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IRANIAN JOURNAL OF OPTIMIZATION

Volume 4, issue 2,spring 2012,336-353
Research Paper

Islamic Azad University
Rasht Branch

Inverse DEA Model with Fuzzy Data for Output Estimation

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Abstract

In this paper, we show that inverse Data Envelopment Analysis (DEA) models can be used to estimate output with fuzzy data for a Decision Making Unit (DMU) when some or all inputs are increased and deficiency level of the unit remains unchanged.

***Keywords**:* Data Envelopment Analysis, Multi-objective Programming, Inverse DEA Model, Fuzzy Numbers

1. Introduction

Data envelopment analysis (DEA) is a non-parametric technique for measuring and evaluating the relative efficiencies of a set of entities, called decision making units (DMUs), with the common inputs and outputs. Since the first introduction of the technique by Charnes et al.[1], known as the CCR model. Since the original publication, DEA has become a popular method for analyzing the efficiency of various organization units [1,3,4]. Interestingly, Charnes and cooper have also had a significant impact on the development of multiple objective linear programming (MOLP) through the development of goal programming. Al though Charnes and Cooper have played a significant role in the development of DEA and MOLP, researchers in these two camps have generally not paid much attention to research performed in the other camps (for more details see[5,6,10]). Recently, Wei et al.

[9] proposed inverse DEA, to answer the questions as follows: if among a group of DMU s, we increase certain input to a particular unit and assume that the DMU maintains its current efficiency level with respect to other DMU s, how much should the outputs of the DMU increase, or if the outputs need to be increased to a certain level and the efficiency of the unit remains unchanged, how much more inputs should be provided to the unit?

In recent years, fuzzy set theory has been proposed as a way to quantify imprecise and vague data in DEA models [11]. The DEA models with fuzzy data (Fuzzy DEA Models) can more realistically represent real-world problems than the conventional DEA models. Fuzzy set theory also allows linguistic data to be used directly within the DEA models. Fuzzy DEA models take the form of fuzzy linear programming models.

The questions discussed in inverse DEA can be considered with fuzzy data, that is, assume that some data are fuzzy numbers and we increase some or all input levels of a given DMU and assume that the DMU maintains its current efficiency level, how much should the outputs of the DMU change? In this paper we consider arbitrary changing in input level with triangular fuzzy numbers and we proposed a fuzzy MOLP model for outputs estimations.

The reminder of the paper organized as follows: In the following section, we review fuzzy sets and fuzzy number linear programming problem. We consider fuzzy DEA problem and its inverse DEA problem in section 3. In section 4, we proposed a fuzzy MOLP for outputs estimate. We consider of weak efficiency case In section 5. In section 6, we use an example to illustrate our computation method. Conclusions are given in section 7.

2. Preliminaries

Since terms like fuzzy sets, fuzzy numbers from fuzzy set theory will be used in sequel; we shall consider a few necessary definitions.

Definition 2.1 If X is a collection of objects denoted generically by x, then a fuzzy set in X is a set of ordered pairs:

$\tilde{A} = \left\{ \left(x, \tilde{A}(x)\right) \,|\, x \in X \right\}$ Where $\tilde{A}(x)$ is called the membership function which associates with each $x \in X$ a number in [0,1] indicating to what degree x is a number.

Definition 2.2 Let $\tilde{A}$ be a fuzzy number, i.e. A convex normalized fuzzy subset of the real line in the senses that:

$(a) \exists x_0 \in R, \tilde{A}(x_0) = 1.$

$(b) \tilde{A}(x_0)$ Is a piecewise continuous function.

The α - level set of $\tilde{A}$ is the $\tilde{A}_{\alpha} = \left\{ x \middle| \tilde{A}(x) \geq \alpha \right\}$ where $\alpha \in [0, 1]$.

Definition 2.3 A triangular fuzzy number denoted by $\tilde{A} = \left(A^{l}, A^{m}, A^{u} \right)$ Where $A^{l} \leq A^{m} \leq A^{u}$ and A^{l}, A^{m} and A^{u} are real numbers.

In this paper we denote the set of all triangular fuzzy numbers by $F(\mathbb{R})$.

Let $\tilde{A} = \left(A^{l}, A^{m}, A^{u} \right)$ and $\tilde{B} = \left(B^{l}, B^{m}, B^{u} \right)$ both be triangular fuzzy numbers. Define:

$$x > 0, x \in R : x\tilde{A} = \left(xA^{l}, xA^{m}, xA^{u} \right), \quad x < 0, x \in R : x\tilde{a} = \left(xA^{u}, xA^{m}, xA^{l} \right),$$

$$\tilde{A} + \tilde{B} = \left(A^{l} + B^{l}, A^{m} + B^{m}, A^{u} + B^{u} \right),$$
$$\tilde{A} - \tilde{B} = \left(A^{l} - B^{u}, A^{m} - B^{m}, A^{u} - B^{l} \right).$$

To introduce a meaningful ordering of fuzzy numbers, we first extended operations min and max on real numbers, to corresponding operations on fuzzy numbers, MIN and MAX. For any two numbers $\tilde{A}$ and $\tilde{B}$, we define

$$MIN(\tilde{A}, \tilde{B})(z) = \sup_{z=\min(x,y)} \min \left[\tilde{A}(x), \tilde{B}(y) \right]$$
$$MAX(\tilde{A}, \tilde{B})(z) = \sup_{z=\max(x,y)} \min \left[\tilde{A}(x), \tilde{B}(y) \right]$$

For all $z \in R$ $(see[7])$.

