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Non-Dominated DEA Cross Efficiency Scores; A Secondary Goal Approach

Saeid Shahghobadi[1*], Abbas Ghomashi[1] and Farhad Moradi [2]

[1]*Department of Applied Mathematics, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran

[2]Department of Applied Mathematics, Sanandaj Branch, Islamic Azad University, Sanandaj, Iran

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Abstract

Data envelopment analysis (DEA) is a non-parametric programming method for evaluating the relative efficiency of a set of peer decision-making units (DMUs) with multiple inputs and multiple outputs. The DEA cross-efficiency method is a well-known method that use to evaluate and ranking a set of peer decision-making units. Whenever a DMU intends to evaluate other DMUs, it faces the problem of non-uniqueness optimal weights of DEA models. Because different weights give us different cross-scores and subsequently different cross-efficiencies scores and this will confuse the decision-maker to make an ultimate decision. The main drawback of this method is the alternate optimal solution set of the DEA model. The main purpose of this study is to propose an approach to this problem to generate non-dominated DEA cross-efficiency scores. We propose a linear programming secondary goal model to select a set of optimal weights for each DMU. Our proposed method is not only simpler than other methods presented with the same purpose, but also does not go beyond the main method.

Keywords:
Data envelopment analysis (DEA)
Cross-efficiency evaluation
Multi-objective optimization

***Correspondence E-mail**: s.shahghobadi@gmail.com

INTRODUCTION

Data envelopment analysis (DEA) is a powerful methodology for evaluating relative efficiency (RE) of a finite set of peer multi-input and multi-output decision-making units (DMU). It was introduced by Charnes, Cooper et al. (1978) for the first time. Since the advent of DEA in 1978, there has been an impressive growth both in theoretical developments and applications of the ideas to practical situations (Cook and Seiford 2009). DEA has two well-known weaknesses. First, its models run individually for each DMU under flexibility in selecting weights in the best advantage. This flexibility in choosing the weights, on the other hand, deters the comparison among DMUs on a common base (Zohrehbandian, Makui et al. 2010, Soltanifar and Shahghobadi 2013, Soltanifar and Shahghobadi 2014, Shahghobadi 2020). Second, DEA can classify DMUs as efficient or inefficient. However, there is often more than one efficient unit, and, in other words, it has a weak discrimination power. Due to the high importance of these two problems, they have received considerable critical attention from researchers. Aldamak and Zolfaghari (2017) reviewed and classified the literature on the ranking of DEA include those approaches published up to 2016.

An innovative approach to solve these issues is using a common set of weights (CSW), which was proposed by Roll, Cook et al. (1991). Chen, Larbani et al. (2009) outlines three essential features of this method; reduce computational complexity/time compared with the traditional model, strong theoretical background, and higher discrimination power. Another essential feature is to provide a common base for ranking the DMUs, both the efficient and inefficient ones (Kao and Hung 2005). A mainstream of researches agreed to use p -distance measure to derive a CSW in DEA (Lotfi, Ebrahimnejad et al. , Roll, Cook et al. 1991, Kao and Hung 2005, Chen, Larbani et al. 2009, Zohrehbandian, Makui et al. 2010, Hosseinzadeh Lotfi, Jahanshahloo et al. 2013, Hosseinzadeh Lotfi, Jahanshahloo et al. 2013, Sun, Wu et al. 2013, Pourhabib Yekta, Kordrostami et al. 2018, Izadikhah and Farzipoor Saen 2019). The idea behind these models is to find a CSW such that its corresponding efficiency vector is closet to a predefined target efficiency vector measured with p -distance measure. This idea proposed by Kao and Hung (2005) for the first time and followed by Chen, Larbani et al. (2009), (Zohrehbandian, Makui et al. 2010). Here, we focus on these models and refer them as p -distance-based CSW models.

A CSW is used to approximate a target efficiency score. Since the generated efficiency score has an essential role in evaluating and ranking the DMUs, it is crucial to the generated efficiency score of a DMU as close as possible to its target efficiency score. Roll, Cook et al. (1991) also believed that a general requirement of such a set (CSW) is that it explains as high a portion as possible of DMU performance. As a summary, a CSW must be able to approximate the efficiency target of DMUs with the least individuals and overall deviations.

