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Shifted Chebyshev Approach for the Solution of Delay Fredholm and Volterra Integro-Differential Equations via Perturbed Galerkin Method

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Abstract

The main idea proposed in this paper is the perturbed shifted Chebyshev Galerkin method for the solutions of delay Fredholm and Volterra integrodifferential equations. The application of the proposed method is also extended to the solutions of integro-differential difference equations. The method is validated using some selected problems from the literature. In all the problems that are considered, the new proposed approach performs better than many other methods.

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INTRODUCTION

Integro-differential equations are resulted from mathematical models in biological, physical, and engineering problems. A lot of interest have been shifted to the methods of the solution of integro-differential equations. In the recent past, different approaches have been used to solve these and related equations such as the delay integro-differential equations (which is the focus of the research work reported in this paper) and delay integro-differential difference equations that are majorly the product of mathematical interpretations of physical world. Taylor collocation method was applied to solve Fredholm-Volterra integro-differential difference equations (Yalçinbas & Sezer, 2006); (Sezer & Gülsu, 2007), Chebyshev polynomial approach was used in (Maleknejad et al., 2007) to solve nonlinear Volterra integral equations of the second kind, Galerkin methods in its various forms were however employed to solve both perturbed and unperturbed linear Volterra-Fredholm integro-differential equations (Biazar & Salehi, 2016); (Fathy et al., 2014); (Issa & Salehi, 2017). Moreover, some other notable Scientists used Tau method to solve Fredholm-Volterra integro-differential equations as presented in Hosseini and Shahmorad (2003); Shahmorad (2005), and error estimate in the numerical scheme used was equally estimated. Legendre method was employed in Khater et al. (2007); Jimoh & Issa (2014);

(Yüzbaşi Ş. , 2017) to solve both integral and integro-differential equations. Other methods employed in the literature to solve such equations and similar ones, include Galerkin finite element method (Jangveladze et al., 2011), Haar wavelet method (Shahsavara, 2010), hybrid methods (Hou & Yang, 2013), variational iteration method (Bildik et al.,, 2010); (Han & Shang, 2010), operational matrix method (Shahmorad & Ostadzad, 2016), etc. Partial integro-differential equation which is a very vibrant model for viscoelasticity was however treated in (Dehghan, 2006) and the references therein. Delay differential equation is an aspect of integro-differential equation, arising in many dynamical systems, nuclear reaction, electrical network and many other areas of mathematical physics. This family of equations has received serious attention from several researchers, and it is also discussed here with a view to arriving at better results.

What defines a time-delay system is the feature that, the system future evolution depends on its present state and period of its history. The study of time-delay system dated back to the early 20th century as a result of modeling of biological system, nuclear reaction, ecological system, just to mention a few (see (Yüzbasi et al., 2013); (Shahmorad & Ostadzad, 2016); (Yüzbaşi & Ismailov, 2014); (Bellen & Zennaro, 2003); (El-Hawary & El-Shami, 2013) and the references therein). Our aim is to improve the accuracy of the existing methods for solving delay Fredholm and Volterra integrodifferential equations, by focusing our attention on Galerkin method and compare the results with the results obtain by operational matrix method, Legendre method, and Taylor collocation method, discussed in the literature (see (Biazar & Salehi, 2016); (Fathy et al., 2014); (Issa & Salehi, 2017); Shahmorad & Ostadzad (2016)).

This paper is organized as follows. In section 2, we review the basic properties of the Chebyshev polynomials. The description of the proposed method for the numerical solution of delay Fredholm integro-differential equation 14 and 15, and delay Volterra integro-differential equation 22 are regarded in section 3. Meanwhile, a brief note on the existence and uniqueness of the solutions of delay integro-differential equation are also addressed in section 3. Section 4 is devoted to presenting some numerical examples to support our findings. And lastly, in section 5, the results of our findings will be discussed and compared, favorably with some existing methods.

PROPERTIES OF CHEBYSHEV POLYNOMIALS

Chebyshev polynomial is an orthogonal polynomial $T_n(u)$ defined on the interval $u\in[-1,1]$ with the weight function $w(u)=\frac{1}{\sqrt{1-u^2}}$.

