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A note on the averaging method for differential equations with maxima

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Abstract

Substantiation of the averaging method for differential equations with maxima is presented. Two theorems on substantiates for differential equations with maxima are established.

Keywords: Averaging method; Differential equations with delay; Differential equations with maxima; Automatic regulation.

1 Introduction. Substantiates theorems

In the papers [3], [5], [6], [10], the differential equations with maximums for the mathematical simulation of some systems with automatic regulation are presented. Application of the averaging method for differential equations with maximums has been studied extensively by many researchers (see [1], [2], [4], [9], [10] and the references therein). In the present paper two theorems on the justification of the averaging method for differential equations with maxima are established.

The differential equation

$$\dot{x}(t) = \varepsilon f(t, x(t), \max_{s \in [g(t), \gamma(t)]} x(s)) \tag{1}$$

with maxima is considered. Here $x \in R^n$ is a phase vector, ε is a small parameter, $f:[0,\infty) \times R^n \times R^n \to R^n$ is n dimensional vector function, $t \geq 0$, $g(t)$ and $\gamma(t)$ are known functions, $0 \leq g(t) \leq \gamma(t) \leq t$ and

$$\max_{s\in[g(t),\gamma(t)]} x(s)=\left(\max_{s\in[g(t),\gamma(t)]} x_1(s),\cdots,\max_{s\in[g(t),\gamma(t)]} x_n(s)\right).$$

Here $\|x\|=\max_i |x_i|$.

Note that if $g(t)=\gamma(t)=t-h,$ then (1) is a differential equation with constant delay and if $g(t)=\gamma(t),$ then (1) is a differential equation with variable delay. Let us consider the following averaged equation

$$\dot{y}(t)=\varepsilon f^0\left(y(t),\max_{s\in[g(t),\gamma(t)]} y(s)\right) \tag{2}$$

for the equation (1). Here

$$f^0(x,y)=\lim_{T\to\infty}\frac{1}{T}\int_0^T f(t,x,y)dt. \tag{3}$$

Theorem 1. In $Q=[0,\infty)\times D\times D,\ D\subset R^n$ the following conditions hold:

1) $f(t,x,y)$ is a continuous function on t and

$$\|f(t,x,y)\|\le M, \tag{4}$$

$$\|f(t,x,y)-f(t,\bar{x},\bar{y})\|\le\lambda[\|x-\bar{x}\|+\|y-\bar{y}\|], \tag{5}$$

2) $g(t)$ and $\gamma(t)$ are evenly continuous functions and $0\le g(t)\le\gamma(t)\le t$;

3) the limit (3) exists evenly with respect to x,y;

4) the solution of the equation (2) at $\varepsilon\in(0,\varepsilon_1],\ t\ge 0,\ y(0)\in D''\subset D$ together with its $\rho-$neighbourhood belongs to D.

Then for any $\eta>0,\ L>0$ there exists $\varepsilon^0(\eta,L)\in(0,\varepsilon_1]$ such that the following estimate holds:

$$\|x(t)-y(t)\|\le\eta, \tag{6}$$

where $x(t),y(t)$ are solutions of systems (1) and (2) accordingly, $x(0)=y(0)\in D''$.

Now, we consider the following partially averaged equation

$$\dot{y}=\varepsilon F\left(t,y(t),\max_{s\in[g(t),\gamma(t)]} y(s)\right) \tag{7}$$

for the equation (1). Here

$$F(t,y,z)=\left\{F^{i}(y,z)=\frac{1}{T}\int_{iT}^{(i+1)T} f(t,y,z)dt,\ t\in[iT,(i+1)T), i=0,1,\cdots\right\}, \quad (8)$$

T is a constant. The step-averaging scheme of (7) for the system (1) will be used.

Theorem 2. Let the conditions (1) and (2) of the theorem 1 is fulfilled and also:

solution of the equation (7) at $\varepsilon\in(0,\varepsilon_1]$, $t\geq 0$ and $y(0)\in D''\subset D$ together with its $\rho-$neighbourhood belongs to D.

Then for any $L>0$, there exist such $C>0$ and $\varepsilon^{0}(L)\in(0,\varepsilon_1]$ that the following estimate is fulfilled:

$$\|x(t)-y(t)\|\leq C\varepsilon, \quad (9)$$

where $x(t), y(t)$ are solutions of systems (1) and (7) accordingly, $x(0)=y(0)\in D''$.