This study aims to examine p -distance-based CSW models from deviations aspect. The key question is, what the main factors are contributing to the reduction of individuals and overall deviations of generated efficiency scores from corresponding target efficiency targets? We found that parameter p and the data set have a direct and significant impact on the variations. We have two simple suggestions for improving the performance of existing models to reduce deviations. Numerical examples and a simulation test revealed that our proposal was very effective in reducing deviations.

The remained of the paper is structured as follows. In section 2, we review the DEA and distance-based CSW models. In section 3, we describe our motivation by a simple numerical example. In section 4, the adjusted models are presented. In section 5, we compare the proposed models with the previous ones via two numerical examples. Also, it contains the result of the correlation test and simulation analysis. Finally, section 7 concludes the study and provides directions for further works.

PRELIMINARIES

First of all, we declare the notations used in this study. the n -dimensional euclidean space is denoted by R^n and non-negative orthant denoted by R_+^n. We symbolize the sets by capital letters, set members by lower-case letters, vectors, and matrices in bold letters: vectors in lower case and matrices in upper case. All vectors are column vectors and the transpose of vectors and matrices displayed by a superscript T. We also use $\mathbf{0}_n$ and $\mathbf{1}_n$ to show n-dimensional vectors with zero and one component, respectively. Also, the set of natural numbers from 1 to n is displayed by $[n]$. Furthermore, the superscript for a variable shows the optimal value of the variable.

DEA

Consider n DMUs with m inputs and s outputs. Let $x_j = (x_{1j}, \dots, x_{mj})^T$ and $y_j = (y_{1j}, \dots, y_{sj})^T$ denote the input and the output vectors of DMU*j* for $j \in [n]$. One version of the CCR model is a two-phase CCR model which is as follows (Charnes, Cooper et al. 1978) :

Phase I

Solve the envelopment form of the CCR model :

$$
\begin{aligned}
& e_o^* = Min\ e \\
& s.t: \\
& \sum_{j\in[n]} \lambda_j x_j + s^- = e x_o \quad i \in [m] \\
& \sum_{j\in[n]} \lambda_j y_j - s_r^+ = y_r \quad r \in [s] \\
& \lambda_j \geq 0, j \in [n]
\end{aligned} \tag{1}
$$

where $s^+ \in R^s$ and $s^- \in R^m$ are the output *shortfalls* and the input *excesses*, respectively.

Phase II

Use e_o^* and solve the following model:

$$
\begin{aligned}
& w_o^* = Max(\mathbf{1}_m^t \boldsymbol{s}^- + \mathbf{1}_s^t \boldsymbol{s}^+) \\
& s.t: \\
& \sum_{j\in[n]} \lambda_j x_j + s^- = e_o^* x_o \quad i \in [m] \\
& \sum_{j\in[n]} \lambda_j y_j - s_r^+ = y_r \quad r \in [s] \\
& \lambda_j \geq 0, j \in [n]
\end{aligned} \tag{2}
$$

Definition 2.1 (Cooper, Seiford et al. 2007) *Let e_o^* and w_o^* are optimal values models* (1) *and* (2), *respectively.*

• DMU*o* is CCR efficient if and only if $e_o^* = 1$ and $w_o^* = 0$.

• DMU*o* is *radial* CCR efficient if $e_o^* = 1$.

• Otherwise, the DMU*o* is called CCR-inefficient.

Let (e_o^*, s^{-*}, s^{+*}) is an optimal solution of the model (1). The *radial projection* and the *CCR projection* of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ are defined equations (3) and (4), respectively as follows :

$$
\begin{cases} \bar{\boldsymbol{x}}_j = e_o^* \boldsymbol{x}_o \\ \bar{\boldsymbol{y}}_o = \boldsymbol{y}_o \quad j \in [n] \end{cases} \tag{3}
$$

$$
\begin{cases} \hat{\boldsymbol{x}}_j = e_o^* \boldsymbol{x}_o - \boldsymbol{s}^{+*} \\ \hat{\boldsymbol{y}}_o = \boldsymbol{y}_o - \boldsymbol{s}^{-*} \quad j \in [n] \end{cases} \tag{4}
$$