can be determined using the recurrence formulae:

$$T_{n+1}(u) = 2uT_n(u) - T_{n-1}(u), n \geq 1, \tag{1}$$

where $T_n(u) = \cos(n\cos^{-1}u)$, $T_0(u) = 1$ and . $T_1(u) = u$

The Chebyshev polynomial can be transform from $u\in[-1,1]$ into $x\in[a,b]$ using the transformation

$$u = \frac{2x-(a+b)}{b-a}, \ a \leq x \leq b. \tag{2}$$

Therefore, the shifted Chebyshev polynomials in x are

$$T_0^*(x) = 1, \ T_1^*(x) = \frac{2x-(a+b)}{b-a} \tag{3}$$

and

$$T_i^*(x) = T_i\left(\frac{2x-(a+b)}{b-a}\right)$$

and the recurrence relation is as follows:

$$T_{n+1}^*(x) = 2\left(\frac{2x-(a+b)}{b-a}\right)T_n^*(x) - T_{n-1}^*(x), \ n \geq 1, \tag{4}$$

where

$$T_n^*(x) = \cos\left[n\cos^{-1}\left(\frac{2x-(a+b)}{b-a}\right)\right],$$

$$n \geq 0, \ a \leq x \leq b. \tag{5}$$

The orthogonality condition over the interval $[-1, 1]$ is:

$$\int_{-1}^{1}\frac{1}{\sqrt{1-u^2}}T_m(u)T_n(u)du = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2}, & m = n, \\ \pi, & m = n = 0. \end{cases} \tag{6}$$

Expressing the approximant $y_n(x)$ in terms of shifted Chebyshev polynomials $T_n^*(x)$ leads to:

$$y_n(x) = \sum_{i=0}^{n}\alpha_i T_i^*(x), \tag{7}$$

where α_i, i= 0,...,n, are the coefficients to be determined.

Similarly, a function F of two or more independent variables defined on $a \leq x_1 \leq x_2 \leq \cdots \leq x_m \leq b$ may be expanded in terms of m shifted Chebyshev polynomials as:

$$F(x_1, x_2, \cdots, x_m) \simeq \sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\cdots\sum_{i_m=0}^{n} \alpha_{i_1 i_2 \cdots i_m} T_{n_{i_1}}^*(x_1)T_{n_{i_2}}^*(x_2)\cdots T_{n_{i_m}}^*(x_n), \tag{8}$$

Review of existence and uniqueness of solutions of delay integro differential equations

The twin terminologies, existence and uniqueness of solutions of either a differential equation, an integral equation, or an integro-differential equation are always essential to be established before proceeding with a method of solution in any case.

The existence and uniqueness of solutions of the delay-differential equations

$$y''(x)=f(x,y(x),y(\hbar(x)),y'(x)), \quad x\in[0,l], \tag{9}$$

subject to the conditions

$$y(x)= \phi(x), x \in[-v,0] \tag{10}$$

$$y[\ell]=\wp \tag{11}$$

where $\ell>0$ and v is a positive number, was established in (Eloe, Raffoul, & Tisdell, 2005), some constructive results were formulated therein. (Interested readers can see (Eloe et al., 2005) for the details of the proofing, including the theory of Petryshyn that was applied).

(Cahlon et al., 1985) on the other hand worked on the existence and uniqueness of the solution of Volterra integral equation with a solution dependent delay. The novelty in Cahlon and Nachman (1985) is its deviation from the earlier researches on the existence and uniqueness of the solution of Volterra integral equations with constant delay (see (Vätä, 1978); Cahlon et al. (1984)). The delay Volterra integral equation considered in (Cahlon & Nachman, 1985) is as follows

$$y(x) = f(x) + \int_0^x F(x, \tau, y(\tau), y(\tau - \lambda(y(\tau))))d\tau, \quad 0 \leq x \leq a \tag{12}$$

$$y(x) = g(x), \quad -\tau_0 \leq x \leq 0, \tag{13}$$

where τ_0 and α are positive constants. Eq. 12 is a nonlinear Volterra integral equation with a delay which depends on the solution. The existence and uniqueness in this case was proved in the space C^x [$-\tau_0$,a] the space of all Holder continuous functions, $0 \leq \alpha \leq 1$, on [$-\tau_0$,a].

The existence and uniqueness of solutions of the family of equations considered in the present work were earlier established in (Maruo & Park, 1994), where existence and uniqueness of solutions for integro-differential equations with time delay in Hilbert space were considered. The authors of (Maruo & Park, 1994) benefited tremendously from the efforts of the earlier scientist. And most recently in 2017, global existence and uniqueness of solutions of integral equations with delay was reported in (Burton & Purnaras, 2017) which exploited the effectiveness of progressive contractions. (Interested readers can read (Maruo & Park, 1994) and (Burton & Purnaras, 2017) for the existence and uniqueness of (14), (16), and (22) in the present work).

SHIFTED CHEBYSHEV PERTURBED GALERKIN METHOD FOR SOLVING DELAY INTEGRO-DIFFERENTIAL EQUATIONS

In this section, we discuss the numerical solution of a class of linear delay integrodifferential equation. The solution of delay Fredholm integro - differential equation is discussed in section 3.1, while section 3.2 is all about the solution of delay Volterra integro - differential equation.

