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An Efficient Technique for Solving Systems of Integral Equations

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Abstract

In this paper, the wavelet method based on the Chebyshev polynomials of the second kind is introduced and used to solve systems of integral equations. Operational matrices of integration, product, and derivative are obtained for the second kind Chebyshev wavelets which will be used to convert the system of integral equations into a system of algebraic equations. Also the error is analyzed and at the end, some examples are presented to demonstrate the efficiency and validity of the proposed method.

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INTRODUCTION

Since few of functional equations can be solved explicitly, it is necessary to develop some methods to obtain approximate solutions of functional equations. In recent years, approximating by using orthogonal functions and polynomials has been developed to estimate the solution in different disciplines. The main idea of using an orthogonal basis is that the problem under study reduces into a system of linear or nonlinear algebraic equations by considering truncated series of orthogonal basis functions with unknown coefficients for the solution of problem and using the operational matrices.

The wavelet theory as an orthogonal system is relatively new and in recent years has been used in different fields of science and engineering. In this paper, a new wavelet, based on orthogonal Chebyshev polynomials of the second kind, is introduced and their operational matrices are obtained and then they are used to solve systems of integral equations. These systems arise from mathematical modeling of many phenomena in science and engineering and some methods, for solving such systems, have been proposed in the literature such as Adomian decomposition (Biazar et al., 2003). Homotopy perturbation (Biazar et al., 2009; Biazar & Ghazvini, 2009), Variational iteration (Biazar & Aminikhah, 2009), Adomian-Pade technique (Dehghan et al, 2009), Runge-Kutta (Maleknejad & Shahrezaee, 2004), Tau (Abbasbandy & Taati, 2009; Pour-Mahmoud et al., 2005), radial basis functions (Golbabai et al., 2009) and etc.

This paper is organized as follows: Section 1 is devoted to introduction; in Section 2 the second kind Chebyshev wavelets are introduced and their operational matrices of integration, product, and derivative are computed; the error bound is obtained in Section 3; some examples are presented in Section 4; Conclusions are given in the final section, 5.

THE SECOND KIND CHEBYSHEV WAVELETS

Wavelets are functions generated from one single function $\psi(x)$, with some vibrations and a zero average, $\int_R \psi(x)dx=0$, that called the mother wavelet, by the operations of dilation and translation (Daubeches, 1992; Hernandez & Weiss, 1996; Chui, 1997; Christensen & Christensen, 2004; Chau, 2004; Walnut, 2004). In other words, wavelets are small waves with rapid decay or compact supports, and thus well localization abilities. When the dilation parameter, a, and the translation parameter, b, vary continuously, we have the following family of continuous wavelets,

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right), \quad a,b\in\mathbb{R}, \quad a\neq 0, \tag{1}$$

If we take the dilation and translation parameters a^{-j}, and kba^{-j}, respectively where $a>1,\ b>0$, j and k are positive integers, then we have the following family of discrete wavelets,

$$\psi_{j,k}(x) = |a|^{\frac{j}{2}}\ \psi(a^j x - kb), \tag{2}$$

These functions generate a wavelet basis for $L^2(\mathbb{R})$, and for special case $a=2$, and $b=1$, the functions $\psi_{(j,k)}(x)$ are an orthonormal basis. As a result, by changing k, the function is shifted on the x-axis and by changing j, the domain of function is varied. The orthogonal wavelets are connected with multiresolution analysis. A multiresolution analysis of $L^2(\mathbb{R})$ is a sequence of closed subspaces V_j of $L^2(\mathbb{R})$ that satisfy the following conditions,

1) $\{0\}\subset\cdots\subset V_{-1}\subset V_0\subset V_1\subset\cdots\subset L^2(\mathbb{R})$,
2) $\cup_{j\in Z}=L^2(R)$,
3) $\cap_{j\in Z}=\{0\}$,
4) $\forall j\ (j\in\mathbb{Z},\ f(x)\in V_j \quad f(2x)\in V_{j+1})$,
5) $\forall k\ (k\in\mathbb{Z}, f(x)\in V_0 \quad f(x-k)\in V_0)$,
6) There are $\phi\in V_0$ such that $\{\phi(x-k)\}_{k\in Z}$ is an orthonormal basis for V_0.

For every $j\in\mathbb{Z}$, define W_j to be the orthogonal complement of V_j in V_{j+1}, denoted by $V_j\perp W_j$ such that $V_{j+1}=V_j\oplus W_j$. In fact, W_j includes the information that V_j converts to V_{j+1}. The sequence $\{\phi_{j,k}(x)|\ k\in\mathbb{Z}\}=\{2^{j/2}\ \phi(2^j\ x-k)|\ k\in\mathbb{Z}\}$ is an orthonormal basis for Vj and $\{\psi_{(j,k)}(x)|\ k\in\mathbb{Z}\}=\{2^{j/2}\ \psi\ 2^j\ x-k)|\ k\in\mathbb{Z}\}$ is an orthonormal basis for W_j.

