Solution of fuzzy differential equations under generalized differentiability by Adomian decomposition method

T. Allahviranloo[1*], L. Jamshidi[1]

[1] Department of Mathematics, Science and Research Branch, Islamic Azad University

,Tehran, 14778, Iran

Correspondence E-mail: T. Allahviranloo, tofigh@allahviranloo.com

Abstract

Adomian decomposition method has been applied to solve many functional equations so far. In this article, we have used this method to solve the fuzzy differential equation under generalized differentiability. We interpret a fuzzy differential equation by using the strongly generalized differentiability. Also one concrete application for ordinary fuzzy differential equation with fuzzy input data are given.

Keywords: Adomian decomposition method; generalizations of the differentiability of fuzzy differential equations

1 Introduction

Usage of fuzzy differential equations is a natural way to model dynamical systems under possibilistic uncertainty. Tainty [1]. Strongly generalized differentiability was introduced in [2] and studied in [3]. we use this differentiability concept in the present paper. Adomian decomposition method has a useful feature that provides the solution in a rapid convergent power series with elegantly computable convergence of the solution.

This paper is organized as follows: In section 2 we bring some basic definitions of fuzzy subsets and distance between fuzzy numbers. In section 3 we define a fuzzy differential equations under generalized differentiability and In section 4 we discuss Adomian decomposition method. In section 5, we have one example of this method and the conclusion and future research is drawn in section 6.

2 Preliminaries

We begin this section with defining the notation we will use in the paper. Let us denote by $\mathfrak{R}_f$ the class of fuzzy subsets of the real axis $u : \mathfrak{R} \to [0,1]$, satisfying the following properties:

(i) u is normal, i.e. $\exists x_0 \in \mathfrak{R}$ with $u(x_0) = 1$;

(ii) u is convex fuzzy set (i.e. $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}, \forall t \in [0,1], x, y \in \mathfrak{R}$);

(iii) u is upper semi continuous on $\mathfrak{R}$;

(iv) $\overline{\{x \in \mathfrak{R} : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Then $\mathfrak{R}_f$ is called the space of fuzzy numbers (see e.g. [4]). Obviously $\mathfrak{R} \subset \mathfrak{R}_f$.

Here $\mathfrak{R} \subset \mathfrak{R}_f$ is understood as $\mathfrak{R} = \{\chi_{\{x\}}; \ x \text{ is usual real number}\}$. For $0 < r \leq 1$, denote $[u]^r = \{x \in \mathfrak{R}; u(x) \geq r\}$ and $[u]^0 = \overline{\{x \in \mathfrak{R}; u(x) > 0\}}$. Then

it is well-known that for any $r \in [0,1], [u]^r$ is a bounded closed interval. For $u, v \in \mathfrak{R}_f$, and $\lambda \in \mathfrak{R}$, the sum $u + v$ and the product $\lambda.u$ are defined by

$$[u + v]^r = [u]^r + [v]^r,$$

$$[\lambda.u]^r = \lambda[u]^r, \forall r \in [0,1],$$

Where

$[u]^r + [v]^r = \{x + y : x \in [u]^r, y \in [v]^r\}$ means the usual addition of two intervals (subsets) of $\mathfrak{R}$ and $\lambda[u]^r = \{\lambda x : x \in [u]^r\}$ means the usual product between a scalar and a subset of $\mathfrak{R}$ (see e.g. [4,5]).

Let $D : \mathfrak{R}_f \times \mathfrak{R}_f \to \mathfrak{R}_+ \cup \{0\}, D(u,v) = \sup_{r \in [0,1]} \max\{u_-^r - v_-^r, u_+^r - v_+^r\}$ be the Hausdorff distance between fuzzy numbers, where $[u]^r = [u_-^r, u_+^r], [v]^r = [v_-^r, v_+^r]$. The following properties are well-known (see e.g. [5,6])

$$D(u + w, v + w) = D(u,v), \forall u,v,w \in \Re_f,$$

$$D(k.u, k.v) = |k| D(u,v), \forall k \in \Re, u,v \in \Re_f,$$

$$D(u + v, w + e) \leq D(u,w) + D(v,e), \forall u,v,w,e \in \Re_f$$

and $(\Re_f, D)$ is a complete metric space.