Delay fredholm integro-differential equation

Consider the following m order delay Fredholm integro-differential equation

$$\begin{cases} L\varphi(x)+\int_0^b A(x,t)\,\varphi(t-\gamma)dt=f(x), x[0,b], \\ \varphi(x)=\theta(x), \qquad x\in[-\gamma,0],) \end{cases} \quad (14)$$

with m independent initial, boundary or mixed conditions

$$\sum_{j=0}^{m-1}\sum_{k=1}^{r}\tau_{sk}^{(j)}\varphi^{(j)}(\beta_k)=\rho_s, \qquad s=1,2,\cdots,m, \quad (15)$$

where $\beta_k\in[0,b],\ k=1,\ldots,r$. Rewriting Eq. 14, we have

$$\begin{cases} L\varphi(x)+\int_0^{\gamma}A(x,t)\theta(t)dt+\int_{\gamma}^{b}A(x,t)\varphi(t-\gamma)dt=f(x), & x\in[0,b], \\ \sum_{j=0}^{m-1}\sum_{k=1}^{r}\tau_{sk}^{(j)}\varphi^{(j)}(\beta_k)=\rho_s,\ s=1,2,\ldots,m, & x\in[-\gamma,0], \end{cases} \quad (16)$$

Replacing $\varphi(x)$ in Eq. 16 by its approximant $\varphi_n(x)$ that can be regarded as the exact solution of the perturbed equation:

$$\begin{cases} L\varphi_n(x)+\int_0^{\gamma}A(x,t)\theta(t)dt+\int_{\gamma}^{b}A(x,t)\varphi_n(t-\gamma)dt-H_n(x)=f(x), & x\in[0,b], \\ \sum_{j=0}^{m-1}\sum_{k=1}^{r}\tau_{sk}^{(j)}\varphi^{(j)}(\beta_k)=\rho_s,\ s=1,2,\ldots,m. & x\in[-\gamma,0], \end{cases} \quad (17)$$

Where $L\equiv\sum_{q=0}^{m}P_q(x)\frac{d^q}{dx^q}$, $P_q(x)$, $f(x)$, $A(x,t)$ are known functions, m is the order of Eq. 14 and

$$H_n(x)=\sum_{i=0}^{m}\beta_{i+1}T^*_{n-m+i+1}(x), \quad (18)$$

is the perturbation term.

Multiplying Eq. 17 by the shifted Chebyshev polynomial $T^*_j(x),\ j=m,\ m+1,\ldots,(n+m+1)$ and integrating the resulting equation over the interval [a,b] gives a set of (n + 2) algebraic equations:

$$\int_a^b\left[\sum_{q=0}^m P_q(x)\sum_{i=0}^n\left(\frac{2}{b-a}\right)^q\alpha_i\frac{d^q}{dx^q}\left(T_i^*(x)\right)+\int_0^\gamma A(x,t)\theta(t)dt+\int_\gamma^b A(x,t)\sum_{i=0}^n\alpha_i T_i^*(t-\gamma)dt-H_n(x)\right]T_j^*(x)dx$$

$$=\int_a^b f(x)T_j^*(x)dx,\ \ j=m,\ m+1,\dots,\ n+m+1. \tag{19}$$

Putting Eq. 19 in matrix form, we have

$$\Theta\vartheta=\Gamma, \tag{20}$$

where Θ is a real $(n+2)\times(n+m+2)$ matrix, ϑ is a column matrix of $(n+m+2)\times 1$ and Γ is column matrix of $(n+2)\times 1$ To obtain $(n+m+2)\times(n+m+2)$ matrix (that is $(n+m+2)$ algebraic equations), we apply m independent initial, boundary or mixed conditions, that is

$$\sum_{j=0}^{m-1}\sum_{k=1}^{r}\tau_{sk}^{(j)}\left(\frac{2}{b-a}\right)^j\alpha_j\frac{d^j}{dx^j}\left(T_j^*(x)\right)\Big|_{x=\beta_k}$$

$$=\rho_s\ \ s=1,\ 2,\dots,m, \tag{21}$$

substituting the values of α_i, i=0,1,...,n in Eq. 7, we obtain the approximate solution of degree *n*.

Delay volterra integro-differential equations

Consider the following delay Volterra integro-differential equation

$$\begin{cases}L\varphi(x)+\int_\gamma^x A(x,t)\varphi(t-\gamma)dt=f(x),\ \ x[0,b],\\ \sum_{j=0}^{m-1}\sum_{k=1}^{r}\tau_{sk}^{(j)}\varphi^{(j)}(\beta_k)=\rho_s,\qquad s=1,2,\dots,m,\end{cases} \tag{22}$$

perturbing Eq. 22, we have

$$\begin{cases}L\varphi(x)+\int_\gamma^x A(x,t)\varphi(t-\gamma)dt-H_n(x)=f(x),\ \ x[0,b],\\ \sum_{j=0}^{m-1}\sum_{k=1}^{r}\tau_{sk}^{(j)}\varphi^{(j)}(\beta_k)=\rho_s,\qquad s=1,2,\dots,m,\end{cases} \tag{23}$$

Again, multiplying Eq. 23 by shifted Chebyshev polynomial $T_i^*(x),\ i=m,\ m+1,\dots,n+m+1,$ integrate the resulting equation over the interval [a,b], together with the given condition(s), we obtain (n+m+2) linear equations, we then solve to determine all unknowns (that is α_i, i=0,...,n and β_i, i=0,...,m+1).