The ($F(\mathbb{R})$, MIN, MAX) can be expressed as the pair ($F(\mathbb{R}), \geq$) where $\geq$ is a partial ordering is defined as:

$$\tilde{A} \leq \tilde{B} \quad iff \quad MIN(\tilde{A}, \tilde{B}) = \tilde{A}$$
$$\tilde{A} \leq \tilde{B} \quad iff \quad MAX(\tilde{A}, \tilde{B}) = B$$

Definition 2.4 A fuzzy number $\tilde{a} = (a^{l}, a^{m}, a^{u}) > 0 (or \geq 0)$ means all its components a^{l}, a^{m} and $a^{u} > 0 (or \geq 0)$.

Theorem 2.1. For any two triangular fuzzy numbers $\tilde{A}$ = (A^l, A^m, A^u) and $\tilde{B}$ = (B^l, B^m, B^u) ***we*** have

$$\tilde{A} \le \tilde{B} \quad iff \quad A^l \le B^l, \quad A^m \le B^m, \quad A^u \le B^u.$$

Theorem 2.2. For any two triangular fuzzy numbers $\tilde{\alpha} = (\alpha^l, \alpha^m, \alpha^u)$ and $\tilde{\beta} = (\beta^l, \beta^m, \beta^u)$ if $\tilde{\beta} > \tilde{\alpha}$ then exists k > 1 such that $\tilde{\beta} \ge k\tilde{\alpha}.$

Proof. $\tilde{\beta} > \tilde{\alpha}$ If and only if

$$\beta^l > \alpha^l \Rightarrow \exists k_1 > 1 : \beta^l \ge k_1 \alpha^l$$
$$\beta^m > \alpha^m \Rightarrow \exists k_2 > 1 : \beta^m \ge k_2 \alpha^m$$
$$\beta^u > \alpha^u \Rightarrow \exists k_3 > 1 : \beta^u \ge k_3 \alpha^u$$

If k = min {k_1, k_2, k_3} then have $\tilde{\beta} \ge k\tilde{\alpha}.$

A linear programming problem (LPP) is defined as:

$$\begin{aligned} &\text{Min} \quad z = \boldsymbol{cx} \\ &\text{S.t} \quad A\boldsymbol{x}=\boldsymbol{b} \\ &\qquad \boldsymbol{x} \ge \boldsymbol{0} \end{aligned} \tag{2.1}$$

Where $\boldsymbol{c} = (c_1, \dots, c_n)$, $\boldsymbol{b} = (b_1, \dots, b_m)$, $A = (a_{ij})_{m \times n}$. In problem (2.1) all of the parameters are crisp[2]. Now, if in the LPP some coefficients of the problem in the objective function, technical coefficients, the right-hand side coefficients be the fuzzy numbers, then we say problem is a fuzzy number linear programming problem. Here, we consider the LPP with fuzzy number in technical and right-hand side coefficients. A fuzzy number linear programming problem (FNLPP) defined as follows:

$$\begin{aligned} &\max \quad z = \sum_{j=1}^{n} c_j x_j \\ &s.t \\ &\qquad \sum_{j=1}^{n} \tilde{A}_{ij} x_j \le \tilde{B}_i \qquad i = 1,2,\dots,m \\ &\qquad x_j \ge 0 \qquad j = 1,2,\dots,n. \end{aligned} \tag{2.2}$$

Where $\widetilde{A}_{ij} = (\widetilde{a}_{ij}^{\,l}, \widetilde{a}_{ij}^{\,m}, \widetilde{a}_{ij}^{\,u}) \in F(\mathbb{R})$ and $\widetilde{B}_{ij} = (\widetilde{b}_{ij}^{\,l}, \widetilde{b}_{ij}^{\,m}, \widetilde{b}_{ij}^{\,u}) \in F(\mathbb{R})$.

By theorem 2.1, problem (2.2) can be rewritten as

$$
\begin{aligned}
\text{Max} \quad & z = \sum_{j=1}^{n} c_j x_j \\
\text{S.t.} \quad & \sum_{j=1}^{n} a_{ij}^{l} x_j \le b_i^{l} \qquad i=1,2,\dots,m \\
& \sum_{j=1}^{n} a_{ij}^{m} x_j \le b_i^{m} \qquad i=1,2,\dots,m \\
& \sum_{j=1}^{n} a_{ij}^{u} x_j \le b_i^{u} \qquad i=1,2,\dots,m
\end{aligned}
\tag{2.3}
$$

However, since all numbers involved in problem (2.3) are crisp, then problem (2.3) is a linear programming problem. In this paper we say problem (2.3) is equivalent to problem (2.2).