Wavelets $\psi_{nm}(x)=\psi(x;k,n,m)$ are defined on the interval [0,1], as follows:

$$\psi_{nm}(x) = \psi(x;k,n,m) = \begin{cases} 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}\,U_m\left(2^k x - 2n + 1\right), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}, \\ 0, & \end{cases} \tag{3}$$

Where k is a positive integer, m is the order of Chebyshev polynomials of the second kind, and $n=1,2,\ldots,2^{k-1}$. $U_m(x)$ is the famous Chebyshev polynomial of the second kind of degree m, which is orthogonal with respect to the weight function $W(x)=\sqrt{(1-x^2)}$ on the interval [-1,1]. These polynomials satisfy the following differential equation,

$$(1-x^2)\, U''_m(x)-3x\, U^{\wedge\prime}_m(x)+n(n+2)\, U_m(x), \quad (4)$$

and can be obtained by recursive formula as follows (Gautschi, 2004),

$$\begin{cases} U_0(x)=1, \\ U_1(x)=2x, \\ U_{m+1}(x)=2xU_m(x)-U_{m-1}(x), \quad m=1,2,\ldots, \end{cases} \quad (5)$$

The set of Chebyshev wavelets of the second kind, $\psi_{nm}(x)$, is an orthogonal set with respect to the weight function $W_n(x)=W(2^k x-2n+1)$. A square-integrable function f(x), defined on the interval [0,1], can be presented as,

$$f(x)=\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}\, \psi_{nm}(x), \quad (6)$$

The series representation of in (6) is called a wavelet series and the wavelet coefficients, c_{nm}, are given by $c_{nm}=(f(x), \psi_{nm}(x))_{W_n(x)}$. The convergence of the series (6), in L^2 ([0,1]), means that,

$$\lim_{S_1,S_2\to\infty} \left\| f(x)-\sum_{n=1}^{S_1}\sum_{m=0}^{S_2} c_{nm}\,\psi_{nm}(x)\right\| = 0, \quad (7)$$

(For more information see reference (Chui, 1997)). Therefore, one can consider the following truncated series for infinity series (6),

$$f(x)\approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\,\psi_{nm}(x) = C^T\,\psi(x), \quad (8)$$

Where C and $\psi(x)$ are $2^{k-1}M\times 1$ matrices given by

$$C=[c_{10}, c_{11},\ldots,c_{1\,M-1},c_{20},c_{21},\ldots,c_{2\,M-1},\ldots,c_{2^{k-1}\,0}),\ldots,c_{2^{k-1}\,M-1})]^T \quad (9)$$
$$=[c_1,c_2,\ldots,c_M, c_{M+1},\ldots,c_{2^{k-1}M}]^T$$

and

$$\psi(x)=[\psi_{10}(x),\psi_{11}(x),\ldots,\psi_{1\,M-1}(x),\psi_{20}(x),\psi_{21}(x),\ldots,\psi_{2\,M-1}(x),\ldots,\psi_{2^{k-1}\,0}(x), \quad (10)$$
$$\ldots,\psi_{2^{k-1}\,M-1}(x)]^T$$
$$=[\psi_1(x),c_2(x),\ldots,c_M(x),c_{(M+1)}(x),\ldots,c_{2^{k-1}M}(x)]^T,$$

The integration of the product of two Chebyshev wavelets vector functions, with respect to the weight vector function $W(x)$, is derived as,

$$\int_0^1 W(x)\,\psi(x)\,\psi^T(x)dx=I, \quad (11)$$

where I is an identity matrix.

A function $f(x,y)$, defined on the domain [0,1]×[0,1], can be approximated as the following,

$$f(x,y)\approx\psi^T(x)\,K\,\psi(y). \quad (12)$$

Here the entries of matrix $K=[k_{ij}]_{2^{k-1}M\times 2^{k-1}M}$ can be obtain by,

$$k_{ij}=\left(\psi_i(x),\left(f(x,y),\psi_j(x)\right)_{W_j(y)}\right)_{W_i(y)}, \quad (13)$$

$i,j=1,2,\ldots,2^{k-1}M$.

The operational matrix of integration

The integration of the vector $\psi(x)$, defined in (10), can be achieved as,

$$\int_0^x \psi(t)\,dt = P\,\psi(x), \quad (14)$$

Where P is the $2^{k-1}M\times 2^{k-1}M$ operational matrix of integration and is determined as follows,

$$P=\frac{1}{2^k}\begin{bmatrix} L & F & F & \ldots & F \\ O & L & F & \ldots & F \\ O & O & L & \ldots & F \\ \vdots & \vdots & \vdots & \ddots & F \\ O & O & O & \ldots & L \end{bmatrix}, \quad (15)$$

where L, F, and O are $M\times M$ matrices given by,

$$L=\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \ldots & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 & \ldots & 0 \\ \frac{1}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ \frac{(-1)^{i+1}}{i} & & \ldots & -\frac{1}{2i} & 0 & \frac{1}{2i} & \ldots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ \frac{(-1)^M}{M-1} & 0 & 0 & \ldots & & -\frac{1}{2(M-1)} & \\ \frac{(-1)^{M+1}}{M} & 0 & 0 & \ldots & -\frac{1}{2M} & & 0 \end{bmatrix}, \quad (16)$$

$$F=\begin{bmatrix} 2 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \frac{2}{3} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1-(-1)^i}{i} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1-(-1)^M}{M} & 0 & \ldots & 0 \end{bmatrix}, \quad (17)$$

$$O = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \qquad (18)$$

The operational matrix of derivative

The operational matrix of derivative, D, is given by,

$$\frac{d\,\psi(x)}{d\,x} = D\,\psi(x), \qquad (19)$$

Where D is the $2^{k-1}M \times 2^{k-1}M$ matrix. This matrix can be determined as follows,

$$D = \begin{bmatrix} \tilde{D} & O & \dots & O \\ O & \tilde{D} & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & \tilde{D} \end{bmatrix}, \qquad (20)$$

in which O is a $M \times M$ null matrix and $\tilde{D}$ is $M \times M$ lower triangular matrix and its entries are derived as follows,

$$\tilde{D}_{i,j} = \begin{cases} \left(1-(-1)^{i+j}\right)j, & i > j, \\ 0, & i \le j, \end{cases} \qquad (21)$$