ILLUSTRATIVE EXAMPLES

In this section, we illustrate the method discussed in section 3 by solving some selected examples from the literature. To achieve our aim, the mathematical packages used are Maple and Matlab. Numerical computation is carried out in Maple while, the graphs are drawn in Matlab. To facilitate comparison problems discussed in Example 4.1 and 4.2 are taken from (Shahmorad & Ostadzad, 2016), problem consider in Example 4.3 is taken from (Saadatmandi & Dehghan, 2010), Example 4.4 from (Kajani, Ghasemi, & Babolian, 2006); (Yusufoglu, 2009)); (Darania & Ebadian, 2007); (Yüzbaşi Ş. , 2017) and Example 4.5 is taken from (El-Hawary & El-Shami, 2013).

Example 4.1

Consider the following delay Fredholm integro-differential equation

$$\begin{cases}\frac{dy}{dx}-2y(x)+\int_0^1 xty\left(t-\frac{1}{2}\right)dt=\left(\frac{1}{4}exp(1)-\frac{61}{48}\right)x-1,\ x\in[0,1]\\ y(x)=2,\qquad\qquad x\in\left[-\frac{1}{2},0\right]\end{cases} \tag{24}$$

with initial condition y(0) = 2 analytic solution y(x) = exp(2x) + x + 1,

Applying the method discussed in section 3.1, by rewriting equation 24 inform of (17), one gets

$$\frac{dy_n}{dx}-2y_n(x)+\int_{0.5}^1 xty_n\left(t-\frac{1}{2}\right)dt+2\int_0^{0.5}xtdt-\beta_1T_n^*(x)-\beta_2T_{n+1}^*(x)$$

$$=\left(\frac{1}{4}\exp(1)-\frac{61}{48}\right)x-1, \tag{25}$$

where

$$H_n(x)=\sum_{i=0}^{m}\beta_{i+1}T^*_{n-m+i+1}(x)=\sum_{i=0}^{1}\beta_{i+1}T^*_{n-m+i+1}(x)$$
$$=\beta_1 T^*_n(x)+\beta_2 T^*_{n+1}(x) \qquad (26)$$

and $y_n(x)=\sum \alpha_i T^*_i(x)$.

Multiplying Eq. 25 by the shifted Chebyshev polynomial $T^*_j(x),\ j=1,\ 2,\ldots,\ (n+2)$ and inte grate the resulting equation over the interval $\left[-\frac{1}{2},\ 1\right]$ gives a set of (n+2) algebraic equations,

and with the attached condition, we now solve (n+3) algebraic equations to find the values of unknowns.

The absolute errors for n=10 are tabulated in Table 1. Table 2 and 3 provide the comparison between the errors obtained from the proposed approach and some others reported in the literature. Fig. 1 is the graphical representation of the results in Table 1. With CPU time of 1.43 secs.

Example 4.2

Consider a third order delay Volterra integro-differential problem with variable coefficients as the following.

$$xy'''(x)+y''(x)+xy'(x)+y(x)+\int_{\frac{1}{2}}^{x} y\left(t-\frac{1}{2}\right)dt$$
$$=\frac{1}{2}\left(x-\frac{1}{2}\right)^2+2\left(x-\frac{1}{2}\right)+2-\cos\left(x-\frac{1}{2}\right),\quad [0,\ 1] \qquad (27)$$

with supplementary conditions $y(0)=0,\ y\left(\frac{1}{2}\right)+y'\left(\frac{1}{2}\right)=\frac{3}{2}+\sin\left(\frac{1}{2}\right)+\cos\left(\frac{1}{2}\right),\ y(0)+y(1)=1+\sin(1),$

and the exact solution is $y(x)=x+\sin(x)$.

Applying the proposed approach results in the following, Table 1 present the absolute errors for n=9, and Table 2 exhibits the maximum errors with n=9 using the present method and those reported in (Shahmorad & Ostadzad, 2016) and the CPU time is 43.35 secs.

Example 4.3

Consider the following third order linear Fredholm integro-differential-difference equation with variable cofficients

$$y'''(x)-xy'(x)+y''(x-1)-\int_{-1}^{1} y(t-1)dt$$
$$=-(x+1)\left[\sin(x-1)+\cos x\right]-\cos(2)+1 \qquad (28)$$

subject to the initial conditions $y(0)=0, y'(0)=1, y''(0)=0$, and the exact solution $y(x)=\sin(x)$.