3. *Fuzzy DEA problem and its fuzzy inverse problem*

Consider n DMUs: $DMU_1 \ldots DMU_n$, with m inputs and s outputs. Inputs and outputs for DMU $_j$ are $\widetilde{X}_j = (\widetilde{x}_{1j}, x_{2j}, \ldots \widetilde{x}_{mj})$ and $\widetilde{Y}_j = (\widetilde{y}_{1j}, \widetilde{y}_{2j}, \ldots, \widetilde{y}_{sj})$ for

$j = 1, 2, \ldots, n$,respectively, where $\widetilde{x}_{ij}, i = 1,2,\ldots, m$ and $\widetilde{y}_{rj}, r = 1,2,\ldots, s$ are fuzzy

Numbers. All $\widetilde{X}_j \in R^m$ and $\widetilde{Y}_j \in R^s$, and also $\widetilde{X}_j > 0$ and $\widetilde{Y}_j > 0$ for

$j = 1, 2, \ldots, n$.

Consider the following output oriented CCR model where inputs and outputs are triangular fuzzy numbers:

$$
\begin{aligned}
\text{Max} \quad & z \\
\text{S.t} \quad & \sum_{j=1}^{n} \lambda_j \widetilde{X}_j \le \widetilde{X}_o \\
& \sum_{j=1}^{n} \lambda_j \widetilde{Y}_j \ge z \widetilde{Y}_o
\end{aligned}
\tag{3.1}
$$

By theorem 2.1, problem (3.1) reduces to the following:

Max z

$$\text{S.t} \quad \sum_{j=1}^{n} \lambda_j x_{ij}{}^{l} \le x_{io}{}^{l} \qquad \text{i=1,2,…,m}$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}{}^{m} \le x_{io}{}^{m} \qquad \text{i=1,2,…m}$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}{}^{u} \le x_{io}{}^{u} \qquad \text{i=1,2,…m} \qquad (3.2)$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}{}^{l} \ge y_{ro}{}^{l} \qquad \text{r=1,2,…,s}$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}{}^{m} \ge y_{ro}{}^{m} \qquad \text{r=1,2,…,s}$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}{}^{u} \ge y_{ro}{}^{u} \qquad \text{r=1,2,…,s}$$

$$\lambda_j \ge 0 \qquad \text{j=1,2,…,n.}$$

The problem (3.2) is an output oriented CCR model where inputs and outputs are real numbers.

Suppose that for DMU_O, z_O is the optimal value of (3.1) ($z_O \ge 1$) and inputs of DMU_O are increased from $\tilde{X}_o to \tilde{\alpha}_o = \tilde{X}_o + \Delta\tilde{X}_o$, where the vector $\Delta\tilde{X}_o \ge 0$ and $\Delta\tilde{X}_o \ne 0$ (i.e. at least one component increases). We need to estimate the corresponding output level $\tilde{\beta}$ when the efficiency index DMU_O remains unchanged, where $\tilde{\beta} = (\tilde{\beta}_1,...,\tilde{\beta}_s)^T$ and $\tilde{\beta}_i = (\tilde{\beta}_i^l, \beta_i^m, \tilde{\beta}_i^u) \in F(\mathbb{R})$

for i=1,2,…,s.

For convenience, suppose DMU_{n+1} represents DMU_O after changing the inputs and outputs. Hence, to measure the efficiency of the DMU_{n+1} , we use the following model:

$$\text{Max} \quad z$$

$$\text{S.t.} \quad \sum_{j=1}^{n} \lambda_j \tilde{X}_j + \lambda_{n+1}\tilde{\alpha}_o \le \tilde{\alpha}_o \qquad (3.3)$$

$$\sum_{j=1}^{n}\lambda_j\tilde{Y}_j+\lambda_{n+1}\tilde{\beta}\geq z\tilde{\beta}$$

$$\lambda_j\geq 0 \qquad j=1,2,\ldots,n,n+1.$$

Where $\tilde{\alpha}_o=\tilde{X}_o+\Delta\tilde{X}_o,\Delta\tilde{X}_o\geq 0,\Delta\tilde{X}_o\neq 0.$

4. The related fuzzy MOLP

To find output maximum, such that efficiency index is remains unchanged, we consider the following fuzzy MOLP:

$$\begin{aligned}
\text{Max}\quad & (\tilde{\beta}_1,\tilde{\beta}_2,\ldots,\tilde{\beta}_s)\\
\text{S.t.}\quad & .\sum_{j=1}^{n}\lambda_j\tilde{X}_j\leq\tilde{\alpha}_o\\
& \sum_{j=1}^{n}\lambda_j\tilde{Y}_j\geq z_o\tilde{\beta} \qquad\qquad (4.1)\\
& \tilde{\beta}\geq\tilde{Y}_o\\
& \lambda_j\geq 0 \qquad j=1,2,\ldots,n.
\end{aligned}$$

Where $\tilde{\alpha}_o$ is defined as before and z_o is given as the optimal value of problem (3.1).

Definition 4.1 Let $(\tilde{\tilde{\beta}},\tilde{\lambda})$ a feasible solution of problem (4.1). If there is no feasible solution $(\tilde{\beta},\lambda)$ of (3.3) such that $\tilde{\beta}>\tilde{\tilde{\beta}},$ then we call $(\tilde{\tilde{\beta}},\tilde{\lambda})$ a weak Pareto solution of problem (4.1).