CSW

Charnes, Cooper et al. (1978) defined the *efficiency function* as $e_j: R_+^m \times R_+^s \to R_+$ and $e_j(u,v) = \frac{u^t y_j}{v^t x_j}$, for DMU$j$. It has showed that DMU*o* is CCR efficient if and only if the optimal value of the following fractional form of the CCR model equals to 1. See (Charnes, Cooper et al. 1978) for details.

$$
\begin{aligned}
& e_k^* = Max \frac{u^t y_k}{v^t x_k} \\
& S.t: \\
& \frac{u^t y_j}{v^t x_j} \leq 1 \qquad j \in [n] \\
& u \geq \varepsilon 1_s, v \geq \varepsilon 1_m
\end{aligned} \tag{5}
$$

Where u and $\boldsymbol{v}$ can be interpreted as the virtual cost and price vectors of inputs and outputs, respectively, and ε is a small non-Archimedes quantity. See (Amin and Toloo 2004) for more details.

Kornbluth (1991) integrated all individual DEA models in one multi-objective linear fractional programming as follows. It is the basic model for generation CSWs.

$$
\begin{aligned}
&Maxe(u,v) = \left(\frac{u^t y_1}{v^t x_1}, \dots, \frac{u^t y_n}{v^t x_n}\right)\\
&S.t:\\
&\frac{u^t y_j}{v^t x_j} \le 1 \quad j \in [n]\\
&u \ge \varepsilon 1_s, v \ge \varepsilon 1_m
\end{aligned}
\qquad (6)
$$

Kao and Hung (2005) considered the CCR efficiency vector, $e^* = (e_1^*, \dots, e_n^*)$, as the target efficiency vector and looked for a CSW such that its corresponding efficiency vector has the minimum $p - distance$ from e^* as follows:

$$
\begin{aligned}
&Min\left(\sum_{j\in[n]} \left(\frac{u^t y_j}{v^t x_j} - e_j^*\right)^p\right)^{\frac{1}{p}}\\
&S.t:\\
&\frac{u^t y_j}{v^t x_j} \le 1 \quad j \in [n]\\
&u \ge \varepsilon 1_s, v \ge \varepsilon 1_m
\end{aligned}
$$

Chen, Larbani et al. (2009) defined $g_k(u,v) := v^t x_k - u^t y_k$ as a basis efficiency measure and 0_n as the target efficiency vector and with the same strategy as Kao and Hung (2005) suggested the following model:

$$
\begin{aligned}
&Min\left(\sum_{j\in[n]} (v^t x_j - u^t y_j)^p\right)^{\frac{1}{p}}\\
&S.t:\\
&v^t x_j - u^t y_j \ge 0 \quad j \in [n]\\
&u \ge \varepsilon 1_n, v \ge \varepsilon 1_n
\end{aligned}
\qquad (8)
$$

Zohrehbandian, Makui et al. (2010) integrated the (Kao and Hung 2005) idea with another innovative idea that was using of *radial projected* data set instead of the original data set, and then proposed the following model:

$$
\begin{aligned}
&Min\left(\sum_{j\in[n]} (v^t(\boldsymbol{e}_j^* x_j) - u^t y_j))^p\right)^{\frac{1}{p}} \qquad (9)\\
&S.t:\\
&v^t(\boldsymbol{e}_j^* x_j) - u^t y_j \ge 0 \quad j \in [n]\\
&\mathbf{1}_s^t \boldsymbol{u} + \mathbf{1}_m^t \boldsymbol{v} = 1\\
&u \ge 0_s, v \ge 0_m
\end{aligned}
$$

We will see how much is effective in reducing deviations the idea of using the projected data set. We call each one of these models by the name of the first author of the corresponding paper.

MOTIVATION

It is easy to verify that these models are specific states of the following more general model.

$$
\begin{aligned}
&Min\left(\sum_{j\in[n]} (r_j)^p\right)^{\frac{1}{p}}\\
&S.t:\\
&t_j - d_j(u,v) = r_j \quad j \in [n]\\
&(u,v) \in W r_j \ge 0, j \in [n]
\end{aligned}
$$

where:

1. *efficiency measure* $d_j: R^s \times R^m \to R$: (7)

$$
d_j(u,v) := \begin{cases} \dfrac{u^t y_j}{v^t x_j}, & in\ model\ (7)\\ u^t y_j - v^t x_j, & in\ model\ (8)\\ u^t y_j - v^t x_j, & in\ model\ (9) \end{cases}
$$