Also are known the following results and concepts.

Definition 1. Let $F : (a,b) \to \Re_f$ and $t_\circ \in (a,b)$. We say that F is differentiable at $t_\circ$; if we have 2 forms as follows:

1- it exists an element $F'(t_0) \in \Re_f$ such that, for all h>0 sufficiently near to 0, there are $F(t_\circ + h) - F(t_\circ), F(t_\circ) - F(t_\circ - h)$ and the limits:

(1) $$\lim_{h \to 0^+} \frac{F(t_\circ + h) - F(t_\circ)}{h} = \lim_{h \to 0^+} \frac{F(t_\circ) - F(t_\circ - h)}{h} = F'(t_\circ)$$

2- It exists an element $F'(t_0) \in \Re_f$ such that, for all $h < 0$ sufficiently near to 0, there are

(2) $$\lim_{h \to 0^-} \frac{F(t_\circ + h) - F(t_\circ)}{h} = \lim_{h \to 0^-} \frac{F(t_\circ) - F(t_\circ - h)}{h} = F'(t_\circ)$$

Theorem 2.1 Let $F : (a,b) \to \Re_f$ be a function and denote $[F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$, for each $\alpha \in [0,1]$. Then

(i) If F is differentiable in the first form (1), then f_α and g_α are differentiable functions and

(3) $$[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]$$

(ii) If F is differentiable in the second form (2), then f_α and g_α are differentiable functions and

(4) $$[F'(t)]^\alpha = [g'_\alpha(t), f'_\alpha(t)]$$

Proof: See [7].

Definition 2. (see e.g. [8]). Let $x, y \in \Re_f$. If there exists $z \in \Re_f$ such that $x = y \oplus z$, then z is   called the H-difference of x and y and it is denoted by $x - y$.

Definition 3. (See e.g. [8].) A function $f : (a, b) \to \Re_f$ is called H-differentiable on $x_0 \in (a, b)$ if for

> $h > 0$ sufficiently small there exist the H-differences $f(x_0 + h) - f(x_0)$,
> $f(x_0) - f(x_0 - h)$ and an element $f'(x_0) \in \Re_f$ such that

$$(5) \qquad 0 =_{h \to 0} \lim D(\frac{f(x_0 + h) - f(x_0)}{h}, f'(x_0)) \quad =$$

$$_{h \to 0} \lim D(\frac{f(x_0) - f(x_0 - h)}{h}, f'(x_0))$$

(Here h at denominator means $\frac{1}{h} \otimes$.)

Definition 4. Let $f : (a, b) \to \Re_f$ and $x_0 \in (a, b)$. We say that f is strongly generalized differentiable at x_0, if there exists an element $f'(x_0) \in \Re_f$, such that

(i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) - f(x_0), f(x_0) - f(x_0 - h)$ and the limits (in the metric D)

$$(6) \qquad _{h \to 0} \lim \frac{f(x_0 + h) - f(x_0)}{h} = \lim \frac{f(x_0) - f(x_0 - h)}{h} = f'(x_0)$$

> or

(ii) for all $h > 0$ sufficiently small, $\exists f(x_0) - f(x_0 + h), f(x_0 - h) - f(x_0)$ and the limits

$$(7) \qquad _{h \to 0} \lim \frac{f(x_0) - f(x_0 + h)}{(-h)} = \lim \frac{f(x_0 - h) - f(x_0)}{(-h)} = f'(x_0)$$

