right]x \quad (26)$$

$$\Longrightarrow \quad f_{Rx} = \frac{8m_R - \beta^2 f_{Rx}(2f_{Dx}+f_{Rx})}{8\rho} + \frac{-2\beta^2 f_{Rx}(2f_{Mx}+f_{Dx}) - 8\delta f_{Rx}}{8\rho}.$$

Thus

$$f_{Rx(\alpha_D>0,\,\alpha_M>0)} = \frac{-\left(4\beta^2(f_{Dx}+f_{Mx}) + 8(\delta+\rho)\right)}{2\beta^2} \pm \frac{\sqrt{\left(4\beta^2(f_{Dx}+f_{Mx}) + 8(\delta+\rho)\right)^2 + 32\beta^2 m_R}}{2\beta^2}.$$

Similarly, (12) becomes

$$f_D(x) = \frac{4\beta^2 f_{Dx}f_{Mx} + 6\beta^2 f_{Dx}^2 + \beta^2 f_{Rx}^2 + 4\beta^2 f_{Dx}f_{Rx}}{16\rho} + \left[\frac{16m_D - 4\beta^2 f_{Dx}f_{Mx} - 6\beta^2 f_{Dx}^2}{16\rho}\right]x + \left[\frac{-\beta^2 f_{Rx}^2 - 4\beta^2 f_{Dx}f_{Rx} - 16\delta f_{Dx}}{16\rho}\right]x \quad (27)$$

$$\Longrightarrow \quad f_{Dx} = \frac{16m_D - 4\beta^2 f_{Dx}f_{Mx} - 6\beta^2 f_{Dx}^2}{16\rho} + \frac{-\beta^2 f_{Rx}^2 - 4\beta^2 f_{Dx}f_{Rx} - 16\delta f_{Dx}}{16\rho}$$

Thus

$$f_{Dx(\alpha_D>0,\,\alpha_M>0)} = \frac{-\left(4\beta^2(f_{Mx}+f_{Rx}) + 16(\delta+\rho)\right)}{12\beta^2} \pm \frac{\sqrt{\left(4\beta^2(f_{Mx}+f_{Rx}) + 16(\delta+\rho)\right)^2 - 24\beta^2(\beta^2 f_{Rx}^2 - 16m_D)}}{12\beta^2}.$$

Also (14) becomes

$$f_M(x) = \frac{-4\beta^2 f_{Dx}f_{Mx}^2 + \beta^2 f_{Dx}^3 + 6\beta^2 f_{Mx}f_{Dx}}{8\rho} + \frac{2\beta^2 f_{Mx}f_{Rx} + 4\beta^2 f_{Mx}^2}{8\rho} + \left[\frac{8m_M + 4\beta^2 f_{Dx}f_{Mx}^2 - \beta^2 f_{Dx}^3 - 6\beta^2 f_{Mx}f_{Dx}}{8\rho}\right]x + \left[\frac{-2\beta^2 f_{Mx}f_{Rx} - 4\beta^2 f_{Mx}^2 - 8\delta f_{Mx}}{8\rho}\right]x, \quad (28)$$

$$\Longrightarrow \quad f_{Mx} = \frac{8m_M + 4\beta^2 f_{Dx}f_{Mx}^2 - \beta^2 f_{Dx}^3 - 6\beta^2 f_{Mx}f_{Dx}}{8\rho} + \frac{-2\beta^2 f_{Mx}f_{Rx} - 4\beta^2 f_{Mx}^2 - 8\delta f_{Mx}}{8\rho}.$$

Thus

$$f_{Mx(\alpha_D>0,\,\alpha_M>0)} = \frac{-W_1}{8\beta^2(1-f_{Dx})} \pm \frac{\sqrt{W_1 - W_2}}{8\beta^2(1-f_{Dx})},$$

$$\text{where } W_1 = 2\beta^2(3f_{Dx} + f_{Rx}) + 8(\delta + \rho),$$

$$W_2 = -16\beta^2(1-f_{Dx})(\beta^2 f_{Dx}^3 - 8m_M).$$

Hence:

Proposition IV Suppose that the distributor transfers the manufacturer's provided subsidy to the retailer, then the advertising efforts are given by

$$a_{R(\alpha_D>0,\,\alpha_M>0)}(x) = \frac{\beta\left(2f_{Dx(\alpha_D>0,\,\alpha_M>0)} + f_{Rx(\alpha_D>0,\,\alpha_M>0)}\right)}{4} \times \frac{\sqrt{1-x(t)}}{4}$$

and

$$a_{D(\alpha_D>0,\,\alpha_M>0)}(x) = \frac{\beta\left(2f_{Mx(\alpha_D>0,\,\alpha_M>0)} + f_{Dx(\alpha_D>0,\,\alpha_M>0)}\right)}{4} \times \frac{\sqrt{1-x(t)}}{4}$$

and the payoffs are given by (26), (27) and (28).

Now, considering the section on unsubsidised advertising we observe that f_{Rx}, f_{Dx} and f_{Mx} are very important to the players since they determine the rate of increase of their respective profits. Obviously, this applies to all the supply chain settings as can also be seen in the last three preceding sections Now for all-four settings discussed above we notice that f_{Rx}, f_{Dx} and f_{Mx} increase as the margins m_R, m_D and m_M increase respectively. We also note (as stated earlier) that if the firms are foresighted, that is, if ρ is very small, and the decay rate is low, then these rates are bound to be very high and will eventually lead to large profits. Based on these obtained rates we will compare individual player's profits and channel profits for unsubsidized advertising, partial subsidies, and total channel subsidy structures using awareness shares, advertising efforts and participation rates.