Theorem 4.1. Let $\left(\tilde{\tilde{\beta}},\bar{\lambda}\right)$ be a weak Pareto solution of problem (4.1) and z_o be the optimal value of problem (3.1). Then z_o is the optimal value of the following problem:

$$\begin{aligned}
\text{Max}\quad & z\\
\text{S.t.}\quad & \sum_{j=1}^{n}\lambda_j\tilde{X}_j\leq\tilde{\alpha}_o
\end{aligned}$$

$$\sum_{j=1}^{n} \lambda_j \tilde{Y}_j \geq z\tilde{\beta} \qquad (4.2)$$

$$\lambda_j \geq 0 \qquad j=1,2,\ldots,n.$$

Proof. Because problem (4.2) has an optimal value, suppose $(z_o^{'}, \lambda^{'})$ is an optimal solution of the problem. As $(\bar{\tilde{\beta}}, \bar{\lambda})$ is a weak Pareto solution of (4.1), it satisfies the conditions:

.

$$\sum_{j=1}^{n} \bar{\lambda}_j \tilde{X}_j \leq \tilde{\alpha}_o$$

$$\sum_{j=1}^{n} \bar{\lambda}_j \tilde{Y}_j \geq z_o \bar{\tilde{\beta}}$$

$$\bar{\tilde{\beta}} \geq \tilde{Y}_o$$

$$\bar{\lambda}_j \geq 0 \qquad j=1,2,\ldots,n.$$

Because $(z_o, \bar{\lambda})$ is a feasible solution (4.2), we have

$z_o^{'} \geq z_o$. *If* $z_o^{'} > z_o$ *then* $\exists \Delta > 0 : z_o^{'} = z_o + \Delta,$ we would have $\sum_{j=1}^{n} \lambda_j^{'} \tilde{Y}_j \geq z_o^{'} \bar{\tilde{\beta}} = (z_o + \Delta)\bar{\tilde{\beta}} = z_o \bar{\tilde{\beta}} + \Delta \bar{\tilde{\beta}} = z_o(\bar{\tilde{\beta}} + \frac{\Delta}{z_o}\bar{\tilde{\beta}}) = z_o \tilde{\beta}_1$ Where $\tilde{\beta}_1 = \bar{\tilde{\beta}} + \frac{\Delta}{z_o}\bar{\tilde{\beta}}$ and $\bar{\tilde{\beta}}_1 = \beta + \frac{\Delta}{z_o}\bar{\tilde{\beta}} > \bar{\tilde{\beta}}.$

So $(\tilde{\beta}_1, \lambda^{'})$ would be a feasible solution of problem (4.1), which is impossible because $(\bar{\tilde{\beta}}, \bar{\lambda})$ is a weak Pareto solution of (4.1). So, we must have $z_o' = z_o$ i.e. z_o is the optimal value of (4.2).

Theorem 4.2. Let $(\bar{\tilde{\beta}}, \bar{\lambda})$ be feasible solution of problem (4.1). If the optimal value of problem (4.2) is z_o, then $(\bar{\tilde{\beta}}, \bar{\lambda})$ must be a weak Pareto solution (4.1).

Proof. If this theorem were not true then there would exist another feasible solution of (4.1), $(\tilde{\beta}^*, \lambda^*)$, such that $\tilde{\beta}^* > \bar{\beta}$. *As* $z_o \geq 1$,

$$\sum_{j=1}^{n} \lambda_j^* \tilde{X}_j \leq \tilde{\alpha}_o$$

$$\sum_{j=1}^{n} \lambda_j^* \tilde{Y}_j \geq z_o \tilde{\beta}^* > z_o \tilde{\bar{\beta}} \qquad (*)$$

$$\lambda_j^* \geq 0 \qquad j = 1,2,\dots,n.$$

From Eq. (*) and theorem 2.2, $\exists k > 1$ such that

$$\sum_{j=1}^{n} \lambda_j^* \tilde{Y}_j \geq z_o k \tilde{\bar{\beta}} = (k z_o) \tilde{\bar{\beta}}$$

So, the optimal value of (4.2) is at least kz_o where $kz_o > z_o$, which is against assumption that z_o is the optimal value of (4.2).

Consider the following problem:

$$\begin{aligned}
&\text{Max} \quad z \\
&\text{S.t} \quad \sum_{j=1}^{n} \lambda_j \tilde{X}_j + \lambda_{n+1} \tilde{\alpha} \leq \tilde{\alpha}_o \\
&\qquad \sum_{j=1}^{n} \lambda_j \tilde{Y}_j + \lambda_{n+1} \tilde{\bar{\beta}} \geq z \tilde{\bar{\beta}}_o \qquad (4.3) \\
&\qquad \lambda_j \geq 0 \qquad j = 1,2,\dots,n,n+1.
\end{aligned}$$

Theorem 4.3. Assume $\tilde{\bar{\beta}} \geq \tilde{Y}_o$. If the optimal value of problem (4.3) is $z_o > 1$, then the optimal value of (4.2) is also z_o; conversely, if the optimal value of problem (4.2) is $z_o > 1$, so is the optimal value of (4.3).