> or

(iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) - f(x_0), f(x_0) - f(x_0 - h)$ and the limits

$$(8) \qquad _{h \to 0} \lim \frac{f(x_0 + h) - f(x_0)}{h} = \lim \frac{f(x_0 - h) - f(x_0)}{(-h)} = f'(x_0)$$

or

(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) - f(x_0 + h), f(x_0) - f(x_0 - h)$ and the limits

$$(9) \quad \lim_{h \to 0} \frac{f(x_0) - f(x_0 + h)}{(-h)} = \lim \frac{f(x_0) - f(x_0 - h)}{h} = f'(x_0)$$

3 A fuzzy differential equation under generalized differentiability

In this section we will study solutions to

$$(1) \quad y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y^{(1)} + a_0(x)y = g(x)$$

Where the $a_i(x), 0 \le i \le n-1$, and $g(x)$ are continuous on some interval I, subject to initial conditions $y(0) = \bar{\gamma}_0, y^{(1)}(0) = \bar{\gamma}_1,..., y^{(n-1)}(0) = \bar{\gamma}_{n-1}$, for fuzzy numbers $\bar{\gamma}_i, 0 \le i \le n-1$, the interval I can be $[0, T]$ for some $T > 0$ or $I = [0, \infty)$ by using the derivative in strongy generalized sense.

Consider three case while coefficient are positive and/or negative.

Case 1: $a_i \ge 0, i = 0,..., n-1$

(i) if the whole of derivatives are differentiable in the first form (1) :

$$[f^{(n)}(x, \alpha), g^{(n)}(x, \alpha)] + a_{n-1}(x)[f^{(n-1)}(x, \alpha), g^{(n-1)}(x, \alpha)] +$$

$$\ldots + a_0(x)[f(x, \alpha), g(x, \alpha) = [g(x), g(x)]$$

(ii) if the whole of derivatives are differentiable in the second form (2) :

$$[g^{(n)}(x, \alpha), f^{(n)}(x, \alpha)] + a_{n-1}(x)[g^{(n-1)}(x, \alpha), f^{(n-1)}(x, \alpha)] +$$

$$\ldots + a_0(x)[g(x, \alpha), f(x, \alpha) = [g(x), g(x)]$$

Then we have:

$$(10) \quad f^{(n)}(x,\alpha)+a_{n-1}(x)f^{(n-1)}(x,\alpha)+...+a_0(x)f(x,\alpha)=g(x)$$

$$(11) \quad g^{(n)}(x,\alpha)+a_{n-1}(x)g^{(n-1)}(x,\alpha)+...+a_0(x)g(x,\alpha)=g(x)$$

$$(12) \quad g^{(n)}(x,\alpha)+a_{n-1}(x)g^{(n-1)}(x,\alpha)+...+a_0(x)g(x,\alpha)=g(x)$$

$$(13) \quad f^{(n)}(x,\alpha)+a_{n-1}(x)f^{(n-1)}(x,\alpha)+...+a_0(x)f(x,\alpha)=g(x)$$

To obtain the Eqs. (10) and(13) and the Eqs.(11) and(12) are the same.

Case 2: $a_i \leq 0, i=0,..., n-1$

(i) if the whole of derivatives are differentiable in the first form (1) :

$$[f^{(n)}(x,\alpha), g^{(n)}(x,\alpha)]+a_{n-1}(x)[g^{(n-1)}(x,\alpha), f^{(n-1)}(x,\alpha)]+$$

$$...+a_0(x)[g(x,\alpha), f(x,\alpha)=[g(x), g(x)]$$

(ii) if the whole of derivatives are differentiable in the second form (2) :

$$[g^{(n)}(x,\alpha), f^{(n)}(x,\alpha)]+a_{n-1}(x)[f^{(n-1)}(x,\alpha), g^{(n-1)}(x,\alpha)]+$$

$$...+a_0(x)[f(x,\alpha), g(x,\alpha)=[g(x), g(x)]$$

Then we have:

$$(14) \quad f^{(n)}(x,\alpha)+a_{n-1}(x)g^{(n-1)}(x,\alpha)+...+a_0(x)g(x,\alpha)=g(x)$$

