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Stability Theorem and Results for Quadrupled Fixed Point of Contractive Type Single Valued Operators

Aniki Samuel Adamariko[1*] and Rauf Kamilu[2]

[1] Department of Mathematics, Faculty of Science, Confluence University of Science and Technology, Osara, Kogi State, Nigeria
[2] Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Ilorin, Nigeria

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Abstract

This paper present the existence and uniqueness of quadrupled fixed point theorems, whose method is quite primarily based definitely on Perov-type fixed point theorem for contraction in metric spaces equipped with vector-valued matrices. Furthermore, the study consist of Ulam-Hyers stability results for quadrupled fixed points of contractive type single valued mappings on complete metric spaces will be obtained.

*** Correspondence E_mail**: smlaniki@yahoo.com

INTRODUCTION

In current times, contraction principle has become a very famous and necessary device in modern analysis, especially in nonlinear analysis which includes its functions to differential and integral equations, variational inequality theory, equilibrium problems, minimization problems, and many others. Banach contraction principle was extended for single-valued contraction on spaces endowed with vector-valued metrics by Perov (1964).

Beside this, one of the most popular generalizations of fixed point theorems is coupled fixed point theorem for continuous and discontinuous operators introduced by Guo and Lakshmikantham (1987), in relation with coupled quasi solutions of an initial value problem for ordinary differential equations. In 2006, Gnana-Bhaskar & Lakshmikantham (2006) introduced the concept of mixed monotone property in partially ordered metric space. Afterward, Lakshmikantham & Ciric (2009) extended these results by giving the definition of the g-monotone property. In (Berinde & Borcut, 2011), the extension and the generalization of the results of Bhaskar & Lakshmikantham (2006) are obtained and delivered the concept of a tripled fixed point and the mixed monotone property of a mapping and (Rauf & Aniki, 2020) extended the work of Berinde & Borcut (2011) to quadrupled fixed point theorems for contractive type mappings in partially ordered Cauchy spaces. For more details on coupled and tripled fixed point results, check (Gupta et al., 2014; Gupta, 2014; Cho et al., 2013; Gupta, 2013) and cited therein.

We summarize in the following the simple notions and results established in view of their generalization.

Definition 1 (Gupta, 2016). Let X be a nonempty set. A mapping $d:X^2\to R$ is referred to as the distance function between u,v in Xif the following properties are satisfied:

i . $d(u,v)\geq 0$ for all $u,v\in X$,

ii. $d(u,v)=0$ if and only $u=v$,

iii. $d(u,v)=d(v,u)$ for all $u,v\in X$,

iv. $d(u,v)\leq d(u,w)+d(w,v)$ for all $u,v,w\in X$,

If $u,v\in R, u=(u_1,\dots,u_m)$ and $v=(v_1,\dots,v_m)$, then, by definition: $u\leq v$ if and only if $u_i\leq v_i$ for $i\in\{1,2,\dots,m\}$.

A set endowed with a vector-valued metric d is called generalized metric space.

We denote with the aid of $M_{mm}(R_+)$ the set of all m×m matrices with positive elements and by I the identity matrix.

Theorem 1 (Gupta, 2016). Let $A\in M_{mm}(R_+)$. The following assertions are equivalent,

i. A is convergent towards the zero matrix,

ii. $A^n\to 0$ as $n\to\infty$

iii. The eigenvalues of A are in the open unit disc, i.e. $|\lambda|<1$, for every $\lambda\in C$ with $\det(A-\lambda I)=0$,

iv. The matrix (I-A) is nonsingular and

$$(I-A)^{-1}=I+A+\cdots+A^n+\cdots \quad (1)$$

v. The matrix (I-A) is nonsingular and $(I-A)^{-1}$ has nonnegative elements.

vi. $A^n q\to 0$ and $qA^n\to 0$ as $n\to\infty$ for each $q\in R^m$.