2. *target* efficiency score $t_j \in R$

$$
t_j(u,v) := \begin{cases} e_{j^*}, & in\ model\ (7)\\ 0, & in\ model\ (8)\\ 0, & in\ model\ (9) \end{cases}
$$

3. *residual* variable r_j

$$
r_j(u,v) := t_j - ef_j
$$

4. Weight restriction set and r_j

$$
W := \begin{cases} w_1 = \{(u, v): u \ge \epsilon 1_s, v \ge \epsilon 1_m\} & in\ model\ (8)\\ w_2 = \{(u, v): u \ge \epsilon 1_s, v \ge \epsilon 1_m\} & in\ model\ (9)\\ \vdots & \vdots\\ w_3 = \{(u, v): 1_s^t u + 1_m^t v = 1, u \ge 0_s, v \ge 0_m\} & in\ model\ (10) \end{cases}
$$

and r_j

Note that we can consider $D' = \sum_{k=1}^{n} (r_j)^p$ instead of $D = (\sum_{k=1}^{n} r_j)^p)^{\frac{1}{p}}$ which is more straightforward in its mathematical expression, and a solution (u, v) minimizes D_p if, and only if, it minimizes D'_p (Kao and Hung 2005).

The ideal value for r_j is zero, and model(10) uses the *penalty function* r_j^p to achieve this goal. Regarding the strictly increasing property and the shape of these functions for $p > 1$, they impose a heavier penalty for *residual*s larger than one compared to *residual*s less than one, and the intensity of this operation increases, if p increases. This leads to rotating the solution towards DMUs with large *residual*s and moving away from DMUs with small *residual*s. In short, r_j^p with $p > 1$ will disrupt the balance and the principle of neutrality in favor of some units; hence, it can be said that these functions have a biased operation. This, together with the fact that the problem for $p = 1$ will tend to have more zero and very small residuals, [4, Page 311], gives us the expectation that the problem for $p = 1$ generate more better solution. To clarify, we explain these cases by a numerical example.

Example 3.1 *In this example, at first, we compare the obtained result from the mentioned models for $p = 1$ and 2. Let we take six DMUs with two inputs and one output as follows:*

$$\boldsymbol{X1} = \begin{pmatrix} 1 & 2 & 2 & 4 & 10 & 2 \\ 4 & 1.75 & 3.5 & 1 & 3 & 4 \end{pmatrix}$$

and

$$\boldsymbol{Y1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The production possibility set (Cooper, Seiford et al. 2007) is depicted in Fig. 1. We employed Model Kao, Model Zohrebandian, and Model Chen with $p = 1$ and $p = 2$, and displayed obtained CSWs and efficiency vectors in Table 2,respectively.

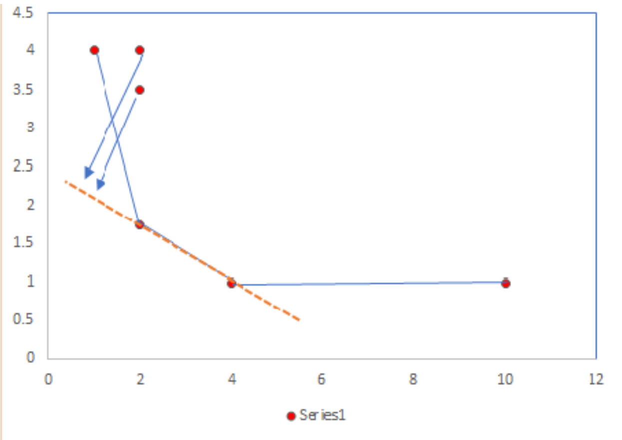

Fig. 1. Farrell frontier for data of example 3.1

Table 1: Generated CSWs by different models

	Chen		Zohrehbandian		Kao	
	$p = 1$	$p = 2$	$p = 1$	$p = 2$	p=1	p=2
v_1	0.000000000001	0.000000000004	0.000000236842	0.000000175448	0.035397000000	3.200400000000
v_2	0.000000000001	0.000000000008	0.000000105263	0.000000172239	0.015848000000	3.200400000000
u	0.000000000003	0.000000000013	0.000000657895	0.000000652313	0.098307000000	3.200400000000