MARKET AWARENESS SHARE

Now, we adopt the following subscript representations for the game scenarios:

- $\alpha_D(x|\alpha_M) = \alpha_M = 0$: both the manufacturer and the distributor do not subsidize advertising;
- $\alpha_D(x|\alpha_M) = 0, \alpha_M > 0$: there is partial channel subsidy from the manufacturer;
- $\alpha_D(x|\alpha_M) > 0, \alpha_M = 0$: there is partial channel subsidy from the distributor;
- $\alpha_D(x|\alpha_M),\ \alpha_M > 0$: there is total channel subsidy from the manufacturer and the distributor.

For instance, the retailer's profit, the awareness share and the channel profit for the distributor's partial channel subsidy situation are expressed as $f_{R(\alpha_D>0,\,\alpha_M=0)}$, $x_{(\alpha_D>0,\,\alpha_M=0)}$, and $f_{(\alpha_D>0,\,\alpha_M=0)}$ respectively. Now, for $\alpha_D(x|\alpha_M) = \alpha_M = 0$, (1) can be expressed as

$$x'(t) = \beta\left[\frac{\beta f_{Rx}\sqrt{1-x}}{2} + \frac{\beta f_{Dx}\sqrt{1-x}}{2}\right]\sqrt{1-x} - \delta = \frac{\beta^2(f_{Rx}+f_{Dx})}{2} - \frac{\beta^2(f_{Rx}+f_{Dx})+\delta}{2}x.$$

Solving this differential equation we have

$$x_{(\alpha_D(x|\alpha_M)=\alpha_M=0)}(t) = \frac{\beta^2(f_{Rx}+f_{Dx})}{\beta^2(f_{Rx}+f_{Dx})+2\delta} + \frac{\{\beta^2(f_{Rx}+f_{Dx})+2\delta\}x_0 - \beta^2(f_{Rx}+f_{Dx})}{\beta^2(f_{Rx}+f_{Dx})+2\delta} \times \exp\left\{-\frac{\beta^2(f_{Rx}+f_{Dx})+2\delta}{2}t\right\}.$$

and the long-run market awareness share is given by

$$x_{(\alpha_D(x|\alpha_M)=\alpha_M=0)}(t) = \frac{\beta^2(f_{Rx}+f_{Dx})}{\beta^2(f_{Rx}+f_{Dx})+2\delta}.$$

Similar argument leads to the following:

For $\alpha_D(x|\alpha_M) = 0,\ \alpha_M > 0$:

$$x_{(\alpha_D(x|\alpha_M)=0,\,\alpha_M>0)}(t) = \frac{\beta^2(2(f_{Rx}+f_{Mx})+f_{Dx})}{\beta^2(2(f_{Rx}+f_{Mx})+f_{Dx})+4\delta} + \left[\frac{[\beta^2(2(f_{Rx}+f_{Mx})+f_{Dx})+4\delta]x_0}{\beta^2(2(f_{Rx}+f_{Mx})+f_{Dx})+4\delta} + \frac{-\beta^2(2(f_{Rx}+f_{Mx})+f_{Dx})}{\beta^2(2(f_{Rx}+f_{Mx})+f_{Dx})+4\delta}\right] \times \exp\left\{-\frac{\beta^2(2(f_{Rx}+f_{Mx})+f_{Dx})+4\delta}{4}t\right\},$$

with a long-run awareness

$$x_{(\alpha_D(x|\alpha_M)=0,\,\alpha_M>0)\infty}(t) = \frac{\beta^2(2(f_{Rx}+f_{Mx})+f_{Dx})}{\beta^2(2(f_{Rx}+f_{Mx})+f_{Dx})+4\delta}.$$

For $\alpha_D(x|\alpha_M) > 0, \alpha_M = 0$:

$$x_{(\alpha_D(x|\alpha_M)>0,\,\alpha_M=0)}(t) = \frac{\beta^2(4f_{Dx}+f_{Rx})}{\beta^2(4f_{Dx}+f_{Rx})+4\delta} + \frac{[\beta^2(4f_{Dx}+f_{Rx})+4\delta]x_0 - \beta^2(4f_{Dx}+f_{Rx})}{\beta^2(4f_{Dx}+f_{Rx})+4\delta} \times \exp\left\{-\frac{\beta^2(4f_{Dx}+f_{Rx})+4\delta}{4}t\right\},$$

with a long-run awareness

$$x_{(\alpha_D(x|\alpha_M)>0,\,\alpha_M=0)}(t) = \frac{\beta^2(4f_{Dx}+f_{Rx})}{\beta^2(4f_{Dx}+f_{Rx})+4\delta}.$$

For $\alpha_D(x|\alpha_M),\ \alpha_M > 0$:

$$x_{(\alpha_D(x|\alpha_M),\,\alpha_M>0)}(t) = \frac{\beta^2\big(2f_{Rx}f_{Dx}(f_{Rx}+3f_{Dx}+2f_{Mx})\big)}{\beta^2\big(2f_{Rx}f_{Dx}(f_{Rx}+3f_{Dx}+2f_{Mx})\big)+8f_{Dx}f_{Rx}\delta} + \left[\frac{\{\beta^2\big(2f_{Rx}f_{Dx}(f_{Rx}+3f_{Dx}+2f_{Mx})\big)+8f_{Dx}f_{Rx}\delta\}x_0}{\beta^2\big(2f_{Rx}f_{Dx}(f_{Rx}+3f_{Dx}+2f_{Mx})\big)+8f_{Dx}f_{Rx}\delta} + \frac{-\beta^2\big(2f_{Rx}f_{Dx}(f_{Rx}+3f_{Dx}+2f_{Mx})\big)}{\beta^2\big(2f_{Rx}f_{Dx}(f_{Rx}+3f_{Dx}+2f_{Mx})\big)+8f_{Dx}f_{Rx}\delta}\right] \times \exp\left\{-\frac{\beta^2\big(2f_{Rx}f_{Dx}(f_{Rx}+3f_{Dx}+2f_{Mx})\big)+8f_{Dx}f_{Rx}\delta}{8f_{Dx}f_{Rx}}\right\},$$

