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Analysis for Quadrupled Fixed Point of Contractive-Type Multi-Valued Operators

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Abstract

This work considered the existence, uniqueness and data dependence of quadrupled fixed point theorems for contractions in metric spaces, equipped with vector-valued metrics whose approach is primarily based on Perov-type fixed point of contractive-type multi-valued mapping in Cauchy spaces. This work obtained results that complement recent and available results in literature.

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INTRODUCTION

For the introductory part of this work, see (Perov, 1964; Guo & Lakshmikantham, 1987; Bhaskar & Lakshmikantham, 2006; Lakshmikantham & Ciric, 2009; Rus, 2009; Berinde & Borcut, 2011; Petru *et al.*, 2011; Cho *et al.*, 2013; Gupta, 2013; Gupta *et al.*, 2014; Gupta, 2014; Gupta, 2016; Rauf & Aniki, 2021). Hence, this work is a continuation to the work of Aniki and Rauf (2020), which complements the results that was recently obtained.

MATERIALS AND METHODS

We summarize in the following the simple notions and results established in view of their generalization.

Definition 1 (Aniki & Rauf, 2020). Let X be a non-empty set. A mapping $d: X^2 \to \mathbb{R}$ is referred to as the distance function between u,v in Xif the following properties are satisfied:

i. $d(u,v) \geq 0$ for all $u,v \in X$,
ii. $d(u,v) = 0$ if and only $u = v$
iii. $d(u,v) = d(v,u)$ for all $u,v \in X$
iv. $d(u,v) \leq d(u,w) + d(w,v)$ for all $u,v,w \in X$

If $u,v \in \mathbb{R}$, $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_m)$, then, by definition: $u \leq v$ if and only if $u_i \leq v_i$ for $i \in \{1,2, \dots, m\}$.

A set endowed with a vector-valued metric d is called generalized metric space.

We denote with the aid of$M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements and by I the identity matrix.

Theorem 1 (Aniki & Rauf, 2020). Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalent,

i. A is convergent towards the zero matrix,
ii. $A^n \to 0$as$n \to \infty$
iii. The eigenvalues of A are in the open unit disc, i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with
$$det(A - \lambda I) = 0$$
iv. The matrix $(I - A)$ is nonsingular and
$$(I - A)^{-1} = I + A + \cdots + A^n + \cdots \qquad (1)$$
v. The matrix $(I - A)$ is nonsingular and $(I - A)^{-1}$ has nonnegative elements.
vi. $A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$ for each $q \in \mathbb{R}^m$.

Theorem 2 (Aniki & Rauf, 2020). Let (X,d) be a complete generalized metric space and the operator $f: X \to X$ with the property that there exists a matrix $A \in M_{mm}(\mathbb{R})$ such that $d\big(f(u),f(v)\big) \leq Ad(u,v)$ for all $u,v \in X$. If A is a matrix convergent towards zero matrix. Then,

i. $Fix(f) = \{u^*\}$
ii. The sequence of successive approximations $(u_n)_{n\in\mathbb{N}}, u_n = f^n(u_0)$ is convergent and has the limit u^*, for all $u_0 \in X$
iii. One has the following estimation
$$d(u_n,u^*) \leq A^n(I - A)^{-1} d(u_0,u_1) \qquad (2)$$
iv. If $g: X \to X$ is an operator such that there exists $v^* \in Fix(g)$ and $\epsilon \in (\mathbb{R}^+_m)$ with $d\big(f(u),g(u)\big) <\in$ for each $u \in X$ then$d(u^*,v^*) \leq (I - A)^{-1}\epsilon$
v. If $g: X \to X$ is an operator such that there exists $\epsilon \in (\mathbb{R}^+_m)^*$ such that $\big(f(u),g(u)\big) \leq \epsilon$ for all $u \in X$ then for the sequence $v_n = g^n(x_0)$ we have the following estimation
$$d(v_n,u^*) \leq (I - A)^{-1}\epsilon + A^n(I - A)^{-1} d(v_0,v_1). \qquad (3)$$

Definition 2 (Aniki & Rauf, 2020). Let (X, d) be a metric space. The system of operational equations
$$u = T_1(u,v,w,x)$$
$$v = T_2(u,v,w,x)$$
$$w = T_3(u,v,w,x)$$
$$x = T_4(u,v,w,x)$$
where $T_1,T_2,T_3,T_4: X^4 \to X$are four mappings. Then, the solution $(u,v,w,x) \in X^4$ of the system is referred to as a quadrupled fixed point for (T_1,T_2,T_3,T_4)