The operational matrix of Product

The product of two vector bases of the Chebyshev wavelets of the second kind is as,

$\psi(x)\ \ \psi^T(x)\ \ C \approx \tilde{C}\ \ \psi(x),$ (22)

where C is the vector defined as (9), and $\tilde{C}$ is a $2^{k-1}M \times 2^{k-1}M$ matrix. This matrix is called the operational matrix of product and is defined as the following,

$$\tilde{C} = \begin{bmatrix} \tilde{C}_1 & O & \dots & O \\ O & \tilde{C}_2 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & \tilde{C}_{2^{k-1}} \end{bmatrix}, \qquad (23)$$

Where O is a $M \times M$ null matrix, and $\tilde{C}_r = 1,2,\dots,2^{k-1}$, are $M \times M$ symmetric matrices as follows,

$$\tilde{C}_r = \begin{bmatrix} S^r_{1,1} & S^r_{1,2} & \dots & S^r_{1,M} \\ S^r_{1,2} & S^r_{2,2} & \dots & S^r_{2,M} \\ \vdots & \vdots & \dots & \vdots \\ S^r_{1,M} & S^r_{2,M} & \dots & S^r_{M,M} \end{bmatrix}, \qquad (24)$$

$$r = 1,2,\dots,2^{k-1},$$

with the following entries,

$$S^r_{i,j} = \left((\psi\,\psi^T\,C)_n, \psi_m\right)_{W_m(x)}, \qquad (25)$$

$$r = 1,2,\dots,2^{k-1}, \quad i,j = 1,2,\dots,M,$$

where $n=(r-1)M+i$ and $m=(r-1)M+j$.

3. Error analysis

Lemma 3.1: The number of vanishing moments for a Chebyshev wavelet of the second kind, $\psi_{n\,m}(x)$, is equal to m, (Nielsen, 1998).

The decay of the wavelet coefficients is given by the following theorem.

Theorem 3.2: Assume that $f(x) \in C^m([0,1])$, and

$$C^T\,\psi(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n\,m}\,\psi_{n\,m}(x)$$

is the approximate solution resulted by using the Chebyshev wavelets method of the second kind. Then the wavelet coefficients, $c_{n\,m}$, decay as follows,

$$|c_{n\,m}| \le C_m\, 2^{-k\left(m+\frac{1}{2}\right)} \max_{\xi_x \in I_{n\,m}} \left|f^{(m)}(\xi_x)\right|, \qquad (26)$$

$$n = 1,2,\dots,2^{k-1}, \quad m = 0,1,\dots,M-1,$$

Where $I_{n\,m} = Supp(\psi_{n\,m}) = \{x \mid \psi_{n\,m}(x) \neq 0\}$ $= \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$ and C_m is a constant independent of n, k, and $f(x)$.

Proof: The wavelet coefficients can be obtained as follows,

$$c_{n\,m} = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} W_n(x)\, f(x)\, \psi_{n\,m}(x) dx, \qquad (27)$$

Consider the Taylor expansion of f(x) around $x = \frac{2n-1}{2^k}$, that is the middle point of subinterval $I_{n\,m}$.

$$f(x) = \sum_{i=0}^{m-1} \frac{\left(x - \frac{2n-1}{2^k}\right)^i}{i!} f^{(i)}\left(\frac{2n-1}{2^k}\right) + \frac{\left(x - \frac{2n-1}{2^k}\right)^m}{m!} f^{(m)}(\xi_x),$$

$$\xi_x \in \left[\frac{2n-1}{2^k}, x\right], \qquad (28)$$

Substituting Eq. (28) into (27) leads to,

$$c_{n\,m} = \sum_{i=0}^{m-1} \frac{f^{(i)}\left(\frac{2n-1}{2^k}\right)}{i!} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} W_n(x) \left(x - \frac{2n-1}{2^k}\right)^i \psi_{n\,m}(x) dx$$

$$+\frac{1}{m}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} W_n(x)\, f^{(m)}(\xi_x)\,\left(x-\frac{2n-1}{2^k}\right)^m \psi_{nm}(x)\,dx,$$

(29)

By regarding the change of variable $y=2^k x-2n+1$ in (29) one has,

$$\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} W_n(x)\left(x-\frac{2n-1}{2^k}\right)^i \psi_{nm}(x)dx$$

$$=2^{k\left(\frac{1}{2}-i\right)}\sqrt{\frac{2}{\pi}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} W_n(x)\left(2^k x-2n+1\right)^i U_m\left(2^k x-2n+1\right)dx$$

$$=2^{-k\left(i+\frac{1}{2}\right)}\sqrt{\frac{2}{\pi}}\int_{-1}^{1}\bar{W}_n(y)\, y^i\, U_m(y)dy,$$

Due to the Lemma 3.1, one has,

$$\int_{-1}^{1}\bar{W}_n(y)\, y^i\, U_m(y)dy=0, \qquad i=0,1,\ldots,m-1, \qquad (30)$$

Therefore,

$$c_{nm}=\frac{2^{-k\left(m+\frac{1}{2}\right)}}{m!}\sqrt{\frac{2}{\pi}}\int_{-1}^{1}\bar{W}_{nm}(y)\, f^{(m)}(\xi_x) y^m\, U_m(y)dy,$$

(31)

and

$$|c_{nm}|\le\frac{\max_{\xi_x\in I_{nm}}\left|f^{(m)}(\xi_x)\right|}{m!}2^{-k\left(m+\frac{1}{2}\right)}\sqrt{\frac{2}{\pi}}\int_{-1}^{1}\left|\bar{W}_{nm}(y)\, y^m\, U_m(y)\right|dy,$$

(32)

By considering $C_m=\frac{\sqrt{\frac{2}{\pi}}\int_{-1}^{1}|\bar{W}_{nm}(y)\, y^m\, U_m(y)|dy}{m!}$, the result will be obtained.