with a long-run awareness

$$x_{(\alpha_D(x|\alpha_M),\,\alpha_M>0)}(t) = \frac{\beta^2\big(2f_{Rx}f_{Dx}(f_{Rx}+3f_{Dx}+2f_{Mx})\big)}{\beta^2\big(2f_{Rx}f_{Dx}(f_{Rx}+3f_{Dx}+2f_{Mx})\big)+8f_{Dx}f_{Rx}\delta}.$$

SUBSIDY LIMITS

It is of managerial and economic importance to the players, especially the leader (or a preceding channel member) to know the subsidy level at which his profit would not be lower than that of the follower(s). This ensures that he is not shortchanged. Now, from (9) and (14), we find that to achieve $f_R \leq f_M$, we must have that

$$m_R x+\frac{\beta^2(f_{Rx})^2(1-x)}{4\left(1-\alpha_D(x|\alpha_M)\right)}+\frac{\beta^2 f_{Rx}f_{Dx}(1-x)}{2(1-\alpha_M)}-\delta f_{Rx}x \leq m_M x-\frac{\alpha_M\beta^2(f_{Dx})^2(1-x)}{4(1-\alpha_M)^2}+\frac{\beta^2 f_{Rx}f_{Mx}(1-x)}{2\left(1-\alpha_D(x|\alpha_M)\right)}+\frac{\beta^2 f_{Mx}f_{Dx}(1-x)}{2(1-\alpha_M)}-\delta f_{Mx}x \quad (29)$$
$$\Rightarrow \quad \alpha_D(x|\alpha_M)\leq\frac{4N+\beta^2 f_{Rx}^2 H-2\beta^2 f_{Rx}f_{Mx}H}{4N},$$

where $H=1-x$ and

$$N=(m_R-m_M-(f_{Rx}-f_{Mx})\delta)x+\frac{\beta^2 f_{Dx}(1-x)}{2(1-\alpha_M)}\left(f_{Rx}+\frac{\alpha_M f_{Dx}}{2(1-\alpha_M)}-f_{Mx}\right).$$

Also from (29) we have

$$(-\alpha_M f_{Dx}+2(1-\alpha_M)f_{Mx}-2(1-\alpha_M)f_{Rx})\beta^2 f_{Dx}H\leq 4(1-\alpha_M)^2 L \Rightarrow \quad \alpha_M\leq\frac{-B\pm\sqrt{4AC+B^2}}{2A},$$

where $A=-4L\alpha_M^2$,

$$B=\beta^2 f_{Dx}^2 H+2\beta^2 f_{Rx}f_{Dx}H-2\beta^2 f_{Dx}f_{Mx}H+8L, C=2\beta^2 f_{Dx}f_{Mx}H-2\beta^2 f_{Dx}f_{Rx}H-4L \quad \text{and}$$
$$L=(m_R-m_M-(f_{Rx}-f_{Mx})\delta)x+\frac{\beta^2 f_{Rx}^2 H}{4\left(1-\alpha_D(x|\alpha_M)\right)}-\frac{\beta^2 f_{Rx}f_{Mx}H}{2\left(1-\alpha_D(x|\alpha_M)\right)}.$$

Clearly, without numerical values it would be difficult or even impossible to ascertain the most appropriate expression for the value of α_M. Recall that $\alpha_M\in[0,1]$. Also considering (12) and (14) we find that to achieve $f_D\leq f_M$ we need to let

$$m_D x+\frac{\beta^2(f_{Dx})^2(1-x)}{4(1-\alpha_M)}-\frac{\alpha_D(x|\alpha_M)\beta^2(f_{Rx})^2(1-x)}{4\left(1-\alpha_D(x|\alpha_M)\right)^2}+\frac{\beta^2 f_{Rx}f_{Dx}(1-x)}{2\left(1-\alpha_D(x|\alpha_M)\right)}-\delta f_{Dx} \leq m_M x-\frac{\alpha_M\beta^2(f_{Dx})^2(1-x)}{4(1-\alpha_M)^2}+\frac{\beta^2 f_{Rx}f_{Mx}(1-x)}{2\left(1-\alpha_D(x|\alpha_M)\right)}+\frac{\beta^2 f_{Mx}f_{Dx}(1-x)}{2(1-\alpha_M)}-\delta f_{Mx}x \quad (30)$$
$$\Rightarrow \quad \alpha_D(x|\alpha_M)\leq\frac{-E\pm\sqrt{4DF+E^2}}{2D},$$

where $D=-4P$,

$$E=\beta^2 f_{Rx}^2 H+2\beta^2 f_{Dx}f_{Rx}H-2\beta^2 f_{Mx}f_{Rx}H+8P,\ F=2\beta^2 f_{Mx}f_{Rx}H-2\beta^2 f_{Dx}f_{Rx}H-4P \quad \text{and}$$
$$P=(m_D-m_M-(f_{Dx}-f_{Mx})\delta)x+\frac{\beta^2 f_{Dx}(1-x)}{2(1-\alpha_M)}\times\left(\frac{f_{Dx}}{2}+\frac{\alpha_M f_{Dx}}{2(1-\alpha_M)}-f_{Mx}\right).$$