In Table 5, we present the absolute errors for n = 6 and n = 7 and Table 3 presents maximum errors using this present method and methods discussed in (Gulsu & Sezer, 2006); (Saadatmandi & Dehghan, 2010). Fig. 2 presents the approximate and exact solutions.

Example 4.4

Consider the Fredholm integro-differential equation

$$y'(x)-y(x)=\frac{1-\exp(x+1)}{x+1}+\int \exp(tx)y(t)dt \qquad (29)$$

with initial condition $y(0)=1$ and exact solution is $y(x)=\exp(x)$. Using the proposed approach leads to the following results, Table 2 presents the absolute errors of the present method together with the methods discussed in (Kajani et al., 2006); (Yusufoglu, 2009); (Darania & Ebadian, 2007); (Yüzbaşi, 2017), that is CAS wavelet method (CASWM), homotopy perturbation method (HPM), the differential transformation method (DTM) and Shifted Legendre method (SLM) respectively, and Fig. 3 is the corresponding figure. And the CPU time is 47.01 secs.

Example 4.5

Let us consider the following delay Volterra integro-differential equation

$$y'(x)-y(x-1)+4y(x)+3\int_{x-1}^{x} y(t)dt=2\exp(1-x),\quad x\geq 0 \qquad (30)$$

with initial condition y(0) = 1, and the exact solution y(x) = exp(-x). The numerical results for the absolute errors are displayed in Table 5 for different values of n. The CPU time is 3.78 secs.

Table 1: Absolute Errors for Example 4.1 and 4.2 for different n

x	**Example 4.1 (with n=10)**	**Example 4.1 (with n=9)**
0	1.0×10^{-9}	0
0.1	1.0×10^{-8}	4.1×10^{-10}
0.2	1.0×10^{-8}	6.2×10^{-10}
0.3	4.0×10^{-9}	7.1×10^{-10}
0.4	2.0×10^{-9}	7.2×10^{-10}
0.5	7.0×10^{-9}	6.8×10^{-10}
0.6	1.7×10^{-8}	5.9×10^{-10}
0.7	1.1×10^{-8}	4.7×10^{-10}
0.8	7.0×10^{-9}	3.3×10^{-10}
0.9	1.7×10^{-8}	1.8×10^{-10}
1.0	1.9×10^{-8}	0

Table 2: Maximum errors for Example 4.1 and 4.2 for specified degree (n)

Examples	***n***	**Operational Matrix Method** (Shahmorad & Ostadzad, 2016)	**Present Method**
Example 4.1	10	4.5×10^{-7}	2.3×10^{-8}
Example 4.2		7.5×10^{-3}	7.2×10^{-10}

Table 3: Absolute Errors for Example 4.3 for different n

x	**TS (n=6)** (Gulsu & Sezer, 2006)	**SL (n=6)** (Saadatmandi & Deghghan, 2010)	**PM (n=6)**	**TS (n=7)** (Gulsu & Sezer, 2006)	**SL (n=7)** (Saadatmandi & Dehghan, 2010)	**PM (n=7)**
-1.0	8.58×10^{-2}	3.84×10^{-2}	1.04×10^{-2}	6.03×10^{-2}	5.05×10^{-3}	4.87×10^{-3}
-0.8	3.93×10^{-3}	1.82×10^{-2}	4.32×10^{-3}	2.28×10^{-2}	2.38×10^{-3}	2.34×10^{-3}
-0.6	1.50×10^{-2}	7.00×10^{-3}	1.39×10^{-3}	6.63×10^{-3}	9.14×10^{-4}	9.13×10^{-4}
-0.4	4.12×10^{-3}	1.86×10^{-3}	2.87×10^{-4}	1.20×10^{-3}	2.42×10^{-4}	2.38×10^{-4}
-0.2	4.85×10^{-4}	2.04×10^{-4}	2.04×10^{-5}	6.90×10^{-5}	2.65×10^{-5}	2.78×10^{-5}
0	0	0	0	0	0	0
0.2	4.59×10^{-4}	1.48×10^{-4}	8.13×10^{-6}	5.30×10^{-5}	1.91×10^{-5}	1.85×10^{-5}
0.4	3.69×10^{-3}	9.67×10^{-4}	1.67×10^{-4}	8.09×10^{-4}	1.25×10^{-4}	1.49×10^{-4}
0.6	1.28×10^{-2}	2.55×10^{-3}	8.77×10^{-4}	3.82×10^{-3}	3.30×10^{-4}	2.86×10^{-4}
0.8	3.17×10^{-2}	4.44×10^{-3}	2.74×10^{-3}	1.14×10^{-2}	5.78×10^{-4}	4.47×10^{-4}
1.0	6.57×10^{-2}	5.76×10^{-3}	6.50×10^{-3}	2.73×10^{-2}	7.53×10^{-4}	4.52×10^{-4}