Theorem 2 (Gupta, 2016). Let (X,d) be a complete generalized metric space and the operator $f:X\to X$ with the property that there exists a matrix $A\in M_{mm}(R)$ such that $d(f(u),f(v))\leq Ad(u,v)$ for all $u,v\in X$. If A is a matrix convergent towards zero matrix. Then,

i. $Fix(f)=\{u^*\}$

ii. The sequence of successive approximations $(u_n)_{n\in N}, u_n=f^n(u_0)$ is convergent and has the limit u^*, for all $u_0\in X$,

iii. One has the following estimation

$$d(u_n,u^*)\leq A^n (I-A)^{-1} d(u_0,u_1), \quad (2)$$

iv. If $g:X\to X$ is an operator such that there exists $v^*\in Fix(g)$ and $\epsilon\in(R_m^+)$ with $d(f(u),g(u))<\epsilon$, for each $u\in X$, then $d(u^*,v^*)\leq(I-A)^{-1}\epsilon$,

v. If $g:X\to X$ is an operator such that there exists $\epsilon\in(R_m^+)\backslash *$ such that $(f(u),g(u))\leq\epsilon$, for all $u\in X$, then for the sequence $v_n=g^n(x_0)$, we have the following estimation

$$d(v_n,u^*)\leq(I-A)^{-1}\epsilon+A^n (I-A)^{-1} d(v_0,v_1). \quad (3)$$

Definition 2 (Gupta, 2016). Let (X,d) be a metric space. The system of operational equations

$$u=T_1(u,v,w,x)$$
$$v=T_2(u,v,w,x)$$

$w=T_3(u,v,w,x)$

$x=T_4(u,v,w,x)$

where $T_1,T_2,T_3,T_4:X^4\to X$ are four mappings. Then, the solution $(u,v,w,x)\in X^4$ of the system is referred to as a quadrupled fixed point for (T_1,T_2,T_3,T_4)

Definition 3 (Gupta, 2016). Let (X,d) be a generalized metric space with the operator $f:X\to X$. Then, the fixed point equation

$$u=f(u) \tag{4}$$

is stated to be generalized Ulam-Hyers stable if there exists an increasing function $\psi:R_+^m\to R_+^m$, continuous at zero with $\psi(0)=0$, such that for any $\epsilon=(\epsilon_1,\ldots,\epsilon_m)$ with $\epsilon_i>0$ for $i\in\{1,\ldots,m\}$ and any solution u^*∈X of the inequality

$$d(v,f(v))\le\epsilon \tag{5}$$

there exists a solution u^* of (4.4) such that

$$d(u_n,u^*)\le\psi(\epsilon) \tag{6}$$

In particular, if $\psi(t)=ct, t\in R_+^m$, (where $c\in M_{mm}(R_+)$), then the fixed point equation (4) is called Ulam-Hyers stable.

Theorem 3 (Gupta, 2016). Let (X,d) be a generalized metric space and the operator $f:X\to X$ with the property that there exists a matrix $A\in M_{mm}(R)$ such that A converges to zero and $d(f(u),f(v))\le Ad(u,v)$ for all $u,v\in X$.

Then, the fixed point equation $u=f(u), u\in X$ is Ulam-Hyers stable.

Proof. From Perov's fixed point theorem, it was proven that $Fix(f)=\{u^*\}$. Let $\epsilon=(\epsilon_1,\ldots,\epsilon_m)$ with $\epsilon_i>0$ for $i\in\{1,\ldots,m\}$ and let v^*∈X be the solution of the inequality

$d(v,f(v))\le\epsilon$.

Then, we successively have that

$$d(u^*,v^*)=d(f(u^*),v^*)$$

$$\le d(f(u^*),f(v^*))+d(f(v^*),v^*)\le Ad(u^*,v^*)+\epsilon$$

by Theorem 2,

$$d(u^*,v^*)\le(I-A)^{-1}\epsilon.$$

Starting from this background, our most important intention in this paper is to extend and generalize the outcomes in (Gupta, 2016), to the case of operators of the form $T_1,T_2,T_3,T_4:X^4\to X$ in the presence of contractive type single value vector-like operators. For related results to Perov's fixed point theorem and for some generalizations and applications of it we refer to (Petru et al., 2011; Precup, 2009; Rus, 2009; Petrusel, 2004; Varga, 2000).

EXISTENCE, UNIQUENESS AND STABILITY RESULTS FOR QUADRUPLED FIXED POINTS

In addition to some notations and results above, the following definition is relevant to the proof of our main theorem.