$$(15) \quad g^{(n)}(x,\alpha)+a_{n-1}(x)f^{(n-1)}(x,\alpha)+...+a_0(x)f(x,\alpha)=g(x)$$

$$(16) \quad g^{(n)}(x,\alpha)+a_{n-1}(x)f^{(n-1)}(x,\alpha)+...+a_0(x)f(x,\alpha)=g(x)$$

$$(17) \quad f^{(n)}(x,\alpha)+a_{n-1}(x)g^{(n-1)}(x,\alpha)+...+a_0(x)g(x,\alpha)=g(x)$$

To obtain the Eqs. (14) and (17) and the Eqs. (15) and (16) are the same.

Case 3: $a_i \leq 0, i=0,..,k, a_i \geq 0, i=k+1,..,n-1$

(i) if the whole of derivatives are differentiable in the first form (1) :

$$[f^{(n)}(x,\alpha),g^{(n)}(x,\alpha)]+a_{n-1}(x)[f^{(n-1)}(x,\alpha),g^{(n-1)}(x,\alpha)]+..+a_{k+1}(x)[f^{(k+1)}(x,\alpha),g^{(k+1)}(x,\alpha)]$$
$$+a_k(x)[g^{(k)}(x,\alpha),f^{(k)}(x,\alpha)]+...+a_0(x)[g(x,\alpha),f(x,\alpha)]=[g(x),g(x)]$$

(ii) if the whole of derivatives are differentiable in the second form (2) :

$$[g^{(n)}(x,\alpha),f^{(n)}(x,\alpha)]+a_{n-1}(x)[g^{(n-1)}(x,\alpha),f^{(n-1)}(x,\alpha)]+...+a_{k+1}(x)[g^{(k+1)}(x,\alpha),f^{(k+1)}(x,\alpha)]$$
$$+a_k(x)[f^{(k)}(x,\alpha),g^{(k)}(x,\alpha)]+...+a_0(x)[f(x,\alpha),g(x,\alpha)]=[g(x),g(x)]$$

Then we have:

$$(18) \quad f^{(n)}(x,\alpha)+a_{n-1}(x)f^{(n-1)}(x,\alpha)+...+a_{k+1}(x)f^{(k+1)}(x,\alpha)$$
$$+a_k(x)g^{(k)}(x,\alpha)+...+a_0(x)g(x,\alpha)=g(x)$$

$$(19) \quad g^{(n)}(x,\alpha)+a_{n-1}(x)g^{(n-1)}(x,\alpha)+...+a_{k+1}(x)g^{(k+1)}(x,\alpha)$$
$$+a_k(x)f^{(k)}(x,\alpha)+...+a_0(x)f(x,\alpha)=g(x)$$

$$(20) \quad g^{(n)}(x,\alpha)+a_{n-1}(x)g^{(n-1)}(x,\alpha)+...+a_{k+1}(x)g^{(k+1)}(x,\alpha)$$
$$+a_k(x)f^{(k)}(x,\alpha)+...+a_0(x)f(x,\alpha)=g(x)$$

$$(21) \quad f^{(n)}(x,\alpha)+a_{n-1}(x)f^{(n-1)}(x,\alpha)+...+a_{k+1}(x)f^{(k+1)}(x,\alpha)$$
$$+a_k(x)g^{(k)}(x,\alpha)+...+a_0(x)g(x,\alpha)=g(x)$$

To obtain the Eqs. (18) and (21) and the Eqs. (19) and (20) are equavalent.

4 Solution of the generalizations of the differentiability of fuzzy functions by Adomain method

In this section we solve the fuzzy differential equestions under generalized differentiability by Adomain decomposition method and restrictions of the method will be discussed.