Also, from (30) we have

$$\beta^2 f_{Dx}H[\alpha_M f_{Dx}+2(1-\alpha_M)f_{Mx}-(1-\alpha_M)f_{Dx}]\leq 4(1-\alpha_M)^2 K \Rightarrow \quad \alpha_M\leq\frac{-I\pm\sqrt{4GJ+I^2}}{2G},$$

where $G=-4K$,

$$I=-2\beta^2 f_{Dx}f_{Mx}H+2\beta^2 f_{Dx}^2 H+8K,\ J=-\beta^2 f_{Dx}^2 H+2\beta^2 f_{Dx}f_{Mx}H-4K \quad \text{and}$$
$$K=(m_D-m_M-(f_{Dx}-f_{Mx})\delta)x+\frac{\beta^2 f_{Rx}(1-x)}{2\left(1-\alpha_D(x|\alpha_M)\right)}\times\frac{\alpha_D f_{Rx}-2\left(1-\alpha_D(x|\alpha_M)\right)(f_{Dx}+f_{Mx})}{2\left(1-\alpha_D(x|\alpha_M)\right)}.$$

As observed earlier, an appropriate expression for α_M can be only determined by using numerical values. Further, from (9) and (12) we see that to achieve $f_R\leq f_D$, we need to let

$$m_R x + \frac{\beta^2 (f_{Rx})^2 (1-x)}{4\big(1-\alpha_D(x|\alpha_M)\big)} + \frac{\beta^2 f_{Rx} f_{Dx}(1-x)}{2(1-\alpha_M)} - \delta f_{Rx} x$$
$$\leq m_D x + \frac{\beta^2 (f_{Dx})^2 (1-x)}{4(1-\alpha_M)} - \frac{\alpha_D(x|\alpha_M)\beta^2 (f_{Rx})^2 (1-x)}{4\big(1-\alpha_D(x|\alpha_M)\big)^2} + \frac{\beta^2 f_{Rx} f_{Dx}(1-x)}{2\big(1-\alpha_D(x|\alpha_M)\big)} - \delta f_{Dx} x \Longrightarrow \quad \alpha_D(x|\alpha_M)$$
$$\leq \frac{-S \pm \sqrt{4RT + S^2}}{2R},$$

where

$$Q = (m_R - m_D - (f_{Rx} - f_{Dx})\delta)x + \frac{\beta^2 f_{Dx}(1-x)}{2(1-\alpha_M)}\left(f_{Rx} - \frac{f_{Dx}}{2}\right),$$
$$R = -4Q,$$
$$S = -2\beta^2 f_{Rx} f_{Dx} H + 8Q \quad \text{and}$$
$$T = -\beta^2 f_{Rx}^2 H + 2\beta^2 f_{Rx} f_{Dx} H - 4Q.$$

Again, just as was stated earlier, the appropriate value of $\alpha_D(x|\alpha_M)$ can only be determined using numerical values. Also, from (31) we have

$$\beta^2 f_{Dx} H(f_{Dx} - 2f_{Rx}) \leq 4(1-\alpha_M)M \qquad \Longrightarrow \alpha_M \leq \frac{\beta^2 f_{Rx} f_{Dx} H - \beta^2 f_{Dx}^2 H + 4M}{4M},$$

where

$$M = (m_R - m_D - (f_{Rx} - f_{Dx})\delta)x + \frac{\beta^2 f_{Rx} H}{2\big(1-\alpha_D(x|\alpha_M)\big)} \times \left(\frac{M_1 - 2\big(1-\alpha_D(x|\alpha_M)\big)f_{Rx}}{2\big(1-\alpha_D(x|\alpha_M)\big)}\right),$$
$$M_1 = \big(1-\alpha_D(x|\alpha_M)\big)f_{Rx} + \alpha_D(x|\alpha_M) f_{Rx}.$$

Numerical Discussion

I. Parameter Values

As stated earlier, the advertising effectiveness parameter $\beta \in [0,1]$, thus, we let $\beta = 0.3$. Also, since the game is on an infinite horizon and the players are considered to be foresighted, we let $\rho = 0.05$. Further, we see that $\beta > \delta$ since the reverse would imply a negative effect. Thus, we let $\delta = 0.1$. The players are engaged in a Stackelberg game with the manufacturer as the leader (and so, he is the prime mover). The distributor is the first follower, and the retailer is the last follower. Thus we have $m_M > m_D > m_R$. As such, we let $m_M = 0.4$, $m_D = 0.3$, $m_R = 0.2$. For simplicity we will write $\alpha_D = \alpha_D(x|\alpha_M)$.

II. The Effect of Subsidies on the Awareness Share

Considering Figure 1 we notice that

$$x_{(\alpha_D,\alpha_M>0)} > x_{(\alpha_D=0,\alpha_M>0)} > x_{(\alpha_D>0,\alpha_M=0)} > x_{(\alpha_D=0,\alpha_M=0)} \; \forall t \in [0,\infty).$$

The awareness share for each channel setting is an increasing function of t which eventually stabilizes in the long run as seen earlier. This implies that the effect of advertising on the awareness is not unlimited. Thus, the supply chain members should be concerned with the determination of the long-run values, and its equivalent advertising spending. They should avoid the emptation of increasing advertising spending beyond this level in the hope of higher awareness levels. temptation of increasing advertising this level in the hope of higher awareness levels.