Definition 4. Let (X,d) be a metric space and let $T_1,T_2,T_3,T_4:X^4\to X$ be four operators. Then, the system of operational equations

$$\begin{aligned} u &= T_1(u,v,w,x)\\ v &= T_2(u,v,w,x)\\ w &= T_3(u,v,w,x)\\ x &= T_4(u,v,w,x)\end{aligned} \tag{7}$$

is said to be Ulam-Hyers stable if there exists $c_1,c_2,\ldots,c_{15},c_{16}>0$ such that for each $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4>0$ and each quadruple $(p^*,q^*,r^*,s^*)\in X^4$ such that

$$\begin{aligned} d(p^*,T_1(p^*,q^*,r^*,s^*)) &\le \epsilon_1\\ d(q^*,T_2(p^*,q^*,r^*,s^*)) &\le \epsilon_2\\ d(r^*,T_3(p^*,q^*,r^*,s^*)) &\le \epsilon_3\\ d(s^*,T_4(p^*,q^*,r^*,s^*)) &\le \epsilon_4\end{aligned} \tag{8}$$

there exists a solution $(u^*,v^*,w^*,x^*)\in X^4$ of (8) such that

$$\begin{aligned} d(p^*,u^*) &\le c_1\epsilon_1+c_2\epsilon_2+c_3\epsilon_3+c_4\epsilon_4\\ d(q^*,v^*) &\le c_5\epsilon_1+c_6\epsilon_2+c_7\epsilon_3+c_8\epsilon_4\\ d(r^*,w^*) &\le c_9\epsilon_1+c_{10}\epsilon_2+c_{11}\epsilon_3+c_{12}\epsilon_4\\ d(s^*,x^*) &\le c_{13}\epsilon_1+c_{14}\epsilon_2+c_{15}\epsilon_3+c_{16}\epsilon_4\end{aligned}$$

The main result is the following, data dependence and Ulam-Hyers stability theorem for the quadrupled fixed point of single-valued mappings (T_1,T_2,T_3,T_4).

Theorem 4. Let (X,d) be a complete metric space, T:X^4→X be a mapping such that

$$d\big(T_1(u,v,w,x),T_1(p,q,r,s)\big) \le k_1 d(u,p) + k_2 d(v,q) + k_3 d(w,r) + k_4 d(x,s)$$

$$d\big(T_2(u,v,w,x),T_2(p,q,r,s)\big) \le k_5 d(u,p) + k_6 d(v,q) + k_7 d(w,r) + k_8 d(x,s)$$

$$d\big(T_3(u,v,w,x),T_3(p,q,r,s)\big) \le k_9 d(u,p) + k_{10} d(v,q) + k_{11} d(w,r) + k_{12} d(x,s)$$

$$d\big(T_4(u,v,w,x),T_4(p,q,r,s)\big) \le k_{13} d(u,p) + k_{14} d(v,q) + k_{15} d(w,r) + k_{16} d(x,s)$$

for all (u,v,w,x),(p,q,r,s)∈X^4. Suppose that

$$A = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \\ k_9 & k_{10} & k_{11} & k_{12} \\ k_{13} & k_{14} & k_{15} & k_{16} \end{pmatrix}$$

converges to zero. Then,

i. There exists a unique element of (u*,v*,w*,x*)∈X^4 such that

$$\begin{aligned} u^* &= T_1(u^*,v^*,w^*,x^*) \\ v^* &= T_2(u^*,v^*,w^*,x^*) \\ w^* &= T_3(u^*,v^*,w^*,x^*) \\ x^* &= T_4(u^*,v^*,w^*,x^*) \end{aligned} \qquad (10)$$

ii. The sequence (T_1^n (u,v,w,x),T_2^n (u,v,w,x),T_3^n (u,v,w,x),T_4^n (u,v,w,x)),n∈N converges to (u*,v*,w*,x*) as n→∞, where

$$T_1^{n+1}(u,v,w,x) = T_1^n\big(T_1(u,v,w,x),T_2(u,v,w,x),T_3(u,v,w,x),T_4(u,v,w,x)\big)$$

$$T_2^{n+1}(v,u,v,x) = T_1^n\big(T_1(u,v,w,x),T_2(u,v,w,x),T_3(u,v,w,x),T_4(u,v,w,x)\big)$$

$$T_3^{n+1}(w,u,v,w) = T_1^n\big(T_1(u,v,w,x),T_2(u,v,w,x),T_3(u,v,w,x),T_4(u,v,w,x)\big)$$