Table 2: Generated efficiency scores by different models with p=1 and p=2

	Chen	Zohrehbandian	Kao	CCR

	$p=1$	$p=2$	$p=1$	$p=2$	p=1	p=2	
DMU_1	0.55	0.3575	1	0.7546	0.9951	0.9297	1
DMU_2	0.7333	0.6128	1	1	0.9978	0.6658	1
DMU_3	0.5	0.3648	0.7812	0.684	0.7786	0.5939	0.781
DMU_4	0.55	0.5957	0.625	0.7463	0.6244	0.3644	1
DMU_5	0.2115	0.2215	0.2451	0.2872	0.2448	0.1447	0.370
DMU_6	0.4583	0.327	0.7353	0.6273	0.7326	0.5761	0.735

Table 3: Deviations of generated efficiency scores from CCR efficiency scores

	Chen		Zohrehbandian		Kao	
	$p=1$	$p=2$	$p=1$	$p=2$	p=1	p=2
$e_1^* - e_1^{CSW}$	0.45	0.6425	0	0.2454	0.0049	0.2371
$e_2^* - e_2^{CSW}$	0.2667	0.3872	0	0	0.0022	0.0042
$e_3^* - e_3^{CSW}$	0.2812	0.4165	0	0.0973	0.0026	0.095
$e_4^* - e_4^{CSW}$	0.45	0.4043	0.375	0.2537	0.3756	0.2655
$e_5^* - e_5^{CSW}$	0.1588	0.1488	0.1253	0.0832	0.1255	0.0873
$e_6^* - e_6^{CSW}$	0.277	0.4083	0	0.108	0.0027	0.105
overall deviation	1.8837	2.4076	0.5003	0.7875	0.5135	0.7941

The deviation of each obtained efficiency score from the corresponding CCR efficiency score is displayed in Table 3.

Regarding Table 2, the results of each model were more desirable with $p=1$ than $p=2$.

Now, we consider models Zohrebandian and Chen with $p=1$. However, these models use the same objective function and efficiency measure; the Zohrebandian model produced a more desirable solution. To find out the cause of this superiority, we focus on their differences; weight restriction set and the data set. It is easy to see that $W_2 \subseteq W_3$ Moreover, this is effective in improving the optimal value. Hence, only one factor remains. For this reason, we have $e_j^* v x_j - u y_j \leq v x_j - u y_j$ for an arbitrary feasible (u, v) and any $j \in [n]$. Equivalently, the residual variables have smaller values in the Zohrebandian model compared with the Chen model, and this reduces the leverage effect of r_j^p. Besides, the projected data is more homogeneous than the original data, and it seems more reasonable to fit them an efficient frontier. Therefore, using a projected data set can be the most effective factor for this superiority.

To see how much-projected data is effective in reducing deviation, imagine a case that there is a weak efficient DMU among observed units. Let we change the input vector of DMU5 changes to $\binom{10}{1}$ in $\boldsymbol{X1}$, and $Y1$ remains unchanged. The new point is displayed by 5^* in Fig. 1. It is on the weak efficient frontier, and when it coincides with its radial projection. Also, the new input matrix is denoted by X_2 and we have:

$$\boldsymbol{X1} = \begin{pmatrix} 1 & 2 & 2 & 4 & 10 & 2 \\ 4 & 1.75 & 3.5 & 1 & 3 & 4 \end{pmatrix}$$

By reusing again of the Zohrebandian model, the generated CSW changes from $(0.2368, 0.1052, 0.6578)^T$ to

$(0.0968, 0.2581, 0.6452)$, the efficiency vector changes from $(1,1,0.781,0.625,0.245,0.733)^t$ to $(0.571,1,0.581,1,0.526,0.526)^t$, the individual deviation vector changes from $(0,0,0,0.375,0.1253,0)$ to$(0.4286,0,0.193,0,0.4737,0.209)$ and subsequently the overall deviation changes from 0.5 to 1.3042. Briefly, this change leads to rotate the common efficient frontier from $1-2$ line toward the dash line as displayed in Fig. 1 .