2 Proofs of theorems 1 and 2

First, we will give the proof of theorem 1. Using the integral equations for (1) and (2), we can write

$$\|x(t)-y(t)\|\leq$$

$$\varepsilon\int_0^t\left\| f(\tau,x(\tau),\max_{s\in[g(\tau),\gamma(\tau)]}x(s))-f(\tau,y(\tau),\max_{s\in[g(\tau),\gamma(\tau)]}y(s))\right\|d\tau$$

$$+\left\|\varepsilon\int_0^t\left[f(\tau,y(\tau),\max_{s\in[g(\tau),\gamma(\tau)]}y(s))-f^{0}(y(\tau),\max_{s\in[g(\tau),\gamma(\tau)]}y(s))\right]d\tau\right\|=I+I^{0}. \quad (10)$$

Note that:

$$\delta(t)=\max_{s\in[0,t]}\|x(s)-y(s)\|$$

is the uniform metric. Then, using this notation and (10), we get:

$$I\leq\varepsilon\lambda\int_0^t\left[\|x(\tau)-y(\tau)\|+\left\|\max_{s\in[g(\tau),\gamma(\tau)]}x(s)-\max_{s\in[g(\tau),\gamma(\tau)]}y(s)\right\|\right]ds\leq$$

$$2\varepsilon\lambda\int_0^t \delta(\tau)d\tau. \tag{11}$$

We consider $t_i = i\Delta,\ i = 0,1,...,m,\ m\Delta = L\varepsilon^{-1}$ and $[0, L\varepsilon^{-1}] = \bigcup_{i=0}^{m-1}[t_i, t_{i+1}]$.

Let $t \in [t_k, t_{k+1})$. Then, using the additive property of the integral, we get:

$$I^0 \le \sum_{i=0}^{k-1}\varepsilon\left\|\int_{t_i}^{t_{i+1}}\left[f(\tau, y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s)) - f^0(y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s))\right]d\tau\right\|$$

$$+\varepsilon\left\|\int_{t_k}^{t}\left[f(\tau, y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s)) - f^0(y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s))\right]d\tau\right\| = \sum_{i=0}^{k-1} I_i + I_k. \tag{12}$$

Let us estimate I_k and I_i for all i, using the triangle inequality, we obtain:

$$I_i \le \varepsilon\left\|\int_{t_i}^{t_{i+1}}\left[f(\tau, y(t_i), \max_{s\in[g(t_i),\gamma(t_i)]} y(s)) - f^0(y(t_i), \max_{s\in[g(t_i),\gamma(t_i)]} y(s))\right]d\tau\right\|$$

$$+\varepsilon\int_{t_i}^{t_{i+1}}\left\|f(\tau, y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s)) - f(\tau, y(t_i), \max_{s\in[g(t_i),\gamma(t_i)]} y(s))\right\|d\tau$$

$$+\varepsilon\int_{t_i}^{t_{i+1}}\left\|f^0(y(t_i), \max_{s\in[g(t_i),\gamma(t_i)]} y(s)) - f^0(y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s))\right\|d\tau = J_i^0 + J_i + J_i^{00}. \tag{13}$$

From (2) and estimate (13) and (1), (2) assumptions of the theorem 1, we get:

$$J_i \le \varepsilon\lambda\int_{t_i}^{t_{i+1}}\left[\left\|y(\tau) - y(t_i)\right\| + \left\|\max_{s\in[g(\tau),\gamma(\tau)]} y(s) - \max_{s\in[g(t_i),\gamma(t_i)]} y(s)\right\|\right]d\tau \le$$

$$\varepsilon\lambda\int_{t_i}^{t_{i+1}}\left[\varepsilon\int_{t_i}^{\tau}\left\|f^0(y(x), \max_{s\in[g(x),\gamma(x)]} y(s))\right\|dx + \varepsilon M max\{\ \omega(\gamma,\Delta), \omega(g,\Delta)\ \}\right]d\tau \le$$

$$\varepsilon^2 \lambda M \Delta\left(\frac{\Delta}{2} + max\{\,\omega(\gamma,\Delta),\omega(g,\Delta)\,\}\right), \tag{14}$$

where $\omega(\alpha,\Delta)$ is a continuity modulus [11] of the function $\alpha(t)$ on the interval $[0,\infty)$, and $\omega(\alpha,\Delta) = \sup\limits_{|t''-t'|\le\Delta} |\alpha(t'') - \alpha(t')|$.

Using the properties of the continuity modulus of the paper [8], we get:

$$\sum_{i=0}^{k-1} J_i \le \varepsilon\lambda ML\left(\frac{L}{2\varepsilon m} + max\left\{\omega\left(\gamma,\frac{L}{\varepsilon m}\right),\omega\left(g,\frac{L}{\varepsilon m}\right)\right\}\right) \le$$

$$\frac{\lambda ML}{m}\left(\frac{L}{2} + max\{\omega(\gamma,L),\omega(g,L)\}\right) + \varepsilon\lambda ML max\{\omega(\gamma,L),\omega(g,L)\}. \tag{15}$$

Similarly to the way the estimate (15) was obtained, it can be proved that

$$\sum_{i=0}^{k-1} J_i^{00} \le \lambda ML\left(\frac{L}{2m} + \left(\frac{1}{m} + \varepsilon\right) max\{\omega(\gamma,L),\omega(g,L)\}\right). \tag{16}$$