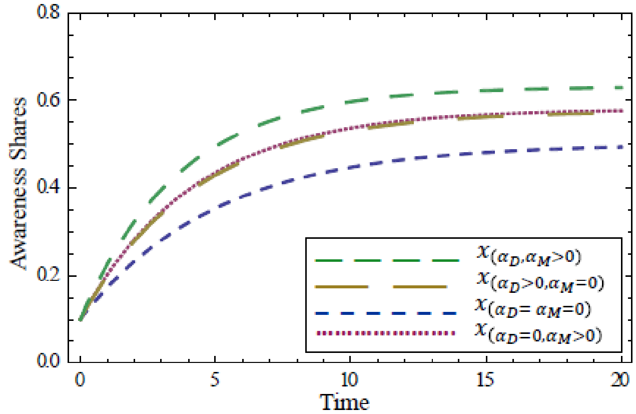

FIGURE 1
A COMPARISON OF THE EFFECT OF SUBSIDIES ON THE AWARENESS SHARE OVER TIME

III. Effect of the Awareness on the Profits

A near-concordance between awareness shares and profits can be seen in Figure 2 and Figure 3 where

$$f_{R(\alpha_D,\alpha_M>0)} > f_{R(\alpha_D>0,\alpha_M=0)} > f_{R(\alpha_D=0,\alpha_M>0)} = f_{R(\alpha_D=0,\alpha_M=0)}$$

and

$$f_{D(\alpha_D,\alpha_M>0)} > f_{D(\alpha_D>0,\alpha_M=0)} > f_{D(\alpha_D=0,\alpha_M>0)} = f_{D(\alpha_D=\alpha_M=0)}$$

respectively.

Clearly, from Figure 2 and Figure 3 we observe that the distributor and the retailer's profits for all scenarios increase with the awareness and are largest with the transfer of subsidy. That is, they perform best with subsidy transfer. Their next best performances are in the intervention scenario. Their performances are worst with the withholding of subsidy. Thus it is in the best interest of the distributor to transfer the subsidy to the retailer. Thus we suggest that where necessary or possible the manufacturer who is the channel leader should condition the distributor to transfer the subsidy, and where none is provided by the manufacturer, then he should be constrained to provide intervention subsidy.

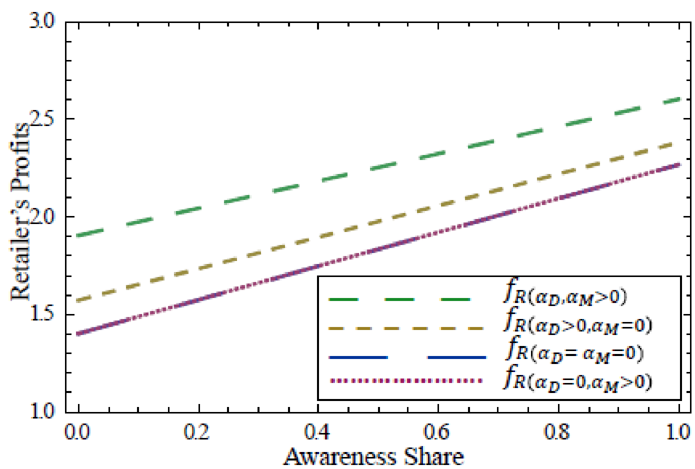

FIGURE 2
EFFECT OF AWARENESS SHARE ON RETAILER'S PROFIT

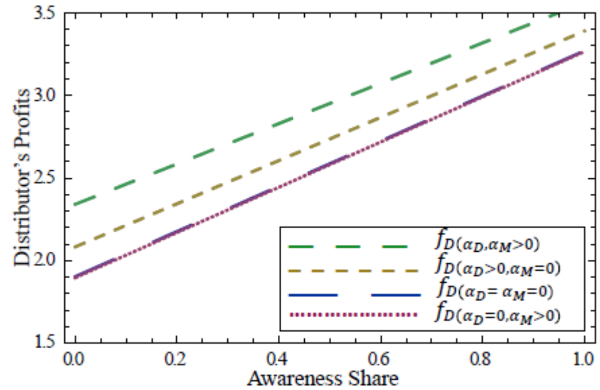

FIGURE 3
EFFECT OF AWARENESS SHARE ON DISTRIBUTOR'S PROFIT

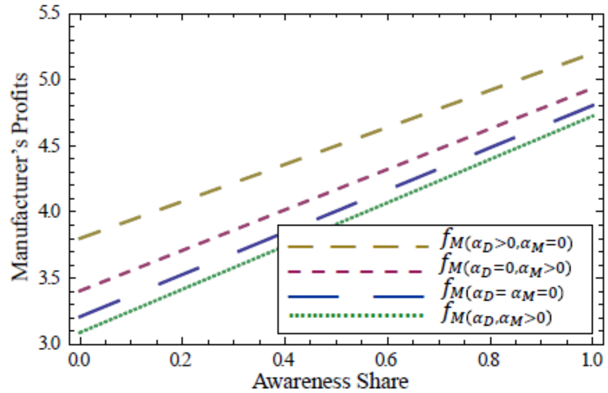

FIGURE 4
EFFECT OF AWARENESS SHARE ON MANUFACTURER'S PROFIT

Again we observe that the manufacturer's profit increases with awareness for all scenarios. However, unlike the cases in Figure 2 and Figure 3 it is clear that the manufacturer's profit as shown in Figure 4 is best with the adoption of the intervention subsidy scenario. This is, followed by the manufacturer's partial subsidy scenario, no-subsidy scenario, and worst with total subsidy provision. Obviously, the manufacturer will prefer intervention to total subsidy. Of course, this is true because with intervention he is relieved of subsidy expenditure. Thus, he may incentivize or where necessary constrain the distributor to provide subsidy for retail advertising. Just as the players' profits, the channel profit increases with awareness. We note that the largeness of the awareness share resulting from the manufacturer's partial channel subsidy and total channel subsidy does not reflect in the entire channel profit as can be seen in Figure 5 where

$$f_{(\alpha_D>0,\alpha_M=0)} > f_{(\alpha_D,\alpha_M>0)} > f_{(\alpha_D=0,\alpha_M>0)} > f_{(\alpha_D=\alpha_M=0)}.$$