AN IMPROVEMENT

As we saw, using $p = 1$ and radial projected data set are effective strategies to produce efficiency scores with a small deviation from CCR efficiency scores. To more reduce residuals, we proposed to use CCR projected data instead of radial projected data in Chen and Zohrebandian models with $p = 1$. By taking $p = 1$ and using (x_j, y_j), these models are as follows, respectively:

$$\begin{aligned} &Min \sum_j r_j \\ &S.t: \\ &v^t\widehat{\boldsymbol{x}}_j - u^t\widehat{\boldsymbol{y}}_j - r_j = 0 \quad j \in [n] \\ &u \geq \varepsilon 1_n, v \geq \varepsilon 1_n \end{aligned}$$

$$\begin{aligned} &Min \sum_j r_j \\ &S.t: \\ &v^t\widehat{\boldsymbol{x}}_j - u^t\widehat{\boldsymbol{y}}_j - r_j = 0 \quad j \in [n] \\ &1_s^t u + 1_m^t v = 1, u \geq 0_s, v \geq 0_m \end{aligned}$$

Because $W_2 \subseteq W_3$ for each ε, so model (11) has fewer residuals than the model (12). In spit of this priority of W_3, it doesn't guaranty the generated CSW to be positive.

Theorem 4.1 *let* (u, v, r) *is an optimal solution for* (11)*. If* $r_o = 0$ *for some* $o \in [n]$*, then*

$$\frac{u^t y_o}{v^t x_o} = \theta_o^* - \varepsilon(1_s^t s_{-*} + 1_m^t s_{+*})$$

where $(\theta^*, s^{-*}, s^{+*})$ is an optimal solution.

Proof. First, we have

$$u^t y_o - v^t x_o = 0$$

Next, consider the dual problem.

$$\begin{aligned} &Max\ u^t y_o \\ &v^t x_o = 1 \\ &S.t: \\ &v^t x_j - u^t y_j - r_j = 0 \quad j \in [n] \\ &u \geq \varepsilon 1_n, v \geq \varepsilon 1_n \end{aligned} \qquad (13)$$

It is easy to see that the vector $\left(u = \frac{u}{vx_o}, v = \frac{v}{vx_o}\right)$, is an optimal solution to this problem. If $(\lambda^*, s^{-*}, s^{+*})$, is also an optimal solution for (11), then we have from Complementary Slackness Theorem [3, Section 6] (Cooper, Seiford et al. 2007) that:

$$\begin{aligned} &(u - \varepsilon 1_s)^t s^{-*} = 0 \\ &(v - \varepsilon 1_m)^t s^{+*} = 0 \end{aligned} \qquad (14)$$

or

$$\begin{aligned} &u^t s^{-*} = \varepsilon 1_s^t s^{-*} \\ &v^t s^{+*} = \varepsilon 1_m^t s^{+*} \end{aligned} \qquad (15) \quad (11)$$

Then

$$u^t(y_o + s_s^{-*}) - v^t(\hat{e}_o^* x_o + s_s^{-*}) = 0$$

and

$$\frac{u^t y_o}{v^t x_o} = \hat{e}_o^* - \frac{u^t s_s^{-*} + v^t s_s^{+*}}{v^t x_o} \qquad (12)$$

This together (15) give

$$\frac{u^t y_o}{v^t x_o} = \theta_o^* - \varepsilon(1_s^t s_{-*} + 1_m^t s_{+*})$$

The above Theorem states that DMUo is CCR efficient, if $\bar{r}_o = 0$ in an optimal solution, (u, v, r), of the model (11).

NUMERICAL EXAMPLE

This section contains two numerical examples that are brought to compare the proposed models with other mentioned distance-based CSW models.

Example 5.1 This example compares the proposed models to the former ones based on the numerical example we had at the Motivation section. Consider again X1 and Y1 as they were. At first, we projected data according to the four equations, and then employed the proposed models, and they are prior once. The obtained CSWs and their efficiency scores are displayed in Table 4 and Table 6, respectively. By comparing columns 1 and 3 with the CCR column in Table 5, we can see that model (11) has succeeded in reducing the individuals and overall deviations compared to the Chen model.

Next, we consider the case of the example that the Zohrebandian model got into trouble with it; $X2$ and $Y1$. After projecting data and employing the proposed models, the obtained CSWs and efficiency scores are shown in Table 6 and Table 7 Interestingly, the adjusted Zohrehbandian model, model, could prevent of rotation answer toward to DMU5* and could reduce deviations again.