We consider the Eqs. (3.1.1) and (3.1.2) , in section 3 and solve by Adomain decomposition method. Assume $\tilde{f}, \tilde{g}$ are fuzzy functions of the independent crisp variables x and t in Eqs. (3.1.1) and (3.1.2)

We have:

$$\tilde{f}^{(n)}(x,\alpha)+a_{n-1}(x)\tilde{f}^{(n-1)}(x,\alpha)+...+a_0(x)\tilde{f}(x,\alpha)=g(x)$$

$$\tilde{g}^{(n)}(x,\alpha)+a_{n-1}(x)\tilde{g}^{(n-1)}(x,\alpha)+...+a_0(x)\tilde{g}(x,\alpha)=g(x)$$

Definition:

$$(22) \quad \tilde{g_1}=\tilde{g}, \tilde{g_2}=\tilde{g}',...,\tilde{g_n}=\tilde{g}^{(n-1)} \quad \text{and} \quad \tilde{f_1}=\tilde{f}, \tilde{f_2}=\tilde{f}',...,\tilde{f_n}=\tilde{f}^{(n-1)}$$

(23) $f_i(0) = \gamma_{i1}(\alpha),$ and $g_i(0) = \gamma_{i2}(0), \ i = 1,2,...,n$

For solving this equation by Adomian decomposition method the equation should be in canonical form which can be derived by rewriting Eq. (22) as follows:

(24) $Lf_1^{\tilde{}} = f_2^{\tilde{}}, Lf_2^{\tilde{}} = f_3^{\tilde{}},...,Lf_n^{\tilde{}} = g(x) - a_{n-1}(x)f_n^{\tilde{}} - ... - a_0(x)f_1^{\tilde{}}$

(25) $Lg_1^{\tilde{}} = g_2^{\tilde{}}, Lg_2^{\tilde{}} = g_3^{\tilde{}},...,Lg_n^{\tilde{}} = g(x) - a_{n-1}(x)g_n^{\tilde{}} - ... - a_0(x)g_1^{\tilde{}}$

where $L_t = \dfrac{\partial}{\partial t}$, with the inverse operator $L_t^{-1} = \int_0^t (.) \ dt$.

Applying the inverse operator, we get

(26) $f_i^{\tilde{}} = f_i(0) + \int_0^x f_{i+1}^{\tilde{}}(x)dx, \ i = 1,...,n-1$

(27) $f_n^{\tilde{}} = f_n(0) + \int_0^x (g(x) - a_{n-1}(x)f_n^{\tilde{}} - ... - a_0(x)f_1^{\tilde{}})dx$

(28) $g_i^{\tilde{}} = g_i(0) + \int_0^x g_{i+1}^{\tilde{}}(x)dx,$

(29) $g_n^{\tilde{}} = g_n(0) + \int_0^x (g(x) - a_{n-1}(x)g_n^{\tilde{}} - ... - a_0(x)g_1^{\tilde{}})dx$

Adomian decomposition method considers the solutions $f^{\tilde{}}, g^{\tilde{}}$ as the sum of a series as:

(30) $f_i^{\tilde{}} = \sum_{j=0}^{\infty} f_{i,j}^{\tilde{}} \quad g_j^{\tilde{}} = \sum_{j=0}^{\infty} g_{i,j}^{\tilde{}}, i = 1,.., n$

So we can calculate the terms of $f_i^{\tilde{}} = \sum_{j=0}^{\infty} f_{i,j}^{\tilde{}} , g_j^{\tilde{}} = \sum_{j=0}^{\infty} g_{i,j}^{\tilde{}}$ term by term as long as we derive desired accuracy, the more terms the more accuracy. Therefore we have :

$$f^{\tilde{}}_{i,0} + f^{\tilde{}}_{i,1} + \ldots = \gamma_{i1}(\alpha) + \int_0^x (f^{\tilde{}}_{i+1,0} + f^{\tilde{}}_{i+1,1} + \ldots) dx,$$

