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IRANIAN JOURNAL OF OPTIMIZATION

Volume 5, issue 1,winter 2013,405-411
Research Paper

Islamic Azad University
Rasht Branch

Adomian Decomposition Method On Nonlinear Singular Cauchy Problem of Euler-Poisson-Darbuox equation

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Abstract

In this paper, we apply Picard's Iteration Method followed by Adomian Decomposition Method to solve a nonlinear Singular Cauchy Problem of Euler-Poisson- Darboux Equation. The solution of the problem is much simplified and shorter to arriving at the solution as compared to the technique applied by Carroll and Showalter (1976)in the solution of Singular Cauchy Problem.

Keywords: Adomian decomposition method, Singular Cauchy problem.

1.ntroduction

The Singular Cauchy Problem

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t}=\Delta u;\ in\ \mathbb{R}^m\times(x,0)\\ &u(x,0)=f(x);u_t(x,0)=0\ on\ \mathbb{R}^m\times\{t=0\}\end{aligned}\right\} \qquad (1)$$

has been studied since the time of Euler(1770).

Some classical results obtained are summarized as follows:

(i). If $k = m-1$, the solution as given by Arsgeirsson(1936) is

$$u(x,t) = \frac{1}{\omega}\int_{\sum_1^m \alpha_i^2 = 1} \cdots \int f(x+\alpha t)\, d\omega_m \qquad (2)$$

where $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_m$; $\omega_m = \dfrac{2\pi^{m/2}}{\Gamma(m/2)}$ is the surface area of the $m-$dimensional unit sphere.

(ii). If $k > m-1$, the solution obtained by Weinstein (1952) is

$$u(x,t) = \frac{\omega_{k+1-m}}{\omega_{k+1}}\int_{\sum_1^m \alpha_i^2 \le 1} \cdots \int f(x+\alpha t)\left(1-\alpha^2\right)^{(k-1-m)/2} d\alpha$$

(3)

where $d\alpha = d\alpha_1 d\alpha_2 \cdots d\alpha_m$

(iii). If $k < m-1$ but not $k \neq -1,-3,-5,\cdots$; Weinstein (1954) improving on his results of (1952) obtained the solution

$$u(x,t) = t^{1-t}\left(\frac{\partial}{t\partial}\right)^n \left[t^{k+2n-1} u^{(k+2n)}(x,t)\right] \qquad (4)$$

where n is a positive integer chosen such that $k+2n \ge m-1$ and $u^{(k+2n)}$ is given by (2) or (3) with f replaced by $f/(k+1)(k+2)\cdots(k+2n-1)$.

The solution of the singular Cauchy problem is unique for $k \ge 0$ whereas for $k < 0$ it is not unique as indicated in the work of Weinstein (1952).

Carroll and Showalter (1976) dealt primarily with the Cauchy problem for singular and degenerate equation of the form

$$A(t)u_{tt} + B(t)u_t + C(t)u = g \qquad (5)$$

where $u(\cdot)$ is a function of t, taking values in a separated locally convex space E, while $A(t), B(t),$ and $C(t)$ are families of linear or nonlinear differential type operators acting in E, some of which become zero or infinite at $t = 0$. They considered appropriate initial data $u(0)$ and $u_t(0)$ at $t = 0$, and g a suitable E-valued function.

2.The Adomian Decomposition Method to Singular Cauchy Problem

The purpose of this paper is to apply Adomian Decomposition method to solve the problem (6).

$$\left.\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2}+\dfrac{k}{t}\dfrac{\partial u}{\partial t}-\Delta u=u^p;\ in\ \mathbb{R}^n\times(x,0) \\ u(x,0)=f(x);\ u_t(x,0)=0\ on\ \mathbb{R}^n\times\{t=0\} \end{array}\right\} \qquad (6)$$

where $p>0, k-$ a parameter and $f(x)$ is smooth with compact support.