Proof. First we assume that the optimal solution of (4.3) is (z_o, λ_o) and $z_o > 1$. Consider the equivalent problem of (4.3):

Max z

$$\text{S.t} \quad \sum_{j=1}^{n} \lambda_j x_{ij}^{l} + \lambda_{n+1}\alpha_{io}^{l} \le \alpha_{io}^{l} \qquad \text{i=1,2,…,m}$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{m} + \lambda_{n+1}\alpha_{io}^{m} \le \alpha_{io}^{m} \qquad \text{i=1,2,…,m}$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{u} + \lambda_{n+1}\alpha_{io}^{u} \le \alpha_{io}^{u} \qquad \text{i=1,2,…,m} \qquad (4.4)$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{l} + \lambda_{n+1}\overline{\beta}^{\,l} \ge z\overline{\beta}^{\,l} \qquad \text{r=1,2,…,s}$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{m} + \lambda_{n+1}\overline{\beta}^{\,m} \ge z\overline{\beta}^{\,m} \qquad \text{r=1,2,…,s}$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{u} + \lambda_{n+1}\overline{\beta}^{\,u} \ge z\overline{\beta}^{\,u} \qquad \text{r=1,2,…,s}$$

$$\lambda_j \ge 0 \qquad \text{j=1,2,…,n}$$

We consider the dual problem of (4.3):

$$\text{Min} \quad \sum_{i=1}^{m} (\alpha_{io}{}^{l} u_i + \alpha_{io}{}^{m} v_i + \alpha_{io}{}^{u} \omega_i)$$

S.t

$$\sum_{i=1}^{m} x_{ij}^{l} u_i + x_{ij}^{m} v_i + x_{ij}^{u} \omega_i - \sum_{r=1}^{s} (y_{rj}^{l} f_r + y_{rj}^{m} g_r + y_{rj}^{u} h_r) \ge 0 \qquad j = 1,2,..., n$$

$$\sum_{r=1}^{s} (\overline{\beta}^{\,l} f_r + \overline{\beta}^{\,m} g_r + \overline{\beta}^{\,u} h_r) = 1 \qquad (4.5)$$

$$\sum_{i=1}^{m} (\alpha_{io}^{l} u_i + \alpha_{io}^{m} v_i + \alpha_{io}^{u} \omega_i) - \sum_{r=1}^{s} (\overline{\beta}^{\,l} f_r + \overline{\beta}^{\,m} g_r + \overline{\beta}^{\,u} h_r) \ge 0 \qquad (**)$$

$$u_i, v_i, \omega_i \ge 0, \qquad i = 1,2,..., m, \qquad f_r, g_r, h_r \ge 0 \qquad r = 1,2,..., s$$

Suppose

$$(u_1^o,..., u_m^o, v_1^o,..., v_m^o, \omega_1^o,..., \omega_m^o)$$

And

$$(f_1^o,..., f_s^o, g_1^o,..., g_s^o, h_1^o,..., h_s^o)$$

Are an optimal solutions of (4.4). So $\sum_{i=1}^{m}(\alpha_{io}^{l}u_i^o + \alpha_{io}^{m}v_i^o + \alpha_{io}^{u}\omega_i^o) = z_o > 1$, and we have

$$\sum_{i=1}^{m}(\alpha_{io}^{l}u_i^o + \alpha_{io}^{m}v_i^o + \alpha_{io}^{u}\omega_i^o) = z_o > 1 = \sum_{r=1}^{s}(\overline{\beta}^{l} f_r^o + \overline{\beta}^{m} g_r^o + \overline{\beta}^{u} h_r^o)$$

According to the complementary slackness condition for LP problem, in any optimal solution of (4.3), the variable ¸λ_{n+1} which corresponds to the constraint (**) in the dual problem (4.5), must be λ_{n+1}= 0. But when ¸ λ_{n+1}= 0, problem (4.3) just becomes (4.2). So, problem (4.2) has same optimal value z_o .

Conversely, if the optimal value of problem (4.2) is z_o > 1, consider the equivalent problem of (4.2):

Max z

$$\begin{aligned}
\text{S.t}\quad & \sum_{j=1}^{n}\lambda_j x_{ij}^{l} \le \alpha_{io}^{l} && i=1,2,\ldots,m \\
& \sum_{j=1}^{n}\lambda_j x_{ij}^{m} \le \alpha_{io}^{m} && i=1,2,\ldots,m \\
& \sum_{j=1}^{n}\lambda_j x_{ij}^{u} \le \alpha_{io}^{u} && i=1,2,\ldots,m \qquad (4.6) \\
& \sum_{j=1}^{n}\lambda_j y_{rj}^{l} \ge z\overline{\beta}^{l} && r=1,2,\ldots,s \\
& \sum_{j=1}^{n}\lambda_j y_{rj}^{m} \ge z\overline{\beta}^{m} && r=1,2,\ldots,s \\
& \sum_{j=1}^{n}\lambda_j y_{rj}^{u} \ge z\overline{\beta}^{u} && r=1,2,\ldots,s \\
& \lambda_j \ge 0 && j=1,2,\ldots,n
\end{aligned}$$

Write the dual problem of (4.6)

$$\text{Min} \quad \sum_{i=1}^{m}(\alpha_{io}{}^{l}u_i + \alpha_{io}{}^{m}v_i + \alpha_{io}{}^{u}\omega_i)$$

S.t

$$\sum_{i=1}^{m} x_{ij}^{l}u_i + x_{ij}^{m}v_i + x_{ij}^{u}\omega_i - \sum_{r=1}^{s}(y_{rj}^{l}f_r + y_{rj}^{m}g_r + y_{rj}^{u}h_r) \geq 0 \qquad j = 1,2,\dots,n$$

$$\sum_{r=1}^{s}(\bar{\beta}^{l}f_r + \bar{\beta}^{m}g_r + \bar{\beta}^{u}h_r) = 1 \tag{4.7}$$

$$u_i, v_i, \omega_i \geq 0, \qquad i = 1,2,\dots,m, \qquad f_r, g_r, h_r \geq 0 \qquad r = 1,2,\dots,s$$

Also has an optimal value z_o.

