Optimization
Iranian Journal of Optimization
Volume 10, Issue 2, 2018, 93-99
Research Paper
Online version is available on: www.ijo.iaurasht.ac.ir

Islamic Azad University
Rashut Branch
ISSN: 2588-5723
E-ISSN:2008-5427

Choosing Weights for a Complete Ranking of DMUs in DEA and Cross-Evaluation

Parichehr Zamani
Assistant Professor, Faculty of Engineering, Qom Branch, Islamic Azad University, Qom, Iran

Received: 22 August 2017
Accepted: 10 November 2017

Keywords:
Data envelopment analysis
Ranking
Pure profit
Cross-efficiency

Abstract

Conventional data envelopment analysis (DEA) assists decision makers in distinguishing between efficient and inefficient decision making units (DMUs) in a homogeneous group. However, DEA does not provide more information about the efficient DMUs. One of the interesting research subjects is to discriminate between efficient DMUs. The aim of this paper is ranking all efficient (extreme and non-extreme) DMUs based on defining the new index which is obtained from basic definitions of models. The proposed method has been able to remove the existing deficiencies in some ranking methods and therefore makes a new contribution to DEA ranking.

***Correspondence E-mail**: zamany2007@yahoo.com

INTRODUCTION

Data envelopment analysis (DEA) is a mathematical programming technique that evaluates the relative efficiency of a homogeneous group of operating decision making units (DMUs), such as schools, hospitals, or sales outlets. The DMUs usually use a set of resources, referred to as input indices, and transform them into a set of outcomes, referred to as output indices. DEA successfully divides them into two categories: efficient DMUs and inefficient DMUs. The DMUs in the efficient category have identical efficiency scores. Therefore, the efficiency measure obtained by DEA can be used for ranking DMUs, but this ranking cannot be applied to efficient units. In other words, DEA does not provide more information about the efficient DMUs. However, it is not appropriate to claim that they have the equivalent performance in actual practice. One of the interesting research subjects is to discriminate between efficient DMUs. Several authors have proposed methods for ranking efficient DMUs. Andersen and Petersen (1993) have presented a model that has named AP for ranking of efficient DMUs. Their proposed method removes the DMU under assessment from the set of DMUs and evaluates the distance (possibly) new efficient frontier as its rank score. Infeasibility of the model in some cases and inability to rank non-extreme efficient DMUs are two of the problems of this method. Jahanshahloo et al. (2007) presented a new ranking system for extreme efficient DMUs based upon the omission of these efficient DMUs from reference set of the inefficient DMUs. In their method, the efficiency change of DEA inefficient units is measured before and after the DEA efficient unit is excluded from their reference set. The DEA efficient unit that can cause the biggest efficiency change of DEA inefficient units when it is removed from their reference set is deemed as the most important DMU. For this reason, this method has named changing the reference set. In this paper, we intend to propose an alternative DEA ranking approach. However, the new proposal enables an efficiency ranking for both extreme and, in particular, non-extreme efficient DMUs. What's more the infeasibility problem that may arise in mentioned methods is eliminated as well. Khodabakhshi and Ariavash (2012) offered a method to rank DMUs. In their method, the maximum and minimum efficiency value of each DMU are measured by considering the sum of all efficiencies equal one. Finally, the rank of each DMU is determined in proportion to a convex combination of its minimum and maximum efficiency values. Jahanshahloo et al. (2017) modified this method by applying the optimistic and pessimistic optimal weights of all DMUs in ranking of the evaluated DMU. Ziari and Ziari (2016) introduced a model for ranking efficient DMUs based on the minimizing the coefficient of variation for inputs-outputs weights. Ziari and Sharifzadeh (2017) proposed a DEA-based approach for benchmarking and ranking extreme efficient units using the idea of super efficiency model and comb ining 1 and ∞ norms with constant and variable returns to scale.

The cross-evaluation method can be utilized to rank DMUs using cross-efficiency scores (Sexton et al., 1986). The main idea of cross-evaluation is to use DEA in a peer evaluation instead of a self-evaluation mode. However, the non-uniqueness of the DEA optimal weights may reduce the usefulness of cross-efficiency as reported in Doyle and Green (1994). Hence choosing weights between alternative optimal solutions as part of a procedure for ranking DMUs is an important problem. See for example Green et al. (1996), Liang et al. (2008), Cooper et al. (2007; 2009), Wu et al. (2009).