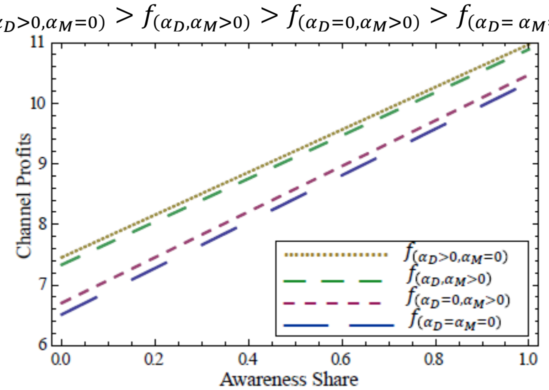

FIGURE 5
EFFECT OF AWARENESS SHARE ON CHANNEL PROFIT

Clearly, the entire channel profit is largest with the distributor's partial channel subsidy (intervention subsidy) against much effort leading to large awareness share from total channel subsidy or manufacturer's partial subsidy. Since the profit from the intervention subsidy is largest, it would be irrational for the manufacturer together with the distributor to subsidize advertising, or for the manufacturer's subsidy provision to be withheld (that is, for the distributor not to extend the manufacturer's subsidy gesture to the retailer) if the distributor can provide intervention subsidy. Thus from the indications in Figure 5 it would be appropriate for the channel members to adopt intervention instead of the other scenarios since the channel profit is larger even with lower awareness. It therefore remains for the players to decide on a sharing formula for the channel profit.

IV. Effect of Retail and Distributor's Advertising Efforts on the Profits

Figure 6 shows that increase in retailer's effort leads to slight increase in his profit, which then plunges badly to zero for all scenarios, with

$$f_{R(\alpha_D = \alpha_M = 0)} > f_{R(\alpha_D > 0, \alpha_M = 0)} > f_{R(\alpha_D = 0, \alpha_M > 0)} > f_{R(\alpha_D, \alpha_M > 0)}$$

for small values of the distributor's effort; while the retailer's profit increases continuously with the distributor's effort for all scenarios. The plunge suggests that the retailer's derivable benefit from his advertising effort is limited. This implies that increasing the retail effort should be done with caution. This is unlike the benefits he (the retailer) enjoys from the distributor's effort. The retailer's profit increases continuously with it (distributor's advertising effort) for all scenarios with

$$f_{R(\alpha_D = \alpha_M = 0)} > f_{R(\alpha_D > 0, \alpha_M = 0)} > f_{R(\alpha_D = 0, \alpha_M > 0)} > f_{R(\alpha_D, \alpha_M > 0)}$$

for small values of the distributor's advertising effort, and the reverse being the case for large values of the effort.

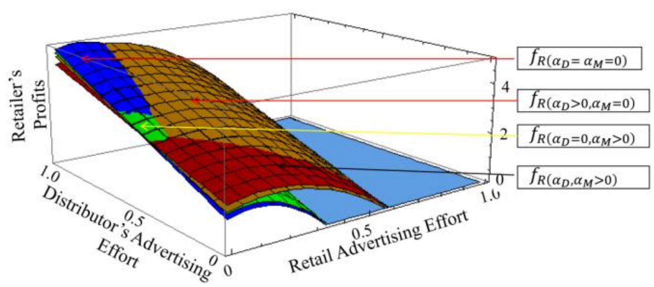

FIGURE 6
EFFECT OF THE DISTRIBUTOR'S ADVERTISING EFFORT TOGETHER WITH THE RETAILER'S

In essence, the retailer benefits a lot from the distributor's effort if he (the retailer) does not engage in advertising the product. However, if he is engaged, then he (the retailer) should major on his optimal effort rather than large effort in the hope of large profit, which cannot be achieved because of the profit plunge to zero for large retail effort. Figure 7 shows that increase in the distributor's effort leads to increase in his profit leading to a maximum. Thereafter, the profit exhibits reductions that are lower than the initial values as the distributor's effort increases. This is the situation for all scenarios. Further, the

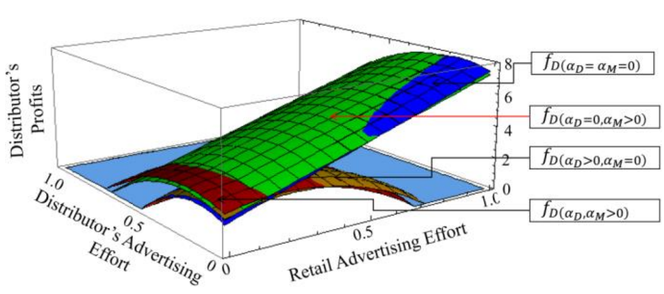

FIGURE 7
EFFECT OF THE DISTRIBUTOR'S ADVERTISING EFFORT TOGETHER WITH THE RETAILER'S ADVERTISING EFFORT ON THE DISTRIBUTOR'S PROFIT FOR ALL SCENARIOS

distributor's profit exhibits similar behavior with increasing retail effort for both intervention and total channel subsidy scenarios, but continuously increases with retail effort for both non-provision of subsidy and manufacturer's partial subsidy scenarios. The implication is that while the distributor benefits more with increasing retail effort, if he does not subsidize such effort, he benefits less, or even run at a loss if he participates either through intervention subsidy or total subsidy. On the other hand, the distributor eventually benefits less or even end up with a loss for large distributor's effort. Thus, the distributor should increase effort with caution, and adopt his optimal strategy regarding the channel adopted scenario to ensure optimal profit. Figure 8 shows that for all scenarios the manufacturer's profit increases continuously with the retail effort, with