The above theorem implies that wavelet coefficients are exponentially decayed with respect to k and by increasing m the decay increases. Therefore, one has $\lim_{n,m\to\infty} c_{nm}=0,$

The error bound of the approximate solution via the Chebyshev wavelets of the second kind series is given by the following theorem.

Theorem 3.3: Suppose that *(x)∈C^M* ([0,1]). The error of approximate solution is bounded as follows, when the Chebyshev wavelets method of the second kind is used,

$$\|f(x)-C^T\psi(x)\|\le\frac{1}{M!\,2^{M(k-1)}}\max_{\xi\in[0,1]}\left|f^{(M)}(\xi)\right|,$$

(33)

Proof: Let's consider the following norm,

$$\|f(x)-C^T\psi(x)\|^2=\int_0^1 W(x)(f(x)-C^T\psi(x))^2dx,$$

(34)

Since the interval [0.1] is divided into the 2^{k-1} subintervals $I_{nm}=\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right]$, that the function f(x) is approximated on each the subinterval by using the Chebyshev wavelets method of the second kind as a polynomial at most of *(M-1)th* degree with the least-square property, therefore, it will be as,

$$\int_0^1 W(x)\left(f(x)-C^T\psi(x)\right)^2dx$$

$$=\sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} W_n(x)\,\left(f(x)-\sum_{m=0}^{M-1}c_{nm}\,\psi_{nm}(x)\right)^2dx$$

$$\le\sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} W_n(x)\,(f(x)-S_M(x))^2dx,$$

where $S_M(x)$ is any polynomial of degree *M-1*, that interpolates *f(x)* on I_{nm} with the following error bound,

$$|f(x)-S_M(x)|\le\frac{2}{M!\,2^{M(k-1)}}\max_{\xi_{nm}\in I_{nm}}\left|f^{(M)}(\xi_{nm})\right|,$$

(35)

Therefore, this error bound can be manipulated to get the results as follows,

$$\|f(x)-C^T\psi(x)\|^2\le\sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} W_n(x)\left(\frac{1}{M!\,2^{M(k-1)}}\max_{\xi_{nm}\in I_{nm}}\left|f^{(M)}(\xi_{nm})\right|\right)^2dx$$

$$\le\sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} W_n(x)\left(\frac{1}{M!\,2^{M(k-1)}}\max_{\xi\in[0,1]}\left|f^{(M)}(\xi)\right|\right)^2dx$$

$$=\int_0^1 W(x)\left(\frac{1}{M!\,2^{M(k-1)}}\max_{\xi\in[0,1]}\left|f^{(M)}(\xi)\right|\right)^2dx$$

$$=\left\|\frac{1}{M!\,2^{M(k-1)}}\max_{\xi\in[0,1]}\left|f^{(M)}(\xi)\right|\right\|^2,$$

NUMERICAL EXAMPLES

In this section, some examples of systems of integral and integro-differential equations are considered and will be solved by wavelet method based on Chebyshev polynomials of the second kind. Here, the computations associated with these examples, are performed by the package Maple.

Example 4.1: Consider the following system of linear Fredholm integro-differential equations,

$$\begin{cases} u''(x) + v'(x) + \int_0^1 2\,x\,t\,(u(t) - 3v(t))\,dt = 3x^2 + \frac{3}{10}x + 8, \\ v''(x) + u'(x) + \int_0^1 3(2x + t^2)\big(u(t) - 2v(t)\big)dt = 21x + \frac{4}{5}, \end{cases}$$

$0 \le x \le 1$,

(36)

with the initial conditions $u(0)=1$, $u'(0)=0$, $v(0)=-1$, and $v'(0)=2$. The exact solutions are $u(x)=3x^2+1$ and $v(x)=x^3+2x-1$, (Pour-Mahmoud et al. 2005).

Set $k=1$ and $M=8$. According the wavelets method, the following approximations are considered,

$u''(x) \approx \psi^T(x)\, C_1$, $\quad v''(x) \approx \psi^T(x)\, C_2$,

$u'(x) \approx \psi^T(x)\, P^T C_1$, $\quad v'(x) \approx \psi^T(x)\, P^T C_2 + 2 = \psi^T(x)\, P^T C_2 + \psi^T(x)\, V_1$,

$u(x) \approx \psi^T(x)\, (P^T)^2 C_1 + 1 = \psi^T(x)\, (P^T)^2 C_1 + \psi^T(x)\, U_0$,

$v(x) \approx \psi^T(x)\, (P^T)^2 C_2 + 2x - 1 = \psi^T(x)\, (P^T)^2 C_2 + \psi^T(x)\, V_0$,

$2xt \approx \psi^T(x)\, K_1\, \psi(t)$, $\quad 3(2x+t^2) \approx \psi^T(x)\, K_2\, \psi(t)$,

$3x^2 + \frac{3}{10}x + 8 \approx \psi^T(x)\, F_1$, $\quad 21x + \frac{4}{5} \approx \psi^T(x)\, F_2$.