In the same way, this paper is concerned with the selection of weights between the alternative optimal solutions of the dual multiplier model for a complete ranking of DMUs. The basic idea in this method is to compare the DMUs by first efficiency score and second a new index which is obtained from basic definitions of models. Hence, this paper is organized as follows. In section 2, we review the cross-efficiency evaluation approach proposed by Sexton. Our new method for complete ranking of DMUs is introduced in section 3. Two numerical examples are documented in section 4. Finally concluding remarks are summarized in the last section.

SEXTON METHOD: CROSS-EFFICIENCY EVALUATION

Suppose that we have n DMUs, where each DMUj, j=1,…,n, produces s different outputs

y_{rj}=(r=1,…,s) using m different inputs x_{ij}=(i=1,…,m) . Also, we assume that all the inputs and outputs are non-negative and at least one input and one output of each DMU is strictly positive.

To estimate a DEA efficiency score of the specific pth DMU, we use the following original DEA model:

$$E^*_{pp} = \max \quad E_{pp} = \frac{\sum_{r=1}^{s} u_{rp} y_{rp}}{\sum_{i=1}^{m} v_{ip} x_{ip}} \tag{1}$$
$$\text{s.t.} \quad E_{pj} = \frac{\sum_{r=1}^{s} u_{rp} y_{rj}}{\sum_{i=1}^{m} v_{ip} x_{ij}} \le 1, \quad j = 1,2,\ldots,n$$
$$u_{rp} \ge 0, \quad r = 1,\ldots,s$$
$$v_{ip} \ge 0, \quad i = 1,\ldots,m$$

where u_rp and v_ip represent the ith input and rth output weights for DMUP. The constraints mean that the ratio of "virtual output" vs. "virtual input" should not exceed 1 for every DMU. The objective is to obtain weights v_ip and u_rpthat maximize the ratio of DMUP, the DMU being evaluated. By virtue of the constraints, the optimal objective value E_pp^*is at most 1.

Table 1: Cross-evaluation matrix

	DMU_1	DMU_2	. . .	DMU_n
DMU_1	E_{11}	E_{12}	. . .	E_{1n}
DMU_2	E_{21}	E_{22}	. . .	E_{2n}
. . .	. . .			
DMU_n	E_{n1}	E_{n2}	. . .	E_{nn}

The cross-efficiency of DMUt, using the weights that DMUp has selected in model (1), is then

$$E_{pt} = \frac{\sum_{r=1}^{s} u^*_{rp} y_{rt}}{\sum_{i=1}^{m} v^*_{ip} x_{it}} \qquad t = 1,2,\ldots,n \tag{2}$$

Where (*) denotes optimal values in model (1). The values obtained from (2) can be organized in a matrix which is called cross-evaluation matrix as illustrated in Table 1.

For DMU_t (t=1,…,n), the average of all E_{pt}(p=1,…,n), $\bar{E}_t = \frac{1}{n}\sum_{p-1}^{n} E_{pt}$,referred to as the cross-effciency score for DMUt. By using this column average of cross-evaluation matrix, Sexton proposed a method for ranking DMUs.

THE PROPOSED METHOD

We point out that DEA model (1) can be transformed equivalently into the following linear programming (see Charnes & Cooper, 1962) where the optimal value of the objective function indicates the relative efficiency of DMU_p. The reformulated linear programming problem, also known as the CCR model (Charnes et al., 1978), is as follows:

$$E^*_{pp} = \max \quad \sum_{r=1}^{s} \mu_{rp} y_{rp}$$
$$\text{s.t.} \quad \sum_{r=1}^{s} \mu_{rp} y_{rj} - \sum_{i=1}^{m} w_{ip} x_{ij} \le 0, \quad j = 1,2,\ldots,n \tag{3}$$
$$\sum_{i=1}^{m} w_{ip} x_{ip} = 1$$
$$\mu_{rp} \ge 0, \quad r = 1,\ldots,s$$
$$w_{ip} \ge 0, \quad i = 1,\ldots,m$$

In evaluating DMU_p, $\sum_{r=1}^{s} u^*_{rp} y_{rj}$ and $\sum_{i=1}^{m} w^*_{ip} x_{ij}$ is referred to as the total revenue and total cost for jth DMU, respectively. Hence first restriction calculates (i.e., $\sum_{r=1}^{s} u^*_{rp} y_{rj} - \sum_{i=1}^{m} w^*_{ip} x_{ij} \le 0$) the pure profit for DMU_j (See Alirezaee & Afsharian, 2007) for a detailed discussion).