$$T_4^{n+1}(x,w,v,u) = T_1^n\big(T_1(u,v,w,x),T_2(u,v,w,x),T_3(u,v,w,x),T_4(u,v,w,x)\big)$$

iii. For all n∈N, the following is obtained,

$$\begin{pmatrix} d\big(u^*,T_1^n(u_0,v_0,w_0,x_0)\big) \\ d\big(u^*,T_2^n(u_0,v_0,w_0,x_0)\big) \\ d\big(u^*,T_3^n(u_0,v_0,w_0,x_0)\big) \\ d\big(u^*,T_4^n(u_0,v_0,w_0,x_0)\big) \end{pmatrix} \le A^n(I-A)^{-1} \begin{pmatrix} d\big(u_0,T_1(u_0,v_0,w_0,x_0)\big) \\ d\big(v_0,T_2(u_0,v_0,w_0,x_0)\big) \\ d\big(w_0,T_3(u_0,v_0,w_0,x_0)\big) \\ d\big(x_0,T_4(u_0,v_0,w_0,x_0)\big) \end{pmatrix}$$

iv. Let $F_1,F_2,F_3,F_4:X^4 \to X$ be a mapping such that, there exists $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4>0$ with

$$\begin{aligned} d\big(T_1(u,v,w,x),F_1(u,v,w,x)\big) &\le \epsilon_1 \\ d\big(T_2(u,v,w,x),F_2(u,v,w,x)\big) &\le \epsilon_2 \\ d\big(T_3(u,v,w,x),F_3(u,v,w,x)\big) &\le \epsilon_3 \\ d\big(T_4(u,v,w,x),F_4(u,v,w,x)\big) &\le \epsilon_4 \end{aligned}$$

for all (u,v,w,x)∈X^4. If $(a^*,b^*,c^*,d^*)\in X^4$ is such that

$$a^* = F_1(a^*,b^*,c^*,d^*)$$
$$b^* = F_2(a^*,b^*,c^*,d^*)$$
$$c^* = F_3(a^*,b^*,c^*,d^*)$$
$$d^* = F_4(a^*,b^*,c^*,d^*)$$

then

$$\begin{pmatrix} d(a^*,u^*) \\ d(b^*,v^*) \\ d(c^*,w^*) \\ d(d^*,x^*) \end{pmatrix} \leq (I-A)^{-1}\epsilon \qquad (11)$$

where $\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix}$.

v. The system of operational equations

$$u = T_1(u,v,w,x)$$
$$v = T_2(u,v,w,x)$$
$$w = T_3(u,v,w,x)$$
$$x = T_4(u,v,w,x)$$

is Ulam-Hyers stable.

Proof. For (i)- (ii), define $T:X^4\to X^4$ by

$$T(u,v,w,x) = \begin{pmatrix} T_1(u,v,w,x) \\ T_2(u,v,w,x) \\ T_3(u,v,w,x) \\ T_4(u,v,w,x) \end{pmatrix} = T_1(u,v,w,x), T_2(u,v,w,x), T_3(u,v,w,x), T_4(u,v,w,x).$$

Consider $\tilde{d}: X^4 \times X^4 \to \mathbb{R}^4_+$,

$$\tilde{d}\big((u,v,w,x),(p,q,r,s)\big) = \begin{pmatrix} d(u,p) \\ d(v,q) \\ d(w,r) \\ d(x,s) \end{pmatrix}$$

Then,

$$\tilde{d}\big(T(u,v,w,x),T(p,q,r,s)\big) = \tilde{d}\left(\begin{pmatrix} T_1(u,v,w,x) \\ T_2(u,v,w,x) \\ T_3(u,v,w,x) \\ T_4(u,v,w,x) \end{pmatrix}, \begin{pmatrix} T_1(p,q,r,s) \\ T_2(p,q,r,s) \\ T_3(p,q,r,s) \\ T_4(p,q,r,s) \end{pmatrix}\right)$$

$$= \begin{pmatrix} d\big(T_1(u,v,w,x),T_1(p,q,r,s)\big) \\ d\big(T_2(u,v,w,x),T_2(p,q,r,s)\big) \\ d\big(T_3(u,v,w,x),T_3(p,q,r,s)\big) \\ d\big(T_4(u,v,w,x),T_4(p,q,r,s)\big) \end{pmatrix}$$