Key: TS is equivalent to Taylor series, SL stands for shifted Legendre, and PM equivalent to present method

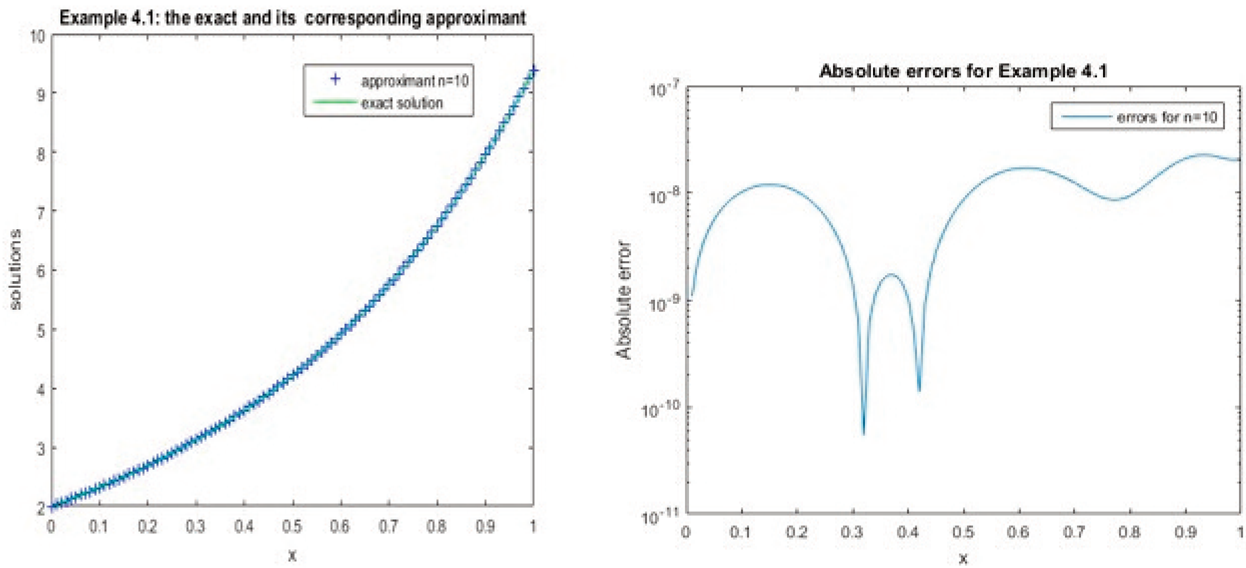

Fig. 1. Approximate solution for Example 4.1 and its absolute errors for n= 10

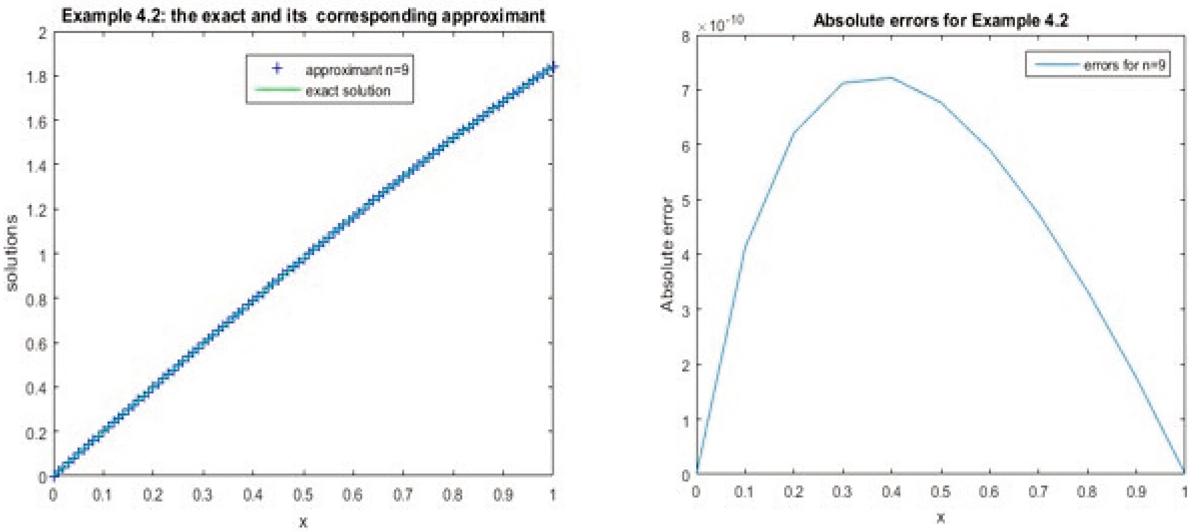

Fig. 2. Approximate, exact solution for Example 4.2 and its absolute errors for n = 9

Table 4: Comparison of maximum errors for Example 4.4 for different approximation (n)