Definition 3 (Aniki & Rauf, 2020). Let (X,d) be a generalized metric space with the operator $f: X \to X$. Then, the fixed point equation
$$u = f(u) \qquad (4)$$

am-Hyers stable if there exists an increasing function $\psi: \mathbb{R}_+^m \to \mathbb{R}_+^m$ continuous at zero with $\psi(0) = 0$ such that for any $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ with $\epsilon_i > 0$ for $i \in \{1, \dots, m\}$ and any solution $u^* \in X$ of the inequality

$$d(v, f(v)) \leq \epsilon \quad (5)$$

there exists a solution u^* of (4) such that

$$d(u_n, u^*) \leq \psi(\epsilon) \quad (6)$$

In particular, if $\psi(t) = ct$, $t \in \mathbb{R}_+^m$, (where $c \in M_{mm}(\mathbb{R}_+)$), then the fixed point equation (4) is called Ulam-Hyers stable.

MAIN RESULTS

Firstly taking note of some notations, let (X,d) be a generalized metric space with $d: X \times X \to \mathbb{R}_+^m$ given by

$$d(u,v) = \begin{pmatrix} d_1(u,v) \\ \vdots \\ d_m(u,v) \end{pmatrix}$$

Then, for $u \in X$ and $A \subseteq X$ denote:

$$D_d(u,A) = \begin{pmatrix} D_{d_1}(uA) \\ \vdots \\ D_{d_m}(u,A) \end{pmatrix} = \begin{pmatrix} \inf_{a \in A} d_1(u,a) \\ \vdots \\ \inf_{a \in A} d_m(u,a) \end{pmatrix}$$

$$P(U) = \{V \subseteq U / V \text{ is nonempty}\}$$

$$P_{cl} = \{V \subseteq U / V \text{ closed}\}.$$

Also denote

$$\mathfrak{D}((u,v,w,x), A \times B \times C \times D) = \begin{pmatrix} \mathfrak{D}_d(u,A) \\ \mathfrak{D}_d(v,B) \\ \mathfrak{D}_d(w,C) \\ \mathfrak{D}_d(x,D) \end{pmatrix}$$

The following result is the existence, uniqueness, data dependence and Ulam-Hyers stability theorem for quadrupled fixed point of multi-valued operators (T_1, T_2, T_3, T_4). for the proof of the result, the following theorem is given.

Theorem 3. Let (X,d) be a complete generalized metric space and let $T: X \to P_{cl}(X)$ be a multi-valued $A-$contraction, i.e. there exists $A \in M_{mm}(\mathbb{R}_+)$ which converges towards zero as $n \to \infty$ and for each $u,v \in X$ and each $p \in T(u)$ there exists $q \in T(v)$ such that $d(p,q) \leq Ad(u,v)$. Then, T is a $MP-$operator, i.e. $Fix(T) \neq \emptyset$ and for each $(u,v) \in Graph(T)$ there exists a sequence $(u_n)_{n \in N}$ of successive approximations for T starting from (u,v) which converges to a fixed point $u^*$of T. Moreover, $d(u,u^*) \leq (I - A)^{-1} d(u,v)$, for all $(u,v) \in Graph(T)$.

Proof. Let $u_0 \in X$ and $u_1 \in T(u_0)$. Then, by the $A-$contraction condition, there exists $u_2 \in T(u_1)$ such that $d(u_1,u_2) \leq Ad(u_0,u_1)$. Now, for $u_2 \in T(u_1)$ there exists $u_3 \in T(u_2)$ such that

$$d(u_2,u_3) \leq Ad(u_1,u_2) \leq A^2 d(u_0,u_1).$$

Now, by an iterative construction, the sequence $(u_n)_{n \in N}$ is gotten such that $u_0 \in X$ and $u_{n+1} \in T(u_n)$, then $d(u_n,u_{n+1}) \leq A^n d(u_0,u_1)$ for all $n \in N$.

Thus, by the relation above, then

$$\begin{aligned} d(u_n,u_{n+p}) &\leq d(u_n,u_{n+1}) + d(u_{n+1},u_{n+2}) \\ &\quad + \cdots + d(u_{n+p-1},u_{n+p}) \\ &\leq Ad(u_0,u_1) + A^2 d(u_0,u_1) + \cdots + A^p d(u_0,u_1) \\ &= A(I + A + \cdots + A^{p-1}) d(u_0,u_1) \end{aligned}$$

On letting $n \to \infty$, then, the sequence $(u_n)_{n \in N}$ is Cauchy. Hence, there exists $u^* \in X$ such that $u^* \in T(u^*)$. Indeed, for $u_n \in T(u_{1-1})$ there exists $p_n \in T(u^*)$ such that

$$d(u_n, u_p) \leq Ad(u_{n-1}, u^*),$$

for all $n \in N$.