$$i = 1, \ldots, n-1$$

$$f^{\tilde{}}_{n,0} + f^{\tilde{}}_{n,1} + \ldots = \gamma_{n1}(\alpha) + \int_0^x g(x) - a_{n-1}(x)(f^{\tilde{}}_{n,0} + f^{\tilde{}}_{n,1} + \ldots) -$$

$$\ldots - a_0(x)(f^{\tilde{}}_{1,0} + f^{\tilde{}}_{1,1} + \ldots) dx.$$

(31)
$$g^{\tilde{}}_{i,0} + g^{\tilde{}}_{i,1} + \ldots = \gamma_{i2}(\alpha) + \int_0^x (g^{\tilde{}}_{i+1,0} + g^{\tilde{}}_{i+1,1} + \ldots) dx, \quad i = 1, \ldots, n-1$$

(32)
$$g^{\tilde{}}_{n,0} + g^{\tilde{}}_{n,1} + \ldots = \gamma_{n2}(\alpha) + \int_0^x g(x) - a_{n-1}(x)(g^{\tilde{}}_{n,0} + g^{\tilde{}}_{n,1} + \ldots) -$$

$$\ldots - a_0(x)(g^{\tilde{}}_{1,0} + g^{\tilde{}}_{1,1} + \ldots) dx.$$

By solving (31) and (32) we can calculate the terms of series above.

Two other case are the same.

5 Examples

Example 1. Consider the following fuzzy equation with the indicated initial conditions:

(33) $y''' + y'' + y = 0$, $y(0) = 0^{\tilde{}}$, $y'(0) = 1^{\tilde{}}$, $y''(0) = 2^{\tilde{}}$.

The exact solution are $f^{\tilde{}} = \gamma_{i1}(\alpha)e^x$ and $g^{\tilde{}} = \gamma_{i2}(\alpha)e^x$ for $\alpha \in [0,1]$.

Since coefficients are positive then we have case 1. of section 3.

(34) $f'''(x,\alpha) + f''(x,\alpha) + f(x,\alpha) = 0$, $g'''(x,\alpha) + g''(x,\alpha) + g(x,\alpha) = 0$,

By Adomian decomposition method we have:

(35)
$$f^{\tilde{}}_1 = f^{\tilde{}}, f^{\tilde{}}_2 = f^{\tilde{}'}, f^{\tilde{}}_3 = f^{\tilde{}''},$$
and
$$g^{\tilde{}}_1 = g^{\tilde{}}, g^{\tilde{}}_2 = g^{\tilde{}'}, g^{\tilde{}}_3 = g^{\tilde{}''},$$
Then:

(36) $Lf^{\tilde{}}_1 = f^{\tilde{}}_2, Lf^{\tilde{}}_2 = f^{\tilde{}}_3, Lf^{\tilde{}}_3 = -f^{\tilde{}}_3 - f^{\tilde{}}_1$

$$Lg_1 = g_2, Lg_2 = g_3, Lg_3 = -g_3 - g_1,$$

Then we have:

(37)
$$\tilde{f}_1 = (0 - \alpha) + \int_0^x \tilde{f}_2 dx,$$

$$\tilde{f}_2 = (1 - \alpha) + \int_0^x \tilde{f}_3 dx,$$

$$\tilde{f}_3 = (2 - \alpha) - \int_0^x (\tilde{f}_3 + \tilde{f}_1) dx.$$

(38)
$$\tilde{g}_1 = (0 + \alpha) + \int_0^x \tilde{g}_2 dx,$$

$$\tilde{g}_2 = (1 + \alpha) + \int_0^x \tilde{g}_3 dx,$$

$$\tilde{g}_3 = (2 + \alpha) - \int_0^x (\tilde{g}_3 + \tilde{g}_1) dx.$$

Considers the sum of series as:

$$\tilde{f_i} = \sum_{j=0}^{\infty} \tilde{f_{i,j}}, \tilde{g_j} = \sum_{j=0}^{\infty} \tilde{g_{i,j}}, i = 1,2,3$$