Table 4: Data for example 5.1

	Chen,p=1	Zohrehbandian,p=1	Model (11)	Model (12)
v_1	0.0001	0.236842	0.0001	0.236842
v_2	0.0001	0.105263	0.0001	0.105263
u	0.000275	0.657895	0.000375	0.657895

Table 5: Efficiency scores of the proposed models and their prior ones for X1 and Y1

	Chen, p=1	Zohrehbandian, p=1	Model (11)	Mode **(12)**	CCR
DMU1	0.55	1	0.75	1	1
DMU2	0.733	1	1	1	1
DMU3	0.5	0.781	0.682	0.781	0.781
DMU4	0.55	0.625	0.75	0.625	1
DMU5	0.212	0.245	0.288	0.245	0.3704
DMU6	0.458	0.735	0.625	0.735	0.7353
Overall deviation	1.88371	0.50027	0.79163	0.50027	

Table 6: Generated CSWs by different models

	Chen,p=1	Zohrehbandian,p=1	Model (11)	Model (12)
v_1	0.0001	0.096774	0.0001	0.236842
v_2	0.0001	0.258065	0.0001	0.105263
u	0.000275	0.645161	0.000375	0.657895

Table 7: Efficiency scores of the proposed models and their prior one for X2 and Y1

	Chen,p=1	Zohrehbandian,p=1	Model (11)	Mode (12)	CCR

DMU1	0.55	0.571	0.750	1	1
DMU2	0.733	1	1	1	1
DMU3	0.5	0.588	0.682	0.781	0.781
DMU4	0.55	1	0.75	0.625	1
DMU5	0.25	0.526	0.341	0.245	1
DMU6	0.458	0.526	0.625	0.735	0.735
Overall deviation	2.47488	1.30425	1.36822	1.10844	

Example 5.2 In this example, we examine a numerical example that came in (Kao and Hung 2005). It contains seventeen DMUs with four inputs and three outputs. Data and attained results are exhibited from Table 8 to Table 10.

This example revealed two points. First, the proposed models, model (12) and model (11), could obtain the first and the second places, respectively, among all CSW models from generating the most similar approximation for CCR efficiency scores point of view. In addition, their efficiency vectors have the highest number of zero residuals. Second, model (12) as expected generated a zero input weight; however, all components of the CSW derived from model (11) were positive.

Table 8: Efficiency scores of the proposed models and their prior one for X2 and Y1

	Inputs				Outputs		
DMUs	i_1	i_2	i_3	i_4	o_1	o_2	o_3
DMU1	51.62	11.23	49.22	33.52	40.49	14.89	166.71
DMU2	85.78	123.98	55.13	108.46	43.51	173.93	6.45
DMU3	66.65	104.18	257.09	13.65	139.74	115.96	0
DMU4	27.87	107.6	14	146.43	25.47	131.79	0
DMU5	51.28	117.51	32.07	84.5	46.2	144.99	0
DMU6	36.05	193.32	59.52	8.23	46.88	190.77	822.92
DMU7	25.83	105.8	9.51	227.2	19.4	120.09	0
DMU8	123.02	82.44	87.35	98.8	43.33	125.84	404.69
DMU9	61.95	99.77	33	86.37	45.43	79.6	52.62
DMU10	80.33	104.65	53.3	79.06	27.28	132.49	42.67
DMU11	205.92	183.49	144.16	59.66	14.09	196.29	16.15
DMU12	82.09	104.94	46.51	127.28	44.87	108.53	0
DMU13	202.21	187.74	149.39	93.65	44.97	184.77	0
DMU14	67.55	82.83	44.37	60.85	26.04	85	23.95
DMU15	72.6	132.73	44.67	173.48	5.55	135.65	24.13
DMU16	84.83	104.28	159.12	171.11	11.53	110.22	49.09
DMU17	71.77	88.16	69.19	123.14	44.83	74.54	6.14

Table 9: Generated CSWs by different models

	Chen,p=1	Zohrebandian, p=1	Model (11)	Mode (12)	Kao,p=1	Wang 1	Wang 2

v_1	0.0001	0.097976	0.006946	0.200409	959.4586	1591.678	0.000657
v_2	0.0001	0.372243	0.011358	0.346254	2313.361	4931.297	0.003527
v_3	0.006333	0	0.0001	0	0.001505	3807.445	0.000225
v_4	0.000828	0.125069	0.001511	0.03428	0.00168	1939.971	0.000516
u_1	0.0001	0.01455	0.001061	0.066351	531.8955	3551.854	0.000321
u_2	0.001675	0.38793	0.012227	0.350344	1988.85	7044.92	0.003639
u_3	0.0001	0.002231	0.0001	0.002362	7.691254	56.7372	2.38E-05