$$\leq \begin{pmatrix} k_1 d(u,p) + k_2 d(v,q) + k_3 d(w,r) + k_4 d(x,s) \\ k_5 d(u,p) + k_6 d(v,q) + k_7 d(w,r) + k_8 d(x,s) \\ k_9 d(u,p) + k_{10} d(v,q) + k_{11} d(w,r) + k_{12} d(x,s) \\ k_{13} d(u,p) + k_{14} d(v,q) + k_{15} d(w,r) + k_{16} d(x,s) \end{pmatrix}$$

$$= \begin{pmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \\ k_9 & k_{10} & k_{11} & k_{12} \\ k_{13} & k_{14} & k_{15} & k_{16} \end{pmatrix} \begin{pmatrix} d(u,p) \\ d(v,q) \\ d(w,r) \\ d(x,s) \end{pmatrix}$$

$$= A\tilde{d}\big((u,v,w,x),(p,q,r,s)\big) \qquad (13)$$

If we denote (u,v,w,x)=α,(p,q,r,s)=β, then $\tilde{d}(T(\alpha),T(\beta))\leq A\tilde{d}(\alpha,\beta)$.

Applying Perov's fixed point Theorem 1, then there exists a unique element $(u^*,v^*,w^*,x^*)\in X^4$ such that

$(u^*,v^*,w^*,x^*)=T(u^*,v^*,w^*,x^*)$

which is equivalent to

$$u^* = T_1(u^*,v^*,w^*,x^*)$$
$$v^* = T_2(u^*,v^*,w^*,x^*)$$
$$w^* = T_3(u^*,v^*,w^*,x^*)$$
$$x^* = T_4(u^*,v^*,w^*,x^*)$$

Moreover, for each $\alpha\in X^4$, then $T(\alpha)\rightarrow\alpha^*$ as $n\rightarrow\infty$,

where

$$T^0(\alpha) = \alpha,$$

$$T^1(\alpha) = T(u,v,w,x) = T_1(u,v,w,x), T_2(u,v,w,x), T_3(u,v,w,x), T_4(u,v,w,x)$$

$$T^2(\alpha) = T\big(T^1(\alpha)\big) = T\big(T_1(u,v,w,x), T_2(u,v,w,x), T_3(u,v,w,x), T_4(u,v,w,x)\big)$$

$$= \big(T_1^2(u,v,w,x), T_2^2(u,v,w,x), T_3^2(u,v,w,x), T_4^2(u,v,w,x)\big)$$

and generally,

$$T_1^{n+1}(\alpha) = T^n\big(T_1(u,v,w,x), T_2(u,v,w,x), T_3(u,v,w,x), T_4(u,v,w,x)\big)$$

$$T_2^{n+1}(\alpha) = T^n\big(T_1(u,v,w,x), T_2(u,v,w,x), T_3(u,v,w,x), T_4(u,v,w,x)\big)$$

$$T_3^{n+1}(\alpha) = T^n\big(T_1(u,v,w,x), T_2(u,v,w,x), T_3(u,v,w,x), T_4(u,v,w,x)\big)$$

$$T_4^{n+1}(\alpha) = T^n\big(T_1(u,v,w,x), T_2(u,v,w,x), T_3(u,v,w,x), T_4(u,v,w,x)\big)$$

Now, it has been obtained that $T(\alpha)\rightarrow\alpha^*=(u^*,v^*,w^*,x^*)$ as $n\rightarrow\infty$, for all $\alpha=(u,v,w,x)\in X^4$. So, for all $(u,v,w,x)\in X^4$, then

$$T_1(u,v,w,x) \rightarrow u^* \text{ as } n \rightarrow \infty$$
$$T_2(u,v,w,x) \rightarrow v^* \text{ as } n \rightarrow \infty$$
$$T_3(u,v,w,x) \rightarrow w^* \text{ as } n \rightarrow \infty$$
$$T_4(u,v,w,x) \rightarrow x^* \text{ as } n \rightarrow \infty$$

iii. By Perov's theorem, then

$$\begin{pmatrix} d(T_1^n(u_0,v_0,w_0,x_0),u^*) \\ d(T_2^n(u_0,v_0,w_0,x_0),v^*) \\ d(T_3^n(u_0,v_0,w_0,x_0),w^*) \\ d(T_4^n(u_0,v_0,w_0,x_0),x^*) \end{pmatrix} = \tilde{d}\big(T^n(u_0,v_0,w_0,x_0),(u^*,v^*,w^*,x^*)\big)$$