Then

(39)
$$\tilde{f}_{2,0} + \tilde{f}_{2,1} + \ldots = 1 - \alpha + \int_0^x (\tilde{f}_{3,0} + \tilde{f}_{3,1} + \ldots) dx,$$

$$\tilde{f}_{1,0} + \tilde{f}_{1,1} + \ldots = -\alpha + \int_0^x (\tilde{f}_{2,0} + \tilde{f}_{2,1} + \ldots) dx,$$

$$\tilde{f}_{3,0} + \tilde{f}_{3,1} + \ldots = 2 - \alpha - \int_0^x (\tilde{f}_{3,0} + \tilde{f}_{3,1} + \ldots + \tilde{f}_{1,0} + \tilde{f}_{1,1} + \ldots) dx,$$

(40)
$$\tilde{g}_{1,0} + \tilde{g}_{1,1} + \ldots = \alpha + \int_0^x (\tilde{g}_{2,0} + \tilde{g}_{2,1} + \ldots) dx,$$

$$g\tilde{}_{2,0} + g\tilde{}_{2,1} + \ldots = 1 + \alpha + \int_0^x (g\tilde{}_{3,0} + g\tilde{}_{3,1} + \ldots) dx,$$

$$g\tilde{}_{3,0} + g\tilde{}_{3,1} + \ldots = 2 + \alpha - \int_0^x (g\tilde{}_{3,0} + g\tilde{}_{3,1} + \ldots + g\tilde{}_{1,0} + g\tilde{}_{1,1} + \ldots) dx,$$

Then we obtain

(41)
$$f\tilde{}_{1,0} = -\alpha, \; f\tilde{}_{2,0} = 1 - \alpha, \; f\tilde{}_{3,0} = 2 - \alpha, \text{ and}$$
$$g\tilde{}_{1,0} = \alpha, \; g\tilde{}_{2,0} = 1 + \alpha, \; g\tilde{}_{3,0} = 2 + \alpha,$$

(42)
$$f\tilde{}_{1,n+1} = \int_0^x f\tilde{}_{2,n} dx,$$

$$f\tilde{}_{2,n+1} = \int_0^x f\tilde{}_{3,n} dx,$$

$$f\tilde{}_{3,n+1} = -\int_0^x (f\tilde{}_{3,n} + f\tilde{}_{1,n}) dx, n = 0,1,\ldots$$

$$g\tilde{}_{1,n+1} = \int_0^x g\tilde{}_{2,n} dx,$$

$$g\tilde{}_{3,n+1} = -\int_0^x (g\tilde{}_{3,n} + g\tilde{}_{1,n}) dx, n = 0,1,\ldots \; g\tilde{}_{2,n+1} = \int_0^x g\tilde{}_{3,n} dx,$$

For $n = 0$ we obtain:

(43)
$$f\tilde{}_{1,1} = \int_0^x f\tilde{}_{2,0} dx = \int_0^x (1 - \alpha) dx = (1 - \alpha)x,$$

$$f\tilde{}_{2,1} = \int_0^x f\tilde{}_{3,0} dx = \int_0^x (2 - \alpha) dx = (2 - \alpha)x,$$

$$f\tilde{}_{3,1} = -\int_0^x (f\tilde{}_{3,0} + f\tilde{}_{1,0}) dx = -\int_0^x (2 - \alpha - \alpha) dx = -(2 - 2\alpha)x,$$

(44)
$$g\tilde{}_{1,1} = \int_0^x g\tilde{}_{2,0} dx = \int_0^x (1 + \alpha) dx = (1 + \alpha)x,$$

$$g\tilde{}_{2,1} = \int_0^x g\tilde{}_{3,0}dx = \int_0^x (2+\alpha)dx = (2+\alpha)x,$$

$$g\tilde{}_{3,1} = -\int_0^x (g\tilde{}_{3,0} + g\tilde{}_{1,0})dx = -\int_0^x (2+\alpha+\alpha)dx = -(2+2\alpha)x,$$