By substituting these assumptions into (36) and multiplying both sides of the system by $W_n(x)\, \psi^T(x)$, and then applying $\int_0^2 (\cdot)\, dx$, the following linear system will be obtained,

$$\begin{cases} C_1 + P^T C_2 + V_1 + K_1(D(P^T)^2 C_1 + U_0 - 3P^T C_2 - 3V_0) = F_1, \\ P^T C_1 + C_2 + K_2(D(P^T)^2 C_1 + U_0 - 2P^T C_2 - 2V_0) = F_2, \end{cases}$$

(37)

By solving system (37), the elements of unknown vectors C_1 and C_2 can be obtained as follows,

$C_1 = [3\sqrt{\pi},0,0,0,0,0,0,0]^T$,

$C_2 = [3/2\,\sqrt{\pi}, 3/4\,\sqrt{\pi},0,0,0,0,0,0]^T$,

Then, the following solutions will be achieved,

$u(x) \approx \psi^T(x)\, (P^T)2\, C_1 + 1 = 3x^2 + 1$,

$v(x) \approx \psi^T(x)\, (P^T)^2\, C_2 + 2x - 1 = x^3 + 2x - 1$,

which are the exact solutions. In (Pour-Mahmoud et al, 2005), this example is solved by Tau method and approximate solutions were obtained while the proposed method leads to exact solution.

Example 4.2: In this example, the following non-linear system of Volterra integro-differential integral equations of the second kind with conditions $u(0)=1, u'(0)=1, v(0)=1$, and $v'(0)=1$, are studied, (Biazar & Aminikhah, 2009),

$$\begin{cases} u'''(x) = x - u'(x) - \int_0^x \left(u''^2(t) + v''^2(t)\right) dt, \\ v'''(x) = \sin x + \frac{1}{2}\sin^2 x + \int_0^x u''(t)v(t)dt, \end{cases}$$

$0 \le x \le 1$, (38)

The exact solutions are $u(x)=\sin x$ and $v(x)=\cos x$.

Set $k=1$ and $M=6$. The elements of vectors C_1 and C_2 are computed by solving the system of non-linear equations for twelve unknowns, via Maple package, as follows,

$c_{10}=-0.7536850066$, $\quad c_{11}=0.1040241385$,
$c_{12}=0.0239268783$, $\quad c_{13}=-0.0010926963$,
$c_{14}=-0.0001252716$, $\quad c_{15}=0.0000034220$,
$c_{20}=0.4117393776$, $\quad c_{21}=0.1904151228$,
$c_{22}=-0.0130713226$, $\quad c_{23}=0.0020001698$,
$c_{24}=0.0000684944$, $\quad c_{25}=0.0000065540$.

Therefore, we have the following approximate solutions,

$u(x)=0.007237318620x^5+0.001675525708x^4-0.1676986585x^3+0.0002907561959x^2+0.99999660244x+0.000001126152312$,

$v(x)=-0.003953768999x^5+0.04607106796x^4-0.002358733495x^3-0.4993893157x^2-0.00006709771441x+1.000002062$.

The plots of the exact and approximate solutions and the absolute error are shown in Fig.1. and some values of the absolute errors are presented in Table 1.

Example 4.3: Consider the following system of non-linear Fredholm integral equations of the second kind,

$$\begin{cases} u(x) = \frac{x}{3} - \frac{1}{12} + e - \frac{1}{2}e^2 + \int_0^1 \left((x-t)u(t)v(t) + v^2(t)\right) dt, \\ v(x) = e^x - \frac{6}{5}x - \frac{3}{2} + e + \int_0^1 \left(x^2 t^2\, u^2(t) - v(t)\right)dt, \end{cases}$$

$0 \le x \le 1$, (39)

with the exact solutions $u(x)=x$ and $v(x)=e^x-x$.

According to the wavelets method, consider $k=2$, $M=6$, and

$$u(x) \approx \sum_{n=1}^{2}\sum_{m=0}^{5} c_{n\,m}^{1}\, \psi_{n\,m}(x) = \psi^T(x)C_1,$$

$$v(x) \approx \sum_{n=1}^{2}\sum_{m=0}^{5} c_{n\,m}^{2}\, \psi_{n\,m}(x) = \psi^T(x)C_2,$$

Applying the Chebyshev wavelets approach of the second kind, the following results will be obtained for vectors C_1 and C_2.

$c_{10}^{1}=0.1566513236$, $\quad c_{1\,1}^{1}=0.07832411924$, $c_{1\,2}^{1}=0$,

$c_{1\,3}^{1}=0,\quad c_{1\,4}^{1}=0,\quad c_{1\,5}^{1}=0,$
$c_{2\,0}^{1}=0.4699477997,\quad c_{2\,1}^{1}=0.07832411913,$
$c_{1\,2}^{1}=0,$
$c_{2\,3}^{1}=0,\quad c_{2\,4}^{1}=0,\quad c_{2\,5}^{1}=0,$
$c_{1\,0}^{2}=0.6543460812,\quad c_{1\,1}^{2}=0.02272977649,$
$c_{1\,2}^{2}=0.006337577837,$
$c_{1\,3}^{2}=0.0002522419,\quad c_{1\,4}^{2}=0.00001051008,$
$c_{1\,5}^{2}=0,$
$c_{2\,0}^{2}=0.8669839458,\quad c_{2\,1}^{2}=0.08831267148,$
$c_{2\,2}^{2}=0.01037344830,$
$c_{2\,3}^{2}=0.0004204031,\quad c_{2\,4}^{2}=0.00001051008,$
$c_{2\,5}^{2}=0.$