Conversely,

$$\begin{aligned} d(u^*,p_n) &\leq d(u^*,u_n) + d(u_n,p_n) \\ &\leq d(u^*,u_n) + Ad(u_{n-1},u^*) \to 0, \text{ as } n \to \infty. \end{aligned}$$

Hence,

$$\lim_{n \to \infty} p_n = u^*.$$

But $p_n \in T(u^*)$, for $n \in N$ and because $T(u^*)$ is closed, then $u^* \in T(u^*)$,

writing

$$\begin{aligned} d(u_n,u_{n+p}) &\leq A(I + A + \cdots + A^{p-1} \\ &\quad + \cdots) d(u_0,u_1) \\ &= A(I - A)^{-1} d(u_0,u_1). \end{aligned}$$

Letting $p \to \infty$, then

$$d(u_n,u^*) \leq A(I - A)^{-1} d(u_0,u_1)$$

for all $n \geq 1$. Thus

$$\begin{aligned} d(u_0,u^*) &\leq d(u_0,u_1) + d(u_1,u^*) \\ &\leq d(u_0,u_1) + A(I - A)^{-1} d(u_0,u_1) \\ &= (I + A(I - A)^{-1}) d(u_0,u_1) \\ &= (I + A + A^2 + \cdots) d(u_0,u_1) \\ &= (I - A)^{-1} d(u_0,u_1). \end{aligned}$$

Definition 4. Let (X,d) be a generalized metric space and $T: X \to P(X)$. Then, the fixed point inclusion

$$u \in F(u), \quad u \in X \quad (7)$$

is called the generalized Ulam-Hyers stable if and only if there exists $\Psi:\mathbb{R}_+^m \to \mathbb{R}_+^m$ increasing, continuous at 0 with $\Psi(0)=0$ such that for each $\epsilon=(\epsilon_1,\dots,\epsilon_m)>0$ and for each ϵ −solution v^* of (7), i.e.

$$\mathfrak{D}_d(v^*,F(v^*))\le \epsilon$$

there exists a solution u^* of the fixed point inclusion (7) such that

$$d(v^*,u^*)\le \Psi(\epsilon).$$

Now, if $\Psi(t)=ct$, for each $t\in\mathbb{R}_+^m$ (where $c\in M_{mm}(\mathbb{R}_+)$), then (7) is said to be Ulam-Hyers stable.

Definition 5. A subset U of a generalized metric space (X,d) is called Proximinal if for each $u\in X$ there exists $p\in U$ such that $d(u,p)=\mathfrak{D}_d(u,U)$.

Theorem 4. Let (X,d) be a complete generalized metric space and let $T:X\to P_{cl}(X)$ be a multi-valued A −contraction with proximinal values. Then, the fixed point inclusion (7) is Ulam-Hyers stable.

Proof. Let $\epsilon=(\epsilon_1,\dots,\epsilon_m)$ with $\epsilon_1>0$, for each $i\in 1,2,\dots,m$ and let $v^*\in X$ an ϵ −solution of (7), i.e.,

$$\mathfrak{D}_d(v^*,T(v^*))\le \epsilon.$$

By the conclusion of Theorem 3, it was shown that for any $(u,v)\in Graph(T)$

$$d(u,u^*(u,v)) \le (I-A)^{-1}d(u,v), \tag{8}$$

where the fixed point of T obtained by Theorem 3 is denoted by $u^*(u,v)$, which is from successive approximations with initial point of (u,v). Since $T(v^*)$ is proximinal, there exists $q\in T(v^*)$ such that

$$d(v^*,q)=\mathfrak{D}_d(v^*,T(v^*)).$$

Hence, by (4)

$$d(v^*,u^*(v^*,q^*))\le (I-A)^{-1}d(v^*,q^*)$$
$$\le (I-A)^{-1}\epsilon.$$

Theorem 5. Let (X,d) be a complete generalized metric space and let $T:X\to P_{cl}(X)$ be a multi-valued A −contraction such that there exists $u^*\in X$ with $T(u^*)=\{u^*\}$. Then, the fixed point inclusion (7) is Ulam-Hyers stable.