The only difference between (4.5) and (4.6) is that (4.5) contains one more constraint, i.e. constraint (**).

But we can show that for any optimal solution

$(u_1^{'},\dots,u_m^{'},v_1^{'},\dots,v_m^{'},\omega_1^{'},\dots,\omega_m^{'})$ And $(f_1^{'},\dots,f_s^{'},g_1^{'},\dots,g_s^{'},h_1^{'},\dots,h_s^{'})$ of problem (4.5), this constraint must holds as strict inequality.

$$\sum_{i=1}^{m}(\alpha_{io}^{l}u_i^{'} + \alpha_{io}^{m}v_i^{'} + \alpha_{io}^{u}\omega_i^{'}) - \sum_{i=1}^{s}(\bar{\beta}^{l}f_r^{'} + \bar{\beta}^{m}g_r^{'} + \bar{\beta}^{u}h_r^{'}) > 0$$

If $\displaystyle\sum_{i=1}^{m}(\alpha_{io}^{l}u_i^{'} + \alpha_{io}^{m}v_i^{'} + \alpha_{io}^{u}\omega_i^{'}) - \sum_{i=1}^{s}(\bar{\beta}^{l}f_r^{'} + \bar{\beta}^{m}g_r^{'} + \bar{\beta}^{u}h_r^{'}) = 0$

Then the optimal value of (4.5) is $\displaystyle\sum_{i=1}^{m}(\alpha_{io}^{l}u_i^{'} + \alpha_{io}^{m}v_i^{'} + \alpha_{io}^{u}\omega_i^{'}) = 1$.

And hence the optimal value of (4.7) would be 1 which contradicts the assumption. So, this additional constraint must be unbinding at any optimal solution and hence problem (4.5) is equivalent to (4.7) which implies that z_o is also optimal value of problems (4.3) and (4.5).

Theorem 4.4. Suppose that the optimal value of problem (3.1) is $z_o > 1$, and the inputs for this DMU are going to increase from $\tilde{X}_o$ *to* $\tilde{\alpha}_o = \tilde{X}_o + \Delta\tilde{X}_o (\Delta\tilde{X}_o \geq 0, \Delta\tilde{X}_o \neq 0)$. Let $(\tilde{\bar{\beta}}, \bar{\lambda})$ be a weak Pareto solution of problem of (4.1), then the optimal solution of (4.3) is still z_o. Conversely, Let $(\tilde{\bar{\beta}}, \bar{\lambda})$ be a feasible solution of problem (4.1). If the optimal value problem (4.3) is z_o, then $(\tilde{\bar{\beta}}, \bar{\lambda})$ must be a weak Pareto solution of (4.1).

Proof. Assume $(\tilde{\tilde{\beta}}, \bar{\lambda})$ is a Pareto solution of (4.1). By theorem 4.1, z_o is the optimal value of (4.2). As $z_o > 1$ and $\tilde{\tilde{\beta}} \geq \tilde{Y}_o$, by theorem 4.3, zo is also the optimal value of (4.3).

Conversely, if the optimal value of (4.3), is z_o, then by theorem 4.3, z_o is also the optimal value of (4.2). Using theorem 4.2, we know that $(\tilde{\tilde{\beta}}, \bar{\lambda}))$ is a weak Pareto solution of (4.1).

To identify some of Pareto solution of (4.1), we convert it to a single-objective programming problem by assuming $p_i > 0$ as the weight of i-th output

$i = 1,\dots,s.$ Therefore we will have:

$$\begin{aligned}
\text{Max} \quad & \sum_{r=1}^{s} p_r \tilde{\beta}_r = \sum_{r=1}^{s} p_r (\beta_r^l + \beta_r^m + \beta_r^u) \\
\text{S.t.} \quad & \sum_{j=1}^{n} \lambda_j \tilde{X}_j \leq \tilde{\alpha}_o \\
& \sum_{j=1}^{n} \lambda_j \tilde{Y}_j \geq z_o \tilde{\beta} \qquad\qquad (4.8) \\
& \tilde{\beta} \geq \tilde{Y}_o \\
& \lambda_j \geq 0 \qquad\qquad j=1,2,\dots,n.
\end{aligned}$$

We know any optimal solution of (4.8) must be a weak Pareto of problem (4.1).

Corollary 1. Suppose the efficiency index of DMU_o under model (3.1) is

$z_o > 1$, and inputs are increased from $\tilde{X}_o$ to $\tilde{\alpha}_o = \tilde{X}_o + \Delta\tilde{X}_o$ ($\Delta\tilde{X}_o \geq 0$, $\Delta\tilde{X}_o \neq 0$). Let $(\tilde{\tilde{\beta}}, \bar{\lambda})$ be an optimal solution of problem (4.8). Then, when the outputs of DMU_o are increased to $\tilde{\tilde{\beta}}$ the efficiency index for the DMU is still z_o.