Therefore, one gets the following approximate solutions,

$$u(x)=\begin{cases}0{,}9998976834x+0{,}000004922548796, & 0\le x<\frac{1}{2},\\ 0{,}9998976858x+0{,}000004922867950, & \frac{1}{2}\le x<1,\end{cases}$$

$$v(x)=\begin{cases}0{,}06869671959x^4+0{,}1373934391x^3+0{,}5152253970x^2\\ -0{,}003936029166x+1{,}000408159, & 0\le x<\frac{1}{2},\\ 0{,}06869671959x^4+0{,}1373934391x^3+0{,}5152253970x^2\\ -0{,}003936027889x+1{,}000408159, & \frac{1}{2}\le x<1,\end{cases}$$

Example 4.4: Consider the following system of non-linear Volterra integral equations of the first kind with the exact solutions $u_1(x)=x^2-x+½$, $u_2(x)=x$, $u_3(x)=-x+1/5$, and $u_4(x)=x^3-¼$, (Biazar et al., 2009),

$$\begin{cases}\int_0^x\left((5+x^2-t^2)\,u_1(t)-x^2\,u_2(t)\,u_4(t)+t^2\,u_3^2(t)\right)dt=f_1(x),\\ \int_0^x\left(x\,t\,u_1(t)\,u_2(t)+x^2\,u_3(t)\,u_4(t)+2\,u_2(t)+(t-x)\,u_3(t)\right)dt=f_2(x),\\ \int_0^x\left(\left(2-\frac{1}{4}x^2\right)u_3(t)+x\,t\,u_1(t)+u_2(t)\,u_4(t)+\,u_2(t)\right)dt=f_3(x),\\ \int_0^x\left((x^2-t^2)(u_2^2(t)+\,u_3^2(t))+(x-t-2)u_4(t)-t^2\,u_1(t)\right)dt=f_4(x),\end{cases}$$

(40)

where, $f_1(x)=-1/5\ x^7-1/6\ x^6+19/75\ x^5-31/200\ x^4+751/375\ x^3-5/2\ x^2+5/2\ x$, $f_2(x)=-1/5\ x^7+1/4\ x^6-1/4\ x^5+7/24\ x^4+7/60\ x^3+9/10\ x^2$, $f_3(x)=1/6\ x^7+1/4\ x^5+1/24\ x^4+1/5\ x^3-x^2+2/5\ x$, $f_4(x)=7/60\ x^5-7/20\ x^4-7/50\ x^3-1/8\ x^2+1/2\ x$.

By taking $k=2$, $M=4$ and applying the Chebyshev wavelets method of the second kind, the following results will be achieved by solving resultant nonlinear system.

$$u_1(x)=\begin{cases}9{,}527886226\times10^{-10}x^3+0{,}9999999994x^2\\ -0{,}99999999994x+0{,}4999999998, & 0\le x<\frac{1}{2},\\ 3{,}368804023\times10^{-9}x^3+0{,}9999999954x^2\\ -0{,}9999999970x+0{,}49999999989, & \frac{1}{2}\le x<1,\end{cases}$$

$$u_2(x)=\begin{cases}-7{,}820737676\times10^{-9}x^3+5{,}712716629\times10^{-9}x^2\\ +0{,}9999999986x+2{,}393653682\times10^{-10}, & 0\le x<\frac{1}{2},\\ 9{,}797346068\times10^{-9}x^3-2{,}079759689\times10^{-8}x^2\\ +1{,}000000014x-2{,}393653682\times10^{-9}, & \frac{1}{2}\le x<1,\end{cases}$$

$$u_3(x)=\begin{cases}2{,}695900942\times10^{-8}x^3-1{,}998487733\times10^{-8}x^2\\ -0{,}9999999954x+0{,}1999999997, & 0\le x<\frac{1}{2},\\ -4{,}834362192\times10^{-9}x^3+5{,}890441163\times10^{-9}x^2\\ -1{,}000000002x+0{,}199999999999, & \frac{1}{2}\le x<1,\end{cases}$$

$$u_4(x)=\begin{cases}0{,}9999999539x^3+3{,}462818993\times10^{-8}x^2\\ -7{,}659691781\times10^{-9}x-0{,}2499999995, & 0\le x<\frac{1}{2},\\ 1{,}00000000x^3-7{,}180961044\times10^{-9}x^2\\ +5{,}585191924\times10^{-9}x-0{,}2500000004, & \frac{1}{2}\le x<1,\end{cases}$$

The exact and approximate solutions are plotted in Figs. 3 and some values of absolute errors are presented in Table 3. This example has been solved by Homotopy perturbations method in (Biazar, et al., 2009) and comparison between the obtained absolute error in (Biazar, et al., 2009) and current paper shows that the absolute error of Chebyshev wavelets method of the second kind are lesser than the absolute error of Homotopy perturbations method.