$$f_{M(\alpha_D,\alpha_M>0)} > f_{M(\alpha_D>0,\alpha_M=0)} > f_{M(\alpha_D=0,\alpha_M>0)} > f_{M(\alpha_D=\ \alpha_M=0)}$$

for small values of effort. That is, the manufacturer benefits maximally from the retail effort for all scenarios, unlike the effect of the distributor's effort on his profit, where it increases continuously for non-provision of subsidy and intervention subsidy settings, but reduces for large values of the distributor's effort for the manufacturer's partial channel subsidy and total channel subsidy. Thus, the manufacturer should ensure that the retailer advertises the product irrespective of the game scenario, while encouraging the distributor to advertise for non-provision of subsidy and intervention scenario. In essence, from the manufacturer's perspective, he should not provide subsidy, and if at all there is going to be provision of subsidy, it should be distributor's intervention subsidy, but not total subsidy by both manufacturer and distributor or non-transfer.

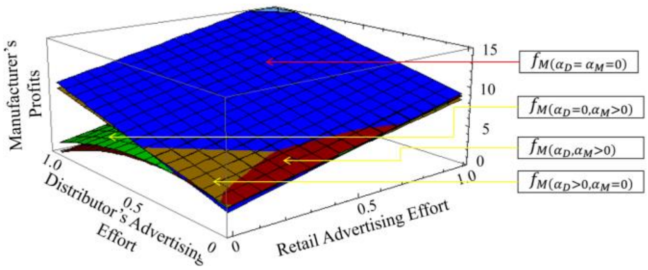

FIGURE 8
EFFECT OF THE DISTRIBUTOR'S ADVERTISING EFFORT TOGETHER WITH THE RETAILER'S ADVERTISING EFFORT ON THE MANUFACTURER'S PROFIT FOR ALL SCENARIOS

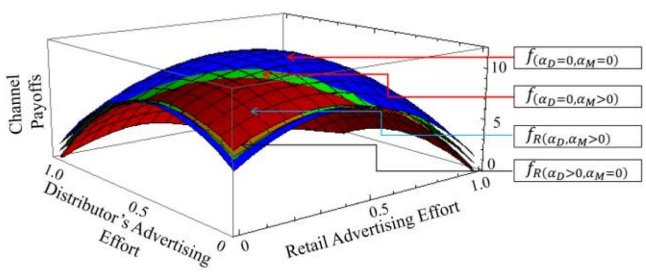

FIGURE 9
EFFECT OF THE DISTRIBUTOR'S ADVERTISING EFFORT TOGETHER WITH THE RETAILER'S ADVERTISING EFFORT ON THE CHANNEL PROFIT FOR ALL SCENARIOS

Considering Figure 9 we notice that the plots for both non-provision and total channel subsidy are well pronounced, while the plots for the manufacturer's partial subsidy and distributor's subsidy are less pronounced. Clearly with low efforts, the entire channel profit $f_{(\alpha_D,\alpha_M>0)}$ is the largest profit, while $f_{(\alpha_D=\alpha_M=0)}$ is the smallest. However, with large efforts, the reverse is the case! In general, for small efforts,

$$f_{(\alpha_D=0,\alpha_M=0)} < f_{(\alpha_D=0,\alpha_M>0)} < f_{(\alpha_D>0,\alpha_M=0)} < f_{(\alpha_D,\alpha_M>0)}. \quad (32)$$

On the other hand, for large efforts

$$f_{(\alpha_D=0,\alpha_M=0)} > f_{(\alpha_D=0,\alpha_M>0)} > f_{(\alpha_D>0,\alpha_M=0)} > f_{(\alpha_D,\alpha_M>0)}. \quad (33)$$

The expression in (32) implies that with subsidy transfer, only a small amount of advertising efforts are needed to achieve large channel profit. This implies that it is in the interest the players that the subsidy gesture by the manufacturer is extended to the retailer if they intend to employ only a small amount of effort. The manufacturer should make policies that will discourage the distributor from being selfish or nonchalant with the manufacturer's subsidy provision. However, in the event that the manufacturer is unable to provide subsidy, considering the benefit of subsidy to the channel, the distributor should intervene being that they want to employ only small amount of effort. On the other hand, (33) shows that with large efforts, subsidy can be overlooked. That is, the channel may not bother about provision or total channel subsidy if large advertising efforts are employed. Thus, considering channel profit we notice that it is possible to switch between provision of subsidy and advertising effort, and that large subsidy is not in tandem with large advertising effort.