For $n=1$ we obtain:

(45)
$$f\tilde{}_{1,2} = \int_0^x f\tilde{}_{2,1}dx = \frac{2-\alpha}{2}x^2,$$

$$f\tilde{}_{2,2} = \int_0^x f\tilde{}_{3,1}dx = \frac{-(2-2\alpha)}{2}x^2,$$

$$f\tilde{}_{3,2} = \frac{1-\alpha}{2}x^2,$$

(46)
$$g\tilde{}_{1,2} = \frac{2+\alpha}{2}x^2,$$

$$g\tilde{}_{2,2} = \frac{-(2+2\alpha)}{2}x^2,$$

$$g\tilde{}_{3,2} = \frac{1+\alpha}{2}x^2,$$

Since $f_1\tilde{} = f\tilde{}$, $g_1\tilde{} = g\tilde{}$ and according to generalized differentiability $[y(t)]_\alpha = [f(t,\alpha), g(t,\alpha)] = [f\tilde{}, g\tilde{}]$ we have:

(47)
$$f\tilde{} = f_1\tilde{} = \sum_{j=0}^\infty f\tilde{}_{1,j} = f\tilde{}_{1,0} + f\tilde{}_{1,1} + ... = -\alpha + (1-\alpha)x + \frac{2-\alpha}{2}x^2 + ...$$

$$g\tilde{} = g_1\tilde{} = \sum_{j=0}^\infty g\tilde{}_{1,j} = g\tilde{}_{1,0} + g\tilde{}_{1,1} + ... = \alpha + (1+\alpha)x + \frac{2+\alpha}{2}x^2 + ...$$

In this example we have derived the exact solution.

Example 2.   Consider the following fuzzy equation:

(48) $y'' - y' + y = x$,
       $y(0) = 0\tilde{}$, $y'(0) = 1\tilde{}$

Solve the fuzzy differential equation under generalized differentiability by apply Adomian decomposition method.

We have:

(49) $f'' - g' + f = x,$

       $g'' - f' + g = x,$

       $\tilde{f}_1 = f, \tilde{f}_2 = f'$ and $\tilde{g}_1 = g, \tilde{g}_2 = g',$

By apply Adomian decomposition method on above equations, we have:

(50) $\tilde{f}_{1,0} = -\alpha, \tilde{f}_{2,0} = 1 - \alpha, \tilde{g}_{1,0} = \alpha, \tilde{g}_{2,0} = 1 + \alpha,$

(51)
$$\tilde{f}_{1,n+1} = \int_0^x \tilde{f}_{2,n}dx, \tilde{f}_{2,n+1} = \int_0^x (x + \tilde{g}_{2,n} - \tilde{f}_{1,n})dx,$$

$$\tilde{g}_{1,n+1} = \int_0^x \tilde{g}_{2,n}dx, \tilde{g}_{2,n+1} = \int_0^x (x + \tilde{f}_{2,n} - \tilde{g}_{1,n})dx, n = 0,1,2,...$$

We obtain:

$$\tilde{f}_1 = \tilde{f} = \tilde{f}_{1,0} + \tilde{f}_{1,1} + \tilde{f}_{1,2} + ...$$

$$= x + \frac{x^2}{2} - \frac{x^3}{6} - \alpha(1+x) + ...$$

(52) $\tilde{g}_1 = \tilde{g} = \tilde{g}_{1,0} + \tilde{g}_{1,1} + \tilde{g}_{1,2} + ...$

$$= x + \frac{1-2\alpha}{2}x^2 + \frac{x^3}{6} + \alpha(1+x) + ...$$

6 Conclusions and future research

In this paper we presented the generalized differentiability for the n-th order, linear, differential equation having fuzzy initial conditions. We apply Adomian decomposition method.

This method has a useful feature in that it provides the solution in a rapid convergent power series with elegantly computable convergence of the solution.

So, future research will be concerned with fuzzy a_i, and fuzzy initial conditions.

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