On taking the Fourier transform of (6) with respect to $x=(x_1,x_2,\cdots,x_n)$;

$$\left.\begin{array}{l} \dfrac{d^2\bar{u}}{dt^2}+\dfrac{k}{t}\dfrac{d\bar{u}}{dt}+|\xi|^2=\bar{u}^p;\ in\ \mathbb{R}^n\times(\xi,0) \\ \bar{u}(\xi,0)=\bar{f}(\xi);\ \bar{u}_t(\xi,0)=0\ on\ \mathbb{R}^n\times\{t=0\} \end{array}\right\} \qquad (7)$$

Then $$t^{1-t}\frac{d}{dt}\left(t^k\frac{d\bar{u}}{dt}\right)+|\xi|^2\bar{u}=\bar{u}^p$$

or

$$L\bar{u}+|\xi|^2\bar{u}=\bar{u}^p;\ L=t^{1-t}\frac{d}{dt}\left(t^k\frac{d}{dt}\right) \qquad (8)$$

Then $$\bar{u}+L^{-1}\left\{|\xi|^2\bar{u}\right\}=L^{-1}\left\{\bar{u}^p\right\};\quad L^{-1}=\int_0^t t^{-k}\int_0^t t^{k-1}[\cdot]dtdt \qquad (9)$$

Solving the homogeneous part of (8) by Picards iteration method with $\bar{u}_0=\bar{f}(\xi)$;

$$\bar{u}_{n+1}=\int_0^t t^{-k}\int_0^t t^{k-1}\left[-|\xi|^2\bar{u}_n\right]dtdt$$

So that we get

$$\bar{u}_1=-|\xi|^2\bar{f}.\frac{t}{k};\quad \bar{u}_2=-|\xi|^4\bar{f}.\frac{t^2}{2!k(k+1)};\quad \bar{u}_3=-|\xi|^6\bar{f}.\frac{t^3}{3!k(k+1)(k+2)};\quad \cdots$$

Therefore

$$\bar{u}(\xi,t)=\bar{f}\left\{1-|\xi|^2\frac{t}{k}+\frac{|\xi|^4}{k(k+1)}.\frac{t^2}{2!}-\frac{|\xi|^6}{k(k+1)(k+2)}.\frac{t^2}{3!}\cdots+(-1)^n\frac{|\xi|^{2n}}{k(k+1)\cdots(k+n-1)}.\frac{t^n}{n!}+\cdots\right\}$$

On taking $k=2m+1$, we get

$$\bar{u}(\xi,t)=\bar{f}(\xi)\cdot 2^m\Gamma(m+1)\cdot\left(\xi\sqrt{t}\right)^{-m}J_m\left(\xi\sqrt{t}\right)$$

$$=\bar{f}(\xi)\Gamma(m+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\xi\sqrt{t}}{2}\right)^{2n}\cdot\frac{1}{\Gamma(m+n+1)}$$

(10)

The nonlinear part is done by the Adomian Decomposition Method, Adomian(1994), where

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^{\infty}\lambda^i\bar{u}_i\right)\right]_{\lambda=0};\quad n=0,1,\cdots \qquad (11)$$

are Adomian polynomials with λ a parameter introduced by convenience. Then Adomian suggests the inductive scheme:

$$\bar{u}_0=\bar{f},\bar{u}_1=A_0,\cdots,\bar{u}_n=A_{n-1} \qquad (12)$$

Thus from (9), we have

$$\bar{u}_{n+1}(t)=\int_0^t A_n\left(\bar{u}_0(s),\bar{u}_1(s),\cdots,\bar{u}_n(s)\right)ds,\ n=0,1,2,\cdots$$

(13)

So we have

$$A_0=\bar{u}_0^p$$
$$A_1=p\bar{u}_0^{p-1}\bar{u}_1$$
$$A_2=\frac{1}{2!}p(p-1)\bar{u}_0^{p-2}\bar{u}_1^2+p\bar{u}_0^{p-1}\bar{u}_2$$
$$A_2=\frac{1}{3!}p(p-1)(p-2)\bar{u}_0^{p-3}\bar{u}_1^3+p(p-1)\bar{u}_0^{p-2}\bar{u}_1\bar{u}_2+p\bar{u}_0^{p-1}\bar{u}_3$$
……

Using (13), we determine few terms of the Adomian series:

$$\bar{u}_0(t) = \bar{f}^{\,p}(\xi)$$
$$\bar{u}_1(t) = \bar{f}^{\,p}(\xi)\cdot t$$
$$\bar{u}_2(t) = p\bar{f}^{\,2p-1}(\xi)\cdot\frac{t^2}{2!}$$
$$\bar{u}_3(t) = \left[p(p-1)\,\bar{f}^{\,3p-1}(\xi) + p^2\,\bar{f}^{\,3p-2}(\xi)\right]\cdot\frac{t^3}{3!}$$
$$\bar{u}_4(t) = \left[p\left(2p^2-1\right)\bar{f}^{\,4p-3}(\xi) + p^2(p-1)\,\bar{f}^{\,4p-1}(\xi)\right]\cdot\frac{t^4}{4!}$$
.....