Methods	***n*=5**	***n*=8**	***n*=12**
Shifted Legendre (SL) (Yüzbaşi Ş. , 2017)	8.55×10^{-4}	1.19×10^{-6}	4.48×10^{-11}
Present Method (PM)	2.68×10^{-6}	4.70×10^{-10}	6.71×10^{-17}

Table 5: Absolute errors for Example 4.5 for different n

x	*n*=4	*n*=8	*n*=12
0	0	0	0
0.1	6.47×10^{-5}	4.93×10^{-7}	1.01×10^{-10}
0.2	6.61×10^{-5}	7.66×10^{-7}	4.71×10^{-10}
0.3	2.89×10^{-4}	8.19×10^{-7}	8.84×10^{-10}
0.4	5.11×10^{-4}	6.99×10^{-7}	1.19×10^{-9}
0.5	6.62×10^{-4}	4.75×10^{-7}	1.30×10^{-9}
0.6	6.96×10^{-4}	2.17×10^{-7}	1.22×10^{-9}
0.7	6.01×10^{-4}	1.33×10^{-8}	9.93×10^{-10}
0.8	4.05×10^{-4}	1.79×10^{-7}	6.79×10^{-10}
0.9	1.78×10^{-4}	2.65×10^{-7}	3.52×10^{-10}
1.0	4.16×10^{-4}	2.75×10^{-7}	7.01×10^{-11}

DISCUSSION OF THE RESULTS AND CONCLUSION

Discussion of results

The absolute errors of equations 4.1 and 4.2 for n = 10 and n = 9 respectively are presented in Table 1. The choice of the values of n is due to provide the possibility of the comparison of our results with some of those existing in the literature. Similar thing is done, although in a more comprehensive manner, in Table 3 for Example 4.3. Table 3 presents results obtained in the literature when Taylor's series (TS) method was used, when sifted Legendre (SL) method was used and those obtained when the present method (PM) is used. This is done to facilitate ease of comparison. Results for different values of n are presented in Table 5 for the absolute errors for Example 4.5. It is easily noticed, in that table, that as the value of n increases, the rate of convergence increases rapidly. On the other hand, Table 2 presents a brief comparison of the maximum errors obtained when operational matrix method was used to solve Example 4.1 and 4.2 with those obtained using the present method.

Table 4 compares the maximum errors from shifted Legendre with those obtained using the present method. The results presented in the tables are depicted graphically as we have them in Figures 1, 2, and 3. The graphs, in virtually all the cases, tally with the existing graphs in the literature, where such graphs are presented.

CONCLUSION

In this paper, we proposed the shifted Chebyshev Galerkin method with perturbation for solving delay Fredholm and Volterra integro-differential equations. Our approach is based on shifted Chebyshev Galerkin method with perturbation terms which reduces the integro-differential equation into a set of (n+m+2) linear equations. The numerical results obtained, shows the accuracy and the effectiveness of our approach. The approach is also applicable to solve some integro-differential-difference equations, as the method was tested on some selected examples from the literature.

REFERENCES

Bellen, A., & Zennaro, M. (2003). *Numerical methods for delay differential equations.* Oxford: Glarendon Press.

Biazar, J., & Salehi, F. (2016). Chebyshev Galerkin method for integro-di□erential equations of the second kind. *Iranian J. of Numer. Analy. and Opt., 6*(1), 31–42.

Bildik, N., Konuralp, A., & Yal¸cinbas, S. (2010). Comparison of legendre polynomial approximation and variational iteration method for the solutions of general linear fredholm integro-differential equations. *Appl. Math. Comput, 59*, 1909–1917.

Burton, T. A., & Purnaras, I. K. (2017). Global existence and uniqueness of solutions of integral equations with delay: progressive contractions. *J. Qual. Theory of Diff. Eq., 49*, 1-6.

Cahlon, B., & Nachman, L. J. (1985). Numerical solutions of volterra integral equations with a solution dependent delay. *J. Math. Anal. and Appl., 112*, 541-562.

Cahlon, B., Schmidt, D., & Nachman, L. J.

(1984). Numerical solution of volterra integral equations with non-constant delay. *J. Integral Eq, 7*(3), 191–208.

Darania, P., & Ebadian, A. (2007). A method for the numerical solution of the integro-differential equations. *Appl. Math. Comput, 188*, 657–668.

Dehghan, M. (2006). Solution of a partial integro-differential equation arising from viscoelasticity. *Int. J. Comput. Math., 83*(1), 123-129.

El-Hawary, H. M., & El-Shami, K. A. (2013). Numerical solution of volterra delayintegro-differential equations via spline/spectral methods. *International Journal of Differential Equations and Applications, 12*, 149-157.

Eloe, P. W., Ra□oul, Y. N., & Tisdell, C. C. (2005). Existence, uniqueness and constructive results for delay di□erential equations. *Electronic J. Diff. Eq.*, 1-11.