$$\leq A^n(I-A)^{-1}.\,\tilde{d}\begin{pmatrix} (u_0,v_0,w_0,x_0), T_1(u_0,v_0,w_0,x_0), T_2(u_0,v_0,w_0,x_0), \\ T_3(u_0,v_0,w_0,x_0), T_4(u_0,v_0,w_0,x_0) \end{pmatrix}$$

$$\leq A^n(I-A)^{-1}.\begin{pmatrix} d\big(u_0, T_1(u_0, v_0, w_0, x_0)\big) \\ d\big(v_0, T_2(u_0, v_0, w_0, x_0)\big) \\ d\big(w_0, T_3(u_0, v_0, w_0, x_0)\big) \\ d\big(x_0, T_4(u_0, v_0, w_0, x_0)\big) \end{pmatrix}.$$

iv. Now, considering $F: X^4 \rightarrow X^4$ such that

$$F(u,v,w,x) = \begin{pmatrix} F_1(u,v,w,x) \\ F_2(u,v,w,x) \\ F_3(u,v,w,x) \\ F_4(u,v,w,x) \end{pmatrix}$$

consider $\tilde{d}: X^4 \times X^4 \rightarrow \mathbb{R}_+^4$,

$$\tilde{d}\big((u,v,w,x),(p,q,r,s)\big) = \begin{pmatrix} d(u,p) \\ d(v,q) \\ d(w,r) \\ d(x,s) \end{pmatrix}$$

Then,

$$\tilde{d}\big((u,v,w,x),(p,q,r,s)\big) = \tilde{d}\left(\begin{pmatrix} T_1(u,v,w,x) \\ T_2(u,v,w,x) \\ T_3(u,v,w,x) \\ T_4(u,v,w,x) \end{pmatrix}, \begin{pmatrix} F_1(p,q,r,s) \\ F_2(p,q,r,s) \\ F_3(p,q,r,s) \\ F_4(p,q,r,s) \end{pmatrix}\right)$$

$$= \begin{pmatrix} d\big(T_1(u,v,w,x), F_1(p,q,r,s)\big) \\ d\big(T_2(u,v,w,x), F_2(p,q,r,s)\big) \\ d\big(T_3(u,v,w,x), F_3(p,q,r,s)\big) \\ d\big(T_4(u,v,w,x), F_4(p,q,r,s)\big) \end{pmatrix} \leq \epsilon \qquad (14)$$

then, applying Perov's fixed point Theorem 2 gives

$d((u^*,v^*,w^*,x^*),(a^*,b^*,c^*,d^*)) \leq (I-A)^{-1}\epsilon$ (15)

v. By (i) and (ii) there exists a unique element $(u^*,v^*,w^*,x^*) \in X^4$ such that (u^*,v^*,w^*,x^*) is a solution for (12) and the sequence $(T_1^n(u,v,w,x), T_2^n(u,v,w,x), T_3^n(u,v,w,x), T_4^n(u,v,w,x)) \rightarrow (u^*,v^*,w^*,x^*)$ as $n \rightarrow \infty$.

Let $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4 > 0$ and $(u^*,v^*,w^*,x^*) \in X^4$ such that

$$d\big(p^*, T_1(p^*,q^*,r^*,s^*)\big) \leq \epsilon_1$$
$$d\big(q^*, T_2(p^*,q^*,r^*,s^*)\big) \leq \epsilon_2$$
$$d\big(r^*, T_3(p^*,q^*,r^*,s^*)\big) \leq \epsilon_3$$
$$d\big(s^*, T_4(p^*,q^*,r^*,s^*)\big) \leq \epsilon_4$$