Thus for nonlinear part, we have

$$\bar{u}(t) = \bar{f}^{\,p}(\xi) + \bar{f}^{\,p}(\xi)\cdot t + p\bar{f}^{\,2p-1}(\xi)\cdot\frac{t^2}{2!} + \left[p(p-1)\,\bar{f}^{\,3p-1}(\xi) + p^2\,\bar{f}^{\,3p-2}(\xi)\right]\cdot\frac{t^3}{3!}$$
$$+\left[p\left(2p^2-1\right)\bar{f}^{\,4p-3}(\xi) + p^2(p-1)\,\bar{f}^{\,4p-1}(\xi)\right]\cdot\frac{t^4}{4!} + \ldots \qquad (14)$$

Using the equations (10) and (14); we write

$$\bar{u}(\xi,t) = \bar{u}_l(\xi,t) + \bar{u}_n(\xi,t) \qquad (15)$$

Therefore

$$u(x,t) = f * \bar{u}_l(\xi,t) + f * \bar{u}_n(\xi,t) \qquad (16)$$

where $\bar{u}_l(\xi,t)$ and $\bar{u}_n(\xi,t)$ are given in equations (10) and (14) respectively.

For simplicity we let $f(x)=\delta$ the delta-Dirac function. Then, $\bar{f}(\xi)=1$. The second term on right hand sides of (15) becomes

$$\bar{u}(t) = 1 + t + p\cdot\frac{t^2}{2!} + p(2p-1)\cdot\frac{t^3}{3!} + p\left(6p^2-7p+2\right)\cdot\frac{t^4}{4!} + O\left(t^5\right)$$
$$= \frac{1}{\left[1-(p-1)t\right]^{1/(p-1)}} \qquad (17)$$

Then

$$u_n(\xi,t) = \int_{\square_n}\cdot 1\cdot d\xi \times \frac{1}{\left[1-(p-1)t\right]^{1/(p-1)}} = \frac{\delta}{\left[1-(p-1)t\right]^{1/(p-1)}}$$

The first term in (15) then will be

$$\bar{u}_l(\xi,t)=\Gamma(m+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\xi\sqrt{t}}{2}\right)^{2n}\cdot\frac{1}{\Gamma(m+n+1)}=2^2\Gamma(m+1)\left(\xi\sqrt{t}\right)^{-m}J_m\left(\xi\sqrt{t}\right)$$

and

$$u_l(x,t)=\int_{\square^n}e^{i\xi x}\cdot\left(\frac{\xi\sqrt{t}}{2}\right)^{-m}J_m\left(\xi\sqrt{t}\right)d\xi\cdot\Gamma(m+1)$$

$$=\int_{\square^n}e^{i\xi x}\left\{\left(\frac{\xi\sqrt{t}}{2}\right)^{-m}\cdot\frac{1}{\sqrt{\pi}}\int_0^{\pi}\cos\left(\xi\sqrt{t}\sin\theta d\theta\right)\right\}d\xi\cdot\Gamma(m+1)\quad(18)$$

Now (18) is not easy to calculate directly since $\bar{u}_l(\xi,t)$ is not an L^1 function. Here we shall consider the use of a cutoff function $e^{-\varepsilon^2\xi^2/2};\varepsilon>0$ (Folland1976) so that we have

$$\bar{u}_l^{\varepsilon}(\xi)=e^{-\varepsilon^2\xi^2/2}\bar{u}(\xi)_l.$$

Clearly $\bar{u}_l^{\varepsilon}(\xi)\to\bar{u}(\xi)_l$.uniformly as $\varepsilon\to 0$, so that $u_t(x,t)$ will be the limit in the topology of tempered distributions of u_l^{ε}, the inverse Fourier transform of $\bar{u}_l^{\varepsilon}$. Moreover, $\bar{u}_l^{\varepsilon}(\xi)\in L^1(\square^n)$, so we can calculate its inverse Fourier transform as an ordinary integral.

3.Conclusion

On comparison of solution of homogeneous part the equation (2)-(3) with that used by Carroll (1951), this is far much simpler for we do not involve transformations which are not easy to identify. Thus Adomian decomposition method offers quite a simpler means of solving nonlinear differential equations.

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