Fathy, M., El-Gamel, M., & El-Azab, M. S. (2014). Legendre-galerkin method for linear fredholm integro-differential equations. *Applied Math. and Comput., 243*, 789–800.

Gulsu, M., & Sezer, M. (2006). Approximations to the solution of linear fredholm integro-differential-difference equation of high order. *J. Franklin Inst., 343*, 720 – 737.

Han, D., & Shang, X. (2010). Application of the variational iteration method for solving nth-order integro-differential equations. *J. of Computational and Applied Mathematics, 234*, 1442–1447.

Hosseini, S. M., & Shahmorad, S. (2003). Tau numerical solution of fredholm integro-di□erential equations with arbitrary polynomial bases. *Applied Mathematical Modelling, 27*, 145-154.

Hosseini, S. M., & Shahmorad, S. (2003a). Numerical solution of a class of integrodifferential equations by the tau method with an error estimation. *Applied Mathematics and Computation, 136*, 559-570.

Hou, J., & Yang, C. (2013). Numerical method in solving fredholm integro-differential equations by using hybrid function operational matrix of derivative. *J. Inf and Computational Sci., 10*(9), 2757–2764.

Issa, K., & Salehi, F. (2017). Approximate solution of perturbed Volterra-Fredholm integrodifferential equations by Chebyshev-Galerkin method. *Journal of Mathematics*, 8213932.

Jangveladze, T., Kiguradze, Z., & Neta, B. (2011). Galerkin finite element method for one nonlinear integro-differential model. *Appl. Math. Comp., 217*(16), 6883–6892.

Jimoh, A. K., & Issa, K. (2014). Comparison of some numerical methods for the solution of fourth order integro-differential equations. *J. Nig. Math. Physics, 28*, 115–122.

Kajani, M. T., Ghasemi, M., & Babolian, E. (2006). Numerical solution of linear integro-di□erential equation by using sine-cosine wavelets. *Appl. Math.Comput., 180*, 569–574.

Khater, A., Shamardan, A., Callebaut, D., & Sakran, M. (2007). Numerical solutions of integral and integro-differential equations using legendre polynomials. *Numer. Alg., 46*, 195–218.

Maleknejad, K., Sohrabi, S., & Rostami, Y. (2007). Numerical solution of nonlinear volterra integral equations of the second kind by using chebyshev polynomials. *Applied Mathematics and Computation, 188*, 123–128.

Maruo, K., & Park, D. G. (1994). Existence and uniqueness of solutions for integrodifferential equations with time delay in hilbert space. *J. Math. Ana. and Appl., 31*, 629–661.

Saadatmandi, A., & Dehghan, M. (2010). Numerical solution of the higher-order linear fredholm integro-differential-difference equation with variable coeffcients. *Computers and Mathematics with Applications, 59*, 2996 - 3004.

Sezer, M., & Gülsu, M. (2007). Polynomial solution of the most general linear fredholm-volterra integrodifferential difference equations by means of taylor collocation method, applied mathematics and computation. *Applied Mathematics and Computation, 185*, 646-657.

Shahmorad, S. (2005). Numerical solution of the linear fredholm-volterra integrodifferential equations by the tau method with an error estimation. *Appl. Math. and Computation, 167*(2), 1418-1429.

Shahmorad, S., & Ostadzad, M. H. (2016). An operational matrix method for solving delay fredholm and volterra integro-differential equations. *International Journal of Computational Methods, 13*(6), 1650040.

Shahsavara, A. (2010). Numerical solution of linear volterra and fredholm integro differential equations using haar wavelets. *Math. Sci. J.,, 6*, 85–96.

Vătă, P. (1978). Convergence theorems for some numerical approximation schemes for the class of non-linear integral equations. *Bul. Univ. Galati Anal., 1*, 25–38.

Yalçinbas, S., & Sezer, M. (2006). A Taylor collocation method for the approximate solution of general linear fredholm volterra integro-di□erence equations with mixed argument. *Appl. Math. Comput., 175*, 675-690.

Yusufoglu, E. (2009). Improved homotopy perturbation method for solving fredholm type integro-di□erential equations. *Chaos Solitons Fractals, 41*, 28–37.

Yüzbaşi, Ş. (2017). Shifted legendre method with residual error estimation for delaylinear fredholm integro-differential equations. *J. of Taibah University for Science, 11*, 344–352.

Yüzbaşi, Ş., & Ismailov, S. (2014). Solving systems of volterra integral and integro-differential equations with proportional delays by differential transformation method. *J. Math., 725648*, 1-5.

Yüzbasi, S., Gök, E., & Sezer, M. (2013). Müntz-legendre polynomial solutions of linear delay fredholm integro-differential equations and residual corrections. *Math. Comput. Appl., 18*, 476–485.