Proof. Let $\epsilon=(\epsilon_1,\dots,\epsilon_m)$ with $\epsilon_1>0$, for each $i\in 1,2,\dots,m$ and let $v^*\in X$ an ϵ −solution of (7), i.e.,

$$\mathfrak{D}_d(v^*,T(v^*))\le \epsilon.$$

By the A −contraction condition, for $u=v^*, v=u^*$ and $p\in T(v^*)$ it is gotten that

$$d(p^*,u^*)\le Ad(v^*,u^*).$$

For any $p\in T(v^*)$, then,

$$d(v^*,u^*)\le d(v^*,p^*)+d(p^*,u^*)$$
$$d(v^*,p^*)+A.\,d(v^*,u^*).$$

Hence,

$$d(v^*,u^*)\le (I-A)^{-1}d(v^*,p^*),$$

for any $p\in T(v^*)$. Thus

$$d(v^*,u^*)\le (I-A)^{-1}\mathfrak{D}_d(v^*,T(v^*))$$
$$\le (I-A)^{-1}\epsilon.$$

Definition 6. Let (X,d) be a metric space. With the following operational inclusions:

$$\left.\begin{array}{l} u\in T_1(u,v,w,x)\\ v\in T_2(u,v,w,x)\\ w\in T_3(u,v,w,x)\\ x\in T_4(u,v,w,x)\end{array}\right\}$$

where $T_1,T_2,T_3,T_4:X^4\to P(X)$ are four given multi-valued operators. By definition, a solution $(u,v,w,x)\in X^4$ of the system (9) is called a quadrupled fixed point for(T_1,T_2,T_3,T_4).

Definition 7. Let (X,d) be a metric space and let $T_1,T_2,T_3,T_4:X^4\to P(X)$ be four multi-valued operators. Then, the operational system (6) is said to be Ulam-Hyers stable if there exists $c_i>0$ for $i=1,2,\dots,16$, such that for each $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4>0$ and for each quadruple $(p^*,q^*,r^*,s^*)\in X^4$ which satisfies the relations

$$\left.\begin{array}{l} d(p^*,\alpha)\le\epsilon_1\ \forall\ \alpha\in T_1(p^*,q^*,r^*,s^*)\\ d(q^*,\beta)\le\epsilon_2\ \forall\ \beta\in T_2(p^*,q^*,r^*,s^*)\\ d(r^*,\gamma)\le\epsilon_3\ \forall\ \gamma\in T_3(p^*,q^*,r^*,s^*)\\ d(s^*,\kappa)\le\epsilon_4\ \forall\ \kappa\in T_4(p^*,q^*,r^*,s^*)\end{array}\right\}$$

there exists a solution $(u^*,v^*,w^*,x^*)\in X^4$ of (9) such that

$$\left.\begin{array}{l} d(p^*,u^*)\le c_1\epsilon_1+c_2\epsilon_2+c_3\epsilon_3+c_4\epsilon_4\\ d(q^*,v^*)\le c_5\epsilon_1+c_6\epsilon_2+c_7\epsilon_3+c_8\epsilon_4\\ d(r^*,w^*)\le c_9\epsilon_1+c_{10}\epsilon_2+c_{11}\epsilon_3+c_{12}\epsilon_4\\ d(s^*,x^*)\le c_{13}\epsilon_1+c_{14}\epsilon_2+c_{15}\epsilon_3+c_{16}\epsilon_4\end{array}\right\}$$

Definition 8. Let (X,d) be a metric space and $S:X^4\to P(X)$ has proximinal values with respect to the first variable if for any $u,v,w,x\in X$ there exists $p,q,r,s\in(u,v,w,x)$ such that

$$d(u,p)=\mathfrak{D}_d(u,S_1(u,v,w,x))$$
$$d(v,q)=\mathfrak{D}_d(v,S_2(u,v,w,x))$$
$$d(w,r)=\mathfrak{D}_d(w,S_3(u,v,w,x))$$
$$d(x,s)=\mathfrak{D}_d(x,S_4(u,v,w,x))$$

which leads to the next main results.