5. Weak efficiency case

We now turn to the case for $z_o = 1$. Consider the following LP problem

$$\text{Max} \quad z$$

$$\text{S.t.} \quad \sum_{j=1}^{n} \lambda_j \tilde{X}_j \le \tilde{\alpha}_o \tag{5.1}$$

$$\sum_{j=1}^{n} \lambda_j \tilde{Y}_j \ge z\tilde{Y}_o$$

$$\lambda_j \ge 0 \qquad j=1,2,\dots,n.$$

Where $\tilde{\alpha}_o = \tilde{X}_o + \Delta\tilde{X}_o$ and $\Delta\tilde{X}_o \ge 0$, $\Delta\tilde{X}_o \ne 0$. Let its optimal value be z*. If we denote the feasible regions of problem (3.1) and (5.1) by S_0 and S_1 respectively, we have: $S_0 \subset S_1$. Therefore, $z^* \ge zo = 1$.

Theorem 5.1. Suppose that the optimal value of problem (3.1) for DMU $_o$ is

$z_o = 1$ and the inputs for this DMU are increased from $\tilde{X}_o$ to $\tilde{\alpha}_o = \tilde{X}_o + \Delta\tilde{X}_o$ ($\Delta\tilde{X}_o \ge 0$, $\Delta\tilde{X}_o \ne 0$). Then, when the outputs of DMU$_o$ *are* increased from $\tilde{Y}_o$ *to* $z^*\tilde{Y}_o$, where z* is the optimal value of problem (5.1), the optimal value of problem (5.1) which corresponds of the new DMU $(\tilde{\alpha}_o, z^*\tilde{Y}_o)$ is still z_o.

Proof. When the inputs and outputs of DMU_o become $\tilde{\alpha} = \tilde{X}_o + \Delta\tilde{X}_o (\Delta\tilde{X}_o \ge 0,\ \Delta\tilde{X}_o \ne 0)$ and $z^*\tilde{Y}_o$, the efficiency index of DMU equals the optimal value of the problem below:

$$\text{Max} \quad z$$

$$\text{S.t.} \quad \sum_{j=1}^{n} \lambda_j \tilde{X}_j + \lambda_{n+1}\tilde{\alpha}_o \le \tilde{\alpha}_o \tag{5.2}$$

$$\sum_{j=1}^{n} \lambda_j \tilde{Y}_j + \lambda_{n+1}(z^*\tilde{Y}_o) \ge z(z^*\tilde{Y}_o)$$

$$\lambda_j \ge 0 \qquad j=1,2,\dots,n.$$

Let the optimal value of problem (5.2) be $\hat{z}$. What we need to prove is that $\hat{z} = 1$. If $\hat{z} \ne= 1$, then $\hat{z} > 1$. We write of the equivalent of problem (5.2):

$$\text{Max} \quad z$$

$$\text{S.t} \quad \sum_{j=1}^{n} \lambda_j x_{ij}^l + \lambda_{n+1}\alpha_{io}^l \le \alpha_{io}^l \qquad i=1,2,\dots,m$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{m} + \lambda_{n+1} \alpha_{io}^{m} \le \alpha_{io}^{m} \qquad i=1,2,\ldots,m$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{u} + \lambda_{n+1} \alpha_{io}^{u} \le \alpha_{io}^{u} \qquad i=1,2,\ldots,m \qquad (5.3)$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{l} + \lambda_{n+1} (z^* y_{ro}^{l}) \ge z(z^* y_{ro}^{l}) \qquad r=1,2,\ldots,s$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{l} + \lambda_{n+1} (z^* y_{ro}^{m}) \ge z(z^* y_{ro}^{m}) \qquad r=1,2,\ldots,s$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^{l} + \lambda_{n+1} (z^* y_{ro}^{u}) \ge z(z^* y_{ro}^{u}) \qquad r=1,2,\ldots,s$$

$$\lambda_j \ge 0 \qquad j=1,2,\ldots,n$$

We write the dual of problem (5.3)

$$\text{Min} \quad \sum_{i=1}^{m} (\alpha_{io}{}^{l} u_i + \alpha_{io}{}^{m} v_i + \alpha_{io}{}^{u} \omega_i)$$

S.t

$$\sum_{i=1}^{m} x_{ij}^{l} u_i + x_{ij}^{m} v_i + x_{ij}^{u} \omega_i - \sum_{r=1}^{s} (y_{rj}^{l} f_r + y_{rj}^{m} g_r + y_{rj}^{u} h_r) \ge 0 \qquad j=1,2,\ldots,n$$

$$\sum_{i=1}^{m} (\alpha_{io}^{l} u_i + \alpha_{io}^{m} v_i + \alpha_{io}^{u} \omega_i) - \sum_{r=1}^{s} (f_r (z^* y_{ro}^{l}) + g_r (z^* y_{ro}^{m}) + h_r (z^* y_{ro}^{u})) \ge 0 \; (*_1)$$

$$\sum_{r=1}^{s} (f_r (z^* y_{ro}^{l}) + g_r (z^* y_{ro}^{m}) + h_r (z^* y_{ro}^{u})) = 1 \qquad (*_2)$$

$$u_i, v_i, \omega_i \ge 0, \qquad i=1,2,\ldots,m, \qquad f_r, g_r, h_r \ge 0 \qquad r=1,2,\ldots,s$$