Table 1: The absolute error values of example 4.2 for some values of *x*

x	*error u(x)*	*error v(x)*
0	1.1262×10^{-6}	2.062×10^{-6}
0.2	3.9300×10^{-8}	7.0700×10^{-8}
0.4	8.9000×10^{-9}	8.7800×10^{-8}
0.6	3.7300×10^{-8}	8.6300×10^{-8}
0.8	6.4700×10^{-8}	2.0700×10^{-8}
1.0	1.1378×10^{-6}	1.9082×10^{-6}

Table 2: The absolute error values of example 4.3 for some values of *x*

x	*error u(x)*	*error v(x)*
0	4.9225×10^{-6}	7.0816×10^{-4}
0.2	1.5541×10^{-5}	3.6273×10^{-5}
0.4	3.6004×10^{-5}	3.0710×10^{-6}
0.6	5.6466×10^{-5}	1.1037×10^{-5}
0.8	7.6928×10^{-5}	5.3720×10^{-5}
1.0	9.7391×10^{-5}	4.9414×10^{-4}

Table 3: The absolute error values of example 4.4 for some values of x

x	error $u_1(x)$	error $u_2(x)$	error $u_3(x)$	error $u_4(x)$
0	2.0000×10^{-10}	2.3937×10^{-10}	3.0000×10^{-10}	5.0000×10^{-10}
0.2	9.6380×10^{-11}	1.2531×10^{-10}	3.6270×10^{-11}	1.5600×10^{-11}
0.4	7.9800×10^{-12}	9.2880×10^{-11}	6.7800×10^{-11}	2.6200×10^{-11}
0.6	2.2834×10^{-10}	6.3520×10^{-10}	2.2370×10^{-10}	1.0139×10^{-9}
0.8	8.0800×10^{-11}	5.1150×10^{-10}	4.0530×10^{-10}	1.0084×10^{-9}
1.0	6.6880×10^{-10}	6.0530×10^{-10}	1.0440×10^{-9}	1.0042×10^{-9}

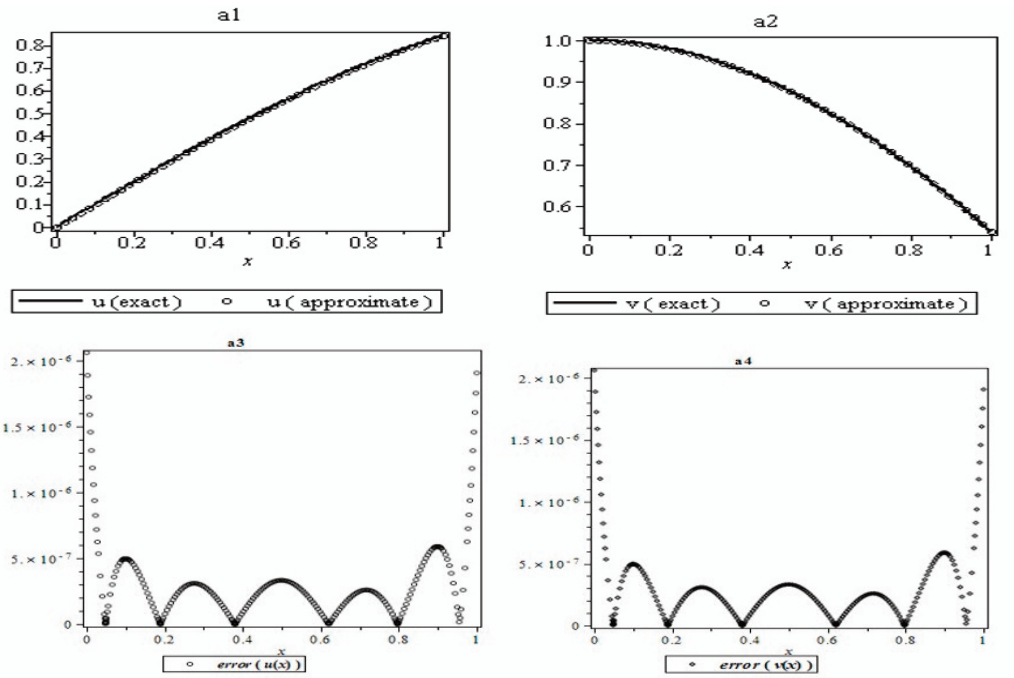

Fig. 1. (*a1*) – (*a2*) The exact and approximate solutions, (*a3*) – (*a4*) Plots of absolute error of example 4.2