V. Effect of Subsidy on Profits

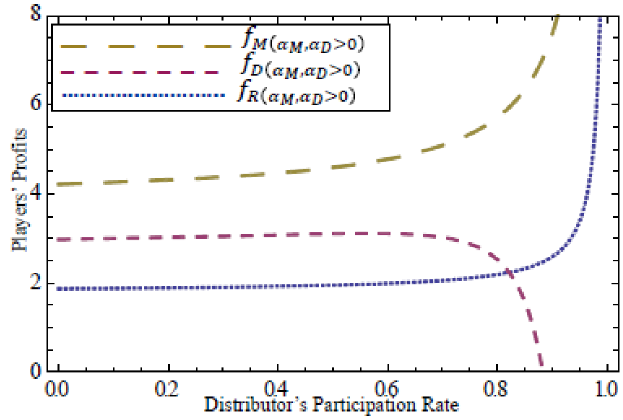

FIGURE 10
EFFECT OF THE DISTRIBUTOR'S SUBSIDY RATE ON PROFITS

Based on our chosen parameter values, the long-run value of the awareness $x_{(\alpha_D,\alpha_M>0)} = 0.6341$. Using this value we observe from Figure 10 that the manufacturer's profit and the retailer's profit increase continuously with the distributor's subsidy, but increases only slightly with the distributor's profit attaining its maximum 3.1050 at $\alpha_D = 0.5639$. Thereafter it exhibits decrease with his (distributor's) own participation, suggesting that irrespective of his level of participation, he has limited benefit. It is very important that he does not incur much loses as a result of his subsidy. Now, since the manufacturer is the Stackelberg leader, the distributor can accept a smaller profit in comparison to the manufacturer. However, being the first follower, the distributor may not naturally accept a lower profit in comparison to the retailer. Thus his profit is expected to be larger than that of the retailer. This is achievable if the distributor's participation rate is less than 0.8215. Any subsidy rate above this level will make the distributor's profit lower than the other players' profits. Thus it is the distributor's optimal (acceptable) subsidy limit. At this subsidy level, the distributor's profit is $f_D = 2.2317$ which his minimum acceptable profit.

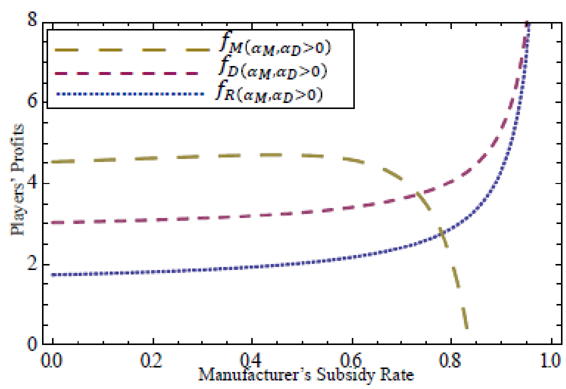

FIGURE 11
EFFECT OF THE MANUFACTURER'S SUBSIDY RATE ON THE PROFITS

Figure 11 shows that the distributor and retailer's profit increases continuously for all values of the manufacturer's subsidy. The manufacturer's profit also increases with his subsidy, attaining a maximum $f_M = 4.6984$ at $\alpha_M = 0.4470$. This implies that the manufacturer is better-off at this subsidy level. After this point there is a continuous decline in his profit. Recall that this is a Stackelberg game, thus it is expected that $f_M > f_D \geq f_R$. To ensure $f_M > f_D$ the manufacturer's subsidy rate must be less than 0.7335. At this level of participation $f_M = 3.7290$. Any subsidy rate above this will lead to $f_M < f_D$, which will lead to shortchanging the manufacturer. As such, this value is the manufacturer's acceptable minimum profit.

PRACTICAL IMPLICATION

It is natural to assume that that the players' ultimate goals are to make large profits. Thus, it is advisable for the players to adopt the distributor's partial subsidy instead of total subsidy since the players and channel perform best with less worry on awareness level in this scenario. Clearly, a player's profit reduces with his participation rate. Thus, a player especially the manufacturer and the distributor should provide subsidy with caution. In short, participation should not exceed the optimal level. Further, since the channel members are at different hierarchical levels, and the manufacturer would not naturally accept a lower profit in comparison to the distributor, neither would the distributor be at home with a lower profit in comparison to the retailer, it would be expedient for the two top hierarchies not to exceed their subsidy limits.

Considering the possible impact of both followers effort we note that this model setting is applicable in a situation where the market is saturated with substitutes of a new product, which may affect its patronage. Further, the benefits of such a model and its impact can be harnessed in a situation where the existence of a product brand is under threat of extinction. This model is an answer to the financial limitations of the followers in a channel. It is useful when neither of the followers, especially the retailer can effectively finance his advertising effort. Unlike the manufacturer, we note that the distributor is better informed about the regional environment, while the retailer is more acquainted with local terrain. This necessitates their direct involvement in advertising with support from the manufacturer. This model is an answer to this setting.

CONCLUDING REMARKS

This work considered a manufacturer-distributor-retailer cooperative advertising supply chain. It used infinite horizon Stackelberg differential game for the first time to model a form of subsidy transfer in which both followers directly advertise the product. The manufacturer subsidizes the distributor's advertising effort, while the distributor subsidizes the retailer's local advertising effort. We obtained the game equilibrium using backward induction by solving the retailer's problem, from which we solve the distributor's problem, and then the manufacturer's problem. We considered four supply chain settings which include: a situation in which there is no subsidy provision; a situation in which only the manufacturer subsidizes advertising; a situation in which only the distributor subsidizes advertising; and a situation in which the manufacturer and the distributor subsidizes advertising. We obtained the optimal advertising policies, subsidy policies, awareness share and profits for the four situations.

The work shows that the awareness share is largest with total subsidy, and least with non-provision of subsidy, but the channel profit is largest with distributor's partial channel subsidy. Since the players' interests are large profits, it is rational to opt for distributor's partial subsidy rather than total subsidy. Also considering the profits, we notice that while the retailer and the distributor are in favor of total subsidy, the manufacturer is at home with the distributor's partial subsidy. Thus with the distributor's partial subsidy the players and channel perform best with less worry on awareness level. This work dealt with a setting in which the manufacturer is the Stackelberg leader. A modification of the work can use the same channel setting to study a situation in which apart from the manufacturer, any of the other players is the Stackelberg leader, or a situation in which the parties play a Nash game without a particular channel leader. Another extension can study a situation where a number of retailers, distributors or manufacturers are in a competition. This can lead to a better grasp of the idea of cooperative advertising.

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