Then,

$$\tilde{d}\big((p^*,q^*,r^*,s^*),(u^*,v^*,w^*,x^*)\big)$$
$$\leq \tilde{d}\left((p^*,q^*,r^*,s^*),\big(T_1(p^*,q^*,r^*,s^*),T_2(p^*,q^*,r^*,s^*),T_3(p^*,q^*,r^*,s^*),T_4(p^*,q^*,r^*,s^*)\big)\right)$$
$$+\tilde{d}\left(\big(T_1(p^*,q^*,r^*,s^*),T_2(p^*,q^*,r^*,s^*),T_3(p^*,q^*,r^*,s^*),T_4(p^*,q^*,r^*,s^*)\big),(u^*,v^*,w^*,x^*)\right)$$

$$=\tilde{d}\left((p^*,q^*,r^*,s^*),\big(T_1(p^*,q^*,r^*,s^*),T_2(p^*,q^*,r^*,s^*),T_3(p^*,q^*,r^*,s^*),T_4(p^*,q^*,r^*,s^*)\big)\right)$$
$$+\tilde{d}\left(\begin{pmatrix}T_1(p^*,q^*,r^*,s^*)\\T_2(p^*,q^*,r^*,s^*)\\T_3(p^*,q^*,r^*,s^*)\\T_4(p^*,q^*,r^*,s^*)\end{pmatrix},\begin{pmatrix}T_1(u^*,v^*,w^*,x^*)\\T_2(u^*,v^*,w^*,x^*)\\T_3(u^*,v^*,w^*,x^*)\\T_4(u^*,v^*,w^*,x^*)\end{pmatrix}\right)$$

$$=\begin{pmatrix}d\big(p^*,T_1(p^*,q^*,r^*,s^*)\big)\\d\big(q^*,T_2(p^*,q^*,r^*,s^*)\big)\\d\big(r^*,T_3(p^*,q^*,r^*,s^*)\big)\\d\big(s^*,T_4(p^*,q^*,r^*,s^*)\big)\end{pmatrix}+\begin{pmatrix}d\big(T_1(p^*,q^*,r^*,s^*),T_1(u^*,v^*,w^*,x^*)\big)\\d\big(T_2(p^*,q^*,r^*,s^*),T_2(u^*,v^*,w^*,x^*)\big)\\d\big(T_3(p^*,q^*,r^*,s^*),T_3(u^*,v^*,w^*,x^*)\big)\\d\big(T_4(p^*,q^*,r^*,s^*),T_4(u^*,v^*,w^*,x^*)\big)\end{pmatrix}$$

$$\leq\begin{pmatrix}\epsilon_1\\\epsilon_2\\\epsilon_3\\\epsilon_4\end{pmatrix}+\tilde{d}\big(T(p^*,q^*,r^*,s^*),T(u^*,v^*,w^*,x^*)\big)$$

$$\leq \epsilon + A\tilde{d}\big((p^*,q^*,r^*,s^*),(u^*,v^*,w^*,x^*)\big)$$

since $(I-A)$ is invertible and $(I-A)^{-1}$ has positive elements,

then,

$$\tilde{d}\big((p^*,q^*,r^*,s^*),(u^*,v^*,w^*,x^*)\big)\leq (I-A)^{-1}\epsilon$$

or equivalently

$$\begin{pmatrix}d(p^*,u^*)\\d(q^*,v^*)\\d(r^*,w^*)\\d(s^*,x^*)\end{pmatrix}=(I-A)^{-1}\epsilon.$$

Denote

$$(I-A)^{-1}=\begin{pmatrix}c_1&c_2&c_3&c_4\\c_5&c_6&c_7&c_8\\c_9&c_{10}&c_{11}&c_{12}\\c_{13}&c_{14}&c_{15}&c_{16}\end{pmatrix}$$

then

$$d(p^*, u^*) \leq c_1\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3 + c_4\epsilon_4$$

$$d(q, v^*) \leq c_5\epsilon_1 + c_6\epsilon_2 + c_7\epsilon_3 + c_8\epsilon_4$$

$$d(r^*, w^*) \leq c_9\epsilon_1 + c_{10}\epsilon_2 + c_{11}\epsilon_3 + c_{12}\epsilon_4$$

$$d(s^*, x^*) \leq c_{13}\epsilon_1 + c_{14}\epsilon_2 + c_{15}\epsilon_3 + c_{16}\epsilon_4$$

This proves that the operational system (12) is Ulam-Hyers stable, which is the conclusion.

CONCLUSIONS

In this research, it was shown that Ulam-Hayers stability was extended from tripled fixed point to quadrupled fixed point. It opens up the space for further extension and generalization of the theorem. Furthermore, Ulam-Hayers stability analysis and results for quadrupled fixed points of contractive type single valued mappings on complete metric spaces was obtained.

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