Suppose $(\hat{u}_1,\ldots,\hat{u}_m,\hat{v}_1,\ldots,\hat{v}_m,\hat{\omega}_1,\ldots,\hat{\omega}_m)$ and $(\hat{f}_1,\ldots,\hat{f}_s,\hat{g}_1,\ldots,\hat{g}_s,\hat{h}_1,\ldots,\hat{h}_s)$ are its optimal solution. By duality result, we know that

$$\sum_{i=1}^{m} (\alpha_{io}^{l} \hat{u}_i + \alpha_{io}^{m} \hat{\upsilon}_i + \alpha_{io}^{u} \hat{\omega}_i) > 1 \qquad (*3)$$

So, from the constraints (*1), (*2) and (*3) we have:

$$\sum_{i=1}^{m}(\alpha_{io}^{l}\hat{u}_i + \alpha_{io}^{m}\hat{\upsilon}_i + \alpha_{io}^{u}\hat{\omega}_i) - \sum_{r=1}^{s}(\hat{f}_r(z^* y_{ro}^{l}) + \hat{g}_r(z^* y_{ro}^{m}) + \hat{h}_r(z^* y_{ro}^{u})) > 0$$ By the complementary slackness theorem, at each optimal solution of (5.2), $\hat{\lambda}_{n+1} = 0$. This means $\hat{z}$ is also the optimal value of the problem

$$\begin{aligned} &\text{Max} \quad z \\ &\text{S.t.} \quad \sum_{j=1}^{n}\lambda_j \tilde{X}_j \le \tilde{\alpha}_o \\ &\sum_{j=1}^{n}\lambda_j \tilde{Y}_j \ge z(z^* \tilde{Y}_o) \\ &\lambda_j \ge 0 \qquad j=1,2,\ldots,n. \end{aligned}$$

And therefore, the optimal value of the problem

$$\begin{aligned} &\text{Max} \quad z \\ &\text{S.t.} \quad \sum_{j=1}^{n}\lambda_j \tilde{X}_j \le \tilde{\alpha}_o \\ &\sum_{j=1}^{n}\lambda_j \tilde{Y}_j \ge (zz^*)\tilde{Y}_o \qquad (5.4) \\ &\lambda_j \ge 0 \qquad j=1,2,\ldots,n. \end{aligned}$$

Is z^*z. But the above problem with replacing the z^*z by z is just problem (5.1), so, the optimal value of problem (5.1) would be z^*z, and we have $\hat{z} > 1$, then $z^*\hat{z} > z^*$, which contradicts that maximum value of (5.1) is z^*.

6. Numerical example

We consider of three DMU with three inputs and two outputs. The data of inputs and outputs are shown in the following table.

DMU_S	I_1	I_2	I_3	O_1	O_2
DMU1	(1,3,4)	(2,5,8)	(1,4,7)	(3,4,9)	(10,18,20)
DMU2	(3,5,10)	(4,7,11)	(8,11,14)	(1,8,20)	(3,11,16)
DMU3	(8,10,24)	(5,11,19)	(4,11,19)	(3,8,14)	(5,11,21)

By evaluating DMU_3 using output-oriented of model (3.1), we have

z_O = 1.1818. We now increase the inputs of $DMU3$ from

$\tilde{X}_3$= ((8, 10, 24), (5, 11, 18), (4, 11, 19)) to

$\tilde{\alpha}_3$ = (11, 15, 34), (8, 16, 28), (7, 16, 29) then by solving the model (4.8) with ($p_1 = p_2 = 1$), we will have

$\tilde{\beta}_3$ = ((8.1231, 10.8308, 24.392), (27.0770, 48.7385, 54.1539)).

In this case, the efficiency of $(\tilde{\alpha}_3, \tilde{\beta}_3)$ and $(\tilde{X}_3, \tilde{Y}_3)$ is equal.

Also, by evaluating DMU_2 using of model (3.1), we have z_o = 1, we now increase the inputs of DMU_2 from $\tilde{X}2 = ((3,5,10),(4,7,11),(8,11,14))$ to

$\tilde{\alpha}_2$= ((5, 12, 19), (6, 14, 20), (10, 18, 23)), then by solving the model (15), we will have z^* = 1.4667 and $z^*\tilde{Y}_2 = \tilde{\beta}_2 =$((1.4667, 11.7334, 29.3334), (4.4, 16.1334, 23.4667)) In this case, the efficiency of $(\tilde{\alpha}_2, \tilde{\beta}_2)$ and $(\tilde{X}_2, \tilde{Y}_2)$ is equal, and both is z_o = 1.

7. Conclusion

In this paper we discuss these problem: In the presence of fuzzy data, how should we control the change in input level of a given DMU such that the efficiency index of the DMU is preserved. To solve the problem we proposed a fuzzy MOLP model. We provide the necessary and sufficient conditions for the input changes under the same efficiency index.

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