Table 10: Efficiency scores of the proposed models and their prior one

	Chen,p=1	Zohrebandian,p=1	Model (11)	Mode (12)	Kao,p=1	Wang 1	Wang 2
DMU1	0.999711	1	1	1	1	1.098501	1.398082
DMU2	.644462	1	1	1	1	1.181471	1.151319
DMU3	.125715	1	.925255	1	1	0.808081	0.980557
DMU4	0.999552	0.842843	1	1	1	1.116814	1.023499
DMU5	0.853745	0.95927	1	1	0.974747	1.251716	1.087137
DMU6	0.999754	1	1	0.965676	0.852368	1.242465	1.007855
DMU7	0.776811	0.666414	0.862937	0.874221	0.9244	0.879898	0.869819
DMU8	0.389958	0.91384	0.833837	0.846816	0.895351	0.943903	1.088737
DMU9	0.887039	0.635705	0.675601	0.678267	0.661897	0.897495	0.751314
DMU10	0.543215	0.914929	0.883372	0.877981	0.872087	1.032459	1.036319
DMU11	0.331456	0.796206	0.667748	0.652726	0.639835	0.755741	0.850491
DMU12	0.445061	0.678381	0.701635	0.717405	0.745557	0.861775	0.818349
DMU13	0.295552	0.713268	0.624594	0.622738	0.622927	0.731393	0.78322
DMU14	0.425523	0.741348	0.709846	0.712437	0.713992	0.862642	0.84214
DMU15	0.514996	0.674491	0.731578	0.721535	0.724498	0.765032	0.805901
DMU16	0.16325	0.627977	0.666354	0.669722	0.699632	0.516827	0.741249
DMU17	0.233724	0.535428	0.566832	0.592378	0.631	0.651293	0.653658
Distance	5.8408	1.7703	1.3229	1.2917	1.5567	2.08808	1.51022

CORRELATION ANALYSIS

Pearson's correlation between each generated efficiency vector and CCR efficiency vectors have computed, and results have displayed in Table 11.

Table 11: Results of Pearson's correlation

	Chen	Zohrehbandian	Model (11)	Mode (12)
Rho	0.6181	0.7782	0.7709	0.7782

Pval	0.0082	0.0002	0.0003	0.0002

It concludes all efficiency vectors are positively correlated to their target efficiency vector. However, both efficiency vectors generated by the proposed models are correlated with $\hat{e}^*$ more robust than other efficiency vectors generated by other models.

In the above examples, we saw that the proposed models performed better than prior models, correspondingly. We simulated to measure how much these events likely to happen. For this reason, we regenerated the data of Example 5.2 uniformly for 100 times. Each time that a data set generated, first models, Phase I, Phase II, Chen, Zohrebandian, and Kao, for $p = 1$, and then models (11) and (12) were employed. Next, we computed the overall deviation of each obtained efficiency vector from the CCR efficiency vector and identified the model, which has a small overall deviation. Finally, by comparing total results, it is revealed that first, 94 percent of the iteration model (11) operated more efficiently than the Chen model. Second, model (12) in 39 percent of iterations and the Zohrebandian model in 24 percent of iteration were successful. In the end, model (11), model Kao with p=1, and model Zohrebandian were able to gain the best overall deviation in 56, 24, and 20 times of iterations.

CONCLUSION

In this paper, it has been shown that a p distance-based CSW model for $p = 1$ generates a solution with smaller individual and overall deviation than when $p > 1$. Also, it has been shown that if the data set is projected on the strongly efficient frontier fist of all, then the deviations associated with the solution will reduce significantly. Since oftentimes, an efficiency vector derived from a CSW used as an alternative for CCR efficiency vector, it is very important to have the least individual and overall deviations as much possible. We suggested improvements for two prior p distance-based CSW models based on those we found out. The proposed models were more successful then corresponding prior models in numerical examples and in a simulation analysis that we have conducted.

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