From assumption (3) of the theorem 1 it follows that there exists a decreasing function $\theta(t) \underset{t\to\infty}{\to} 0$ such that

$$\left\|\varepsilon\int_0^{t_i}\left[f(\tau, y(t_i), \max_{s\in[g(t_i),\gamma(t_i)]} y(s)) - f^0(y(t_i), \max_{s\in[g(t_i),\gamma(t_i)]} y(s))\right]d\tau\right\| \le$$

$$\varepsilon t_i \theta(t_i) \le \tau_i \theta(\frac{\tau_i}{\varepsilon}).$$

Therefore, for any η_1 exists $\varepsilon_0(\eta_1) > 0$, such that for any $\varepsilon \le \varepsilon_0(\eta_1)$, the following inequality holds:

$$J_i^0 \le \varepsilon\left\|\int_0^{t_{i+1}} f(\tau, y(t_i), \max_{s\in[g(t_i),\gamma(t_i)]} y(s)) - f^0(y(t_i), \max_{s\in[g(t_i),\gamma(t_i)]} y(s))d\tau\right\|$$

$$+\varepsilon\left\|\int_0^{t_i} f(\tau, y(t_i), \max_{s\in[g(t_i),\gamma(t_i)]} y(s)) - f^0(y(t_i), \max_{s\in[g(t_i),\gamma(t_i)]} y(s))d\tau\right\| \le 2\eta_1. \tag{17}$$

From (4) it follows that

$$I_k \le \frac{2ML}{m}. \tag{18}$$

Using estimates (14)-(18), we get

$$I^0 \le \frac{2ML}{m}\left(\lambda\frac{L}{2} + \lambda max\left\{\omega\left(\gamma, L\right), \omega\left(g, L\right)\right\} + 1\right)$$

$$+2\varepsilon\lambda ML max\left\{\omega\left(\gamma, L\right), \omega\left(g, L\right)\right\} + 2m\eta_1(\varepsilon) = \nu(m,\varepsilon). \tag{19}$$

So, for $t \in [0,\tau]$, we can write

$$\|x(t) - y(t)\| \le 2\varepsilon\lambda\int_0^t \delta(s)ds + \nu(m,\varepsilon) \le 2\varepsilon\lambda\int_0^\tau \delta(s)ds + \nu(m,\varepsilon). \tag{20}$$

Then, using the definition of $\delta(\tau)$, we get

$$\delta(\tau) = \max_{0\le s\le\tau} \| x(s) - y(s) \| \le 2\varepsilon\lambda\int_0^\tau \delta(s)ds + \nu(m,\varepsilon). \tag{21}$$

Applying the Gronwall-Bellman lemma, we get

$$\delta(t) \le \nu(m,\varepsilon)e^{2\varepsilon\lambda t} \le \nu(m,\varepsilon)e^{2\varepsilon\lambda L} < \eta.$$

Note that by appropriate choice of sufficiently large m and sufficiently small ε, the value $\nu(m,\varepsilon)$, can be made as small as possible.

Theorem 1 is proved.

Second, we will give the proof of theorem 2. It is similarly to proof of the theorem 1, only in the estimate (14) $\Delta = t_{i+1} - t_i = T$, and Δ is not depend on ε.

So, from (14) it follows that

$$\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} \left\| f(\tau, y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s)) - F(y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s)) \right\| d\tau$$

$$\le \varepsilon\lambda M\left(\frac{\Delta}{2} + max\{\ \omega(\gamma,\Delta), \omega(g,\Delta)\ \}\right), \tag{22}$$

Similarly to the way the estimate (22) was obtained, it can be proved that

$$\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left\| f^0(y(\tau),\max_{s\in[g(\tau),\gamma(\tau)]}y(s))-F(y(t_i),\max_{s\in[g(t_i),\gamma(t_i)]}y(s))\right\| d\tau$$

$$\leq \lambda ML\left(\frac{L}{2m}+\left(\frac{1}{m}+\varepsilon\right)max\{\omega(\gamma,L),\omega(g,L)\}\right). \qquad (23)$$

Using (8), we get

$$\int_{t_i}^{t_{i+1}}\left[f(\tau,y(t_i),\max_{s\in[g(t_i),\gamma(t_i)]}y(s))-F(y(t_i),\max_{s\in[g(t_i),\gamma(t_i)]}y(s))\right] d\tau=0,$$

$$\varepsilon\left\|\int_{t_k}^{\tau}\left[f(\tau,y(\tau),\max_{s\in[g(\tau),\gamma(\tau)]}y(s))-F(y(\tau),\max_{s\in[g(\tau),\gamma(\tau)]}y(s))\right] d\tau\right\|$$

$$\leq \varepsilon 2MT$$

From (23), (24) the inequality (9) holds. Theorem 2 is proved.

3 Conclusion

The theoretical results concerning of the averaging method for the differential equation with maxima are presented. We have used two averaging schemes: complete averaging scheme and step-averaging scheme. For the second scheme the estimation of proximity of solutions are given and averaged systems are more exact.

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