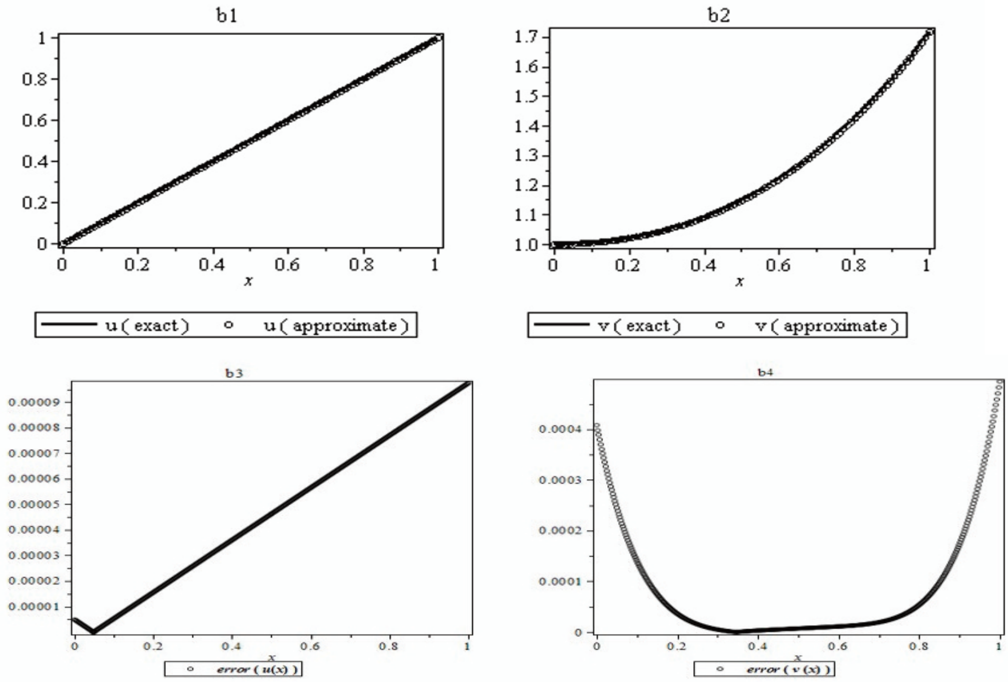

Fig. 2. (*b1*) – (*b2*) The exact and approximate solutions, (*b3*) – (*b4*) Plots of absolute error of example 4.3

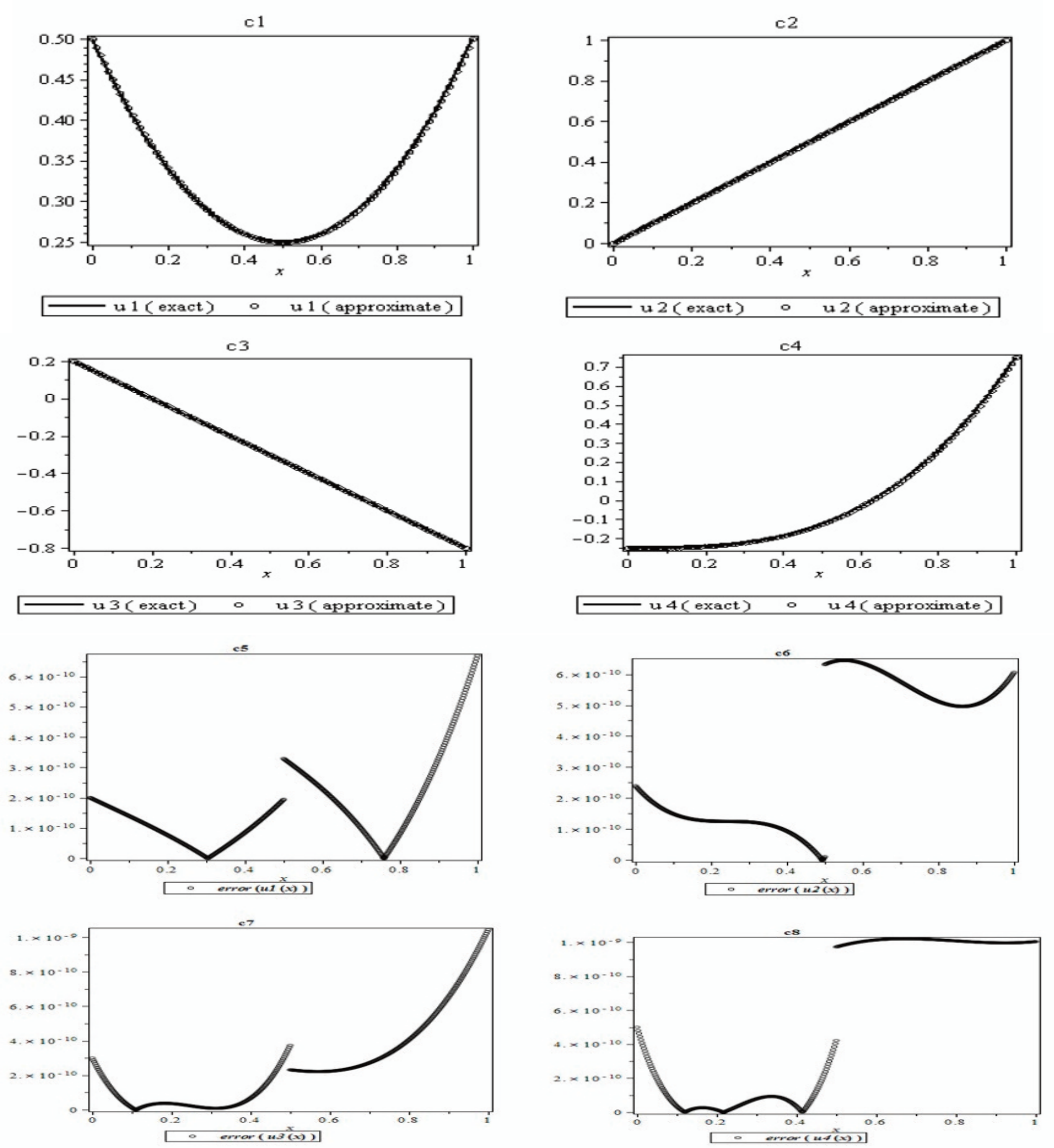

Fig. 3. (*c1*) - (*c4*) The exact and approximate solutions, (*c5*) - (*c8*) Plots of absolute error of example 4.4.

CONCLUSION

The aim of this paper was to introduce the Chebyshev wavelets method of the second kind for obtaining the solutions of systems of integral equations. Considering the results based on these wavelets and using the operational matrices, the system of integral equations is converted into an algebraic system. According to the results obtained in the illustrative examples, it is concluded that the proposed method is a very effective and useful technique for finding approximate solutions of these systems. Finding the applications of this method and other orthogonal functions is one of the objectives of the further investigations.

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