Theorem 6. Let (X,d) be a metric space and let $T_1,T_2,T_3,T_4: X^4 \to P_{cl}(X)$ be four multi-valued operators. Suppose that T_1,T_2,T_3,T_4 has proximinal values with respect to the first, second, third and fourth variable. For each $(u,v,w,x),(p,q,r,s) \in X^4$ and each $\alpha_1 \in T_1(u,v,w,x)$, $\alpha_2 \in T_2(u,v,w,x)$, $\alpha_3 \in T_3(u,v,w,x)$, $\alpha_4 \in T_4(u,v,w,x)$, there exists $\beta_1 \in T_1(p,q,r,s)$, $\beta_2 \in T_2(p,q,r,s)$, $\beta_3 \in T_3(p,q,r,s)$, $\beta_4 \in T_4(p,q,r,s)$ satisfying

$$d(\alpha_1,\beta_1) \leq k_1 d(u,p) + k_2 d(v,q) + k_3 d(w,r) + k_4 d(x,s)$$
$$d(\alpha_2,\beta_2) \leq k_5 d(u,p) + k_6 d(v,q) + k_7 d(w,r) + k_8 d(x,s)$$
$$d(\alpha_3,\beta_3) \leq k_9 d(u,p) + k_{10} d(v,q) + k_{11} d(w,r) + k_{12} d(x,s)$$
$$d(\alpha_4,\beta_4) \leq k_{13} d(u,p) + k_{14} d(v,q) + k_{15} d(w,r) + k_{16} d(x,s)$$

suppose that

$$A = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \\ k_9 & k_{10} & k_{11} & k_{12} \\ k_{13} & k_{14} & k_{15} & k_{16} \end{pmatrix}$$

converges to zero. Then,

i. there exists $(u^*,v^*,w^*,x^*) \in X^4$ a solution for (9).
ii. the system of operational inclusions (9) is Ulam-Hyers stable.

Proof. Defining $T: X^4 \to P_{cl}(X) \times P_{cl}(X) \times P_{cl}(X) \times P_{cl}(X)$ by

$$T(u,v,w,x) = T_1(u,v,w,x) \times T_2(u,v,w,x) \times T_3(u,v,w,x) \times T_4(u,v,w,x).$$

Denote

$\Gamma = X \times X \times X \times X$ and consider $\tilde{d}: \Gamma \times \Gamma \to \mathbb{R}^4_+$,

$$\tilde{d}\big((u,v,w,x),(p,q,r,s)\big) = \begin{pmatrix} d(u,p) \\ d(v,q) \\ d(w,r) \\ d(x,s) \end{pmatrix}.$$

Going by the hypothesis of the theorem, $s = (u,v,w,x)$, $t = (p,q,r,s) \in X^4$ and each $\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in T(u,v,w,x)$, there exists $\beta = (\beta_1,\beta_2,\beta_3,\beta_4) \in T(p,q,r,s)$ satisfying the relation $\tilde{d}\left((\alpha,\beta) \leq A\tilde{d}(s,t)\right)$, which proves that T is a multi-valued $A-$contraction. Since $T_1(u,v,w,x) \subset X$, $T_2(u,v,w,x) \subset X$, $T_3(u,v,w,x) \subset X$, $T_4(u,v,w,x) \subset X$, is proximinal with respect to the first, second, third and fourth variables respectively, for any $u,v,w,x \in X$ there exists $p \in T_1(u,v,w,x)$, $q \in T_2(u,v,w,x)$, $r \in T_3(u,v,w,x)$, $s \in T_4(u,v,w,x)$ such that

$$d(u,p) = \mathfrak{D}_d\big(u,T_1(u,v,w,x)\big),$$
$$d(v,q) = \mathfrak{D}_d\big(v,T_2(u,v,w,x)\big),$$
$$d(w,r) = \mathfrak{D}_d\big(w,T_3(u,v,w,x)\big),$$
$$d(x,s) = \mathfrak{D}_d\big(x,T_4(u,v,w,x)\big).$$

Then, the set $T(u,v,w,x) = T_1(u,v,w,x) \times T_2(u,v,w,x) \times T_3(u,v,w,x) \times T_4(u,v,w,x)$ is proximinal, since for any $u,v,w,x \in X$ there exists $(p,q,r,s) \in T(u,v,w,x)$ such that

$$\tilde{d}\big((u,v,w,x),(p,q,r,s)\big) = \mathfrak{D}_{\tilde{d}}\big((u,v,w,x),T(u,v,w,x)\big).$$

which is the conclusion.

Remark 1. Notice that, if (X,d) is a metric space and $T: X^4 \to P(X)$ is a multi-valued operator and define

$$T_1(u,v,w,x) = T(u,v,w,x),\ T_2(u,v,w,x) = T(v,u,v,x),\ T_3(u,v,w,x) = T(w,u,v,w),\ T_4(u,v,w,x) = T(x,w,v,u),$$

then, the approach leads to some quadrupled fixed point theorem in the classical sense.

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