$$H_{pt} = \max \quad \mu^t_p y_t - w^t_p x_t \tag{4}$$
$$\text{s.t.} \quad (\mu_p, w_p) \in A^*_P$$

The model (4) can be rewritten as follows:

$$H_{pt} = \max \quad \sum_{r=1}^{s} \mu_{rp} y_{rt} - \sum_{i=1}^{m} w_{ip} x_{it}$$
$$\text{s.t.} \quad \sum_{r=1}^{s} \mu_{rp} y_{rp} = E^*_{pp}$$
$$\sum_{i=1}^{m} w_{ip} x_{ip} = 1 \tag{5}$$
$$\sum_{r=1}^{s} \mu_{rp} y_{rj} - \sum_{i=1}^{m} w_{ip} x_{ij} \le 0, \qquad j = 1,\ldots,n$$
$$\mu_{rp} \ge 0, \quad r = 1,\ldots,s$$
$$w_{ip} \ge 0, \quad i = 1,\ldots,m$$

where E_{pp}^* is the optimal value of the objective in (3).

Now, we employ a new index of DMUp in the cross-evaluation matrix as shown in Table 2. For each DMUp (p=1,…,n) , the sum of row quantities of other DMUs, $\overline{H}_p = \sum_{t=1, t\neq 1}^{n} H_{pt}$, is used for ranking. Hence among the DEA-efficient DMUs, the higher the $\overline{H}_p$ value, the better the rank of DMUp. Specially, DMUl has a better rank than DMUk, if Both DMUs are the same in efficiency score and $\overline{H}_l > \overline{H}_k$.

Table 2: The new cross-evaluation matrix

	DMU_1	DMU_2	. . .	DMU_n
DMU_1	H_{11}	H_{12}	. . .	H_{1n}
DMU_2	H_{21}	H_{22}	. . .	H_{2n}
. . .	. . .			
DMU_n	H_{n1}	H_{n2}	. . .	H_{nn}

Since in real-world applications the probability that two real numbers with decimal digits become equal, is zero, no DMUs receive the same indexes. This only happens if all DMUs lie on the same efficient hyperplane, which is not a realistic situation.

While the results from the AP method can be infeasible, the proposed model is always feasible. This method can also be used to rank all efficient (extreme and non-extreme) DMUs. For this usage, the method is less problematic than other ranking methods (Andersen and Petersen, 1993; Jahanshahloo et al., 2007)

NUMERICAL EXAMPLE

In this section we are going to ranking the data listed in Table 3. we are also going to rank the Iranian bank branches by our proposed method.

Fictional data

Table 3 shows 5 DMUs with 2 inputs and 1 output where the output value is scaled to 1 for each DMU. Also, Fig. 1 describes these 5 DMUs. The initial results are given in Table 4, where in the lines we have DMUs and in the columns we have the efficiency of the CCR model and the optimal weights of (3). The new index and order of efficient DMU are calculated through (5) and they are listed in Table 5.

Table 3: DMUs'data (extracted from Alirezaee and Afsharian, 2007)

DMU	A	B	C	D	E
Input 1	4	7	8	4	2
Input2	3	3	1	2	4
Output	1	1	1	1	1

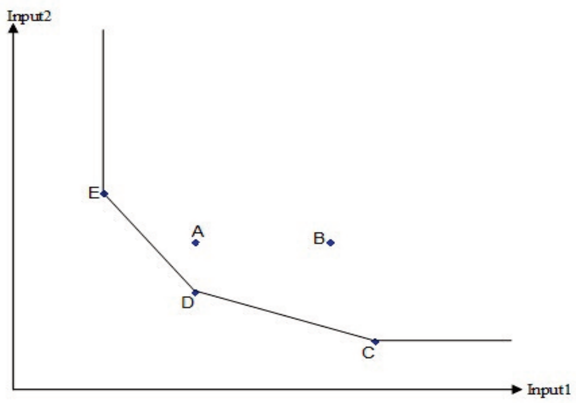

Fig. 1. The Farrel Frontier

Table 4: The optimal weights of model (3)

DMU	Efficiency	w_1	w_2	μ
A	0.8571	0.1429	0.1429	0.8571
B	0.6316	0.0526	0.2105	0.6316
C	1.0000	0.0833 0	0.3333 1.0000	1.0000 1.0000
D	1.0000	0.1667 0.0833	0.1667 0.3333	1.0000 1.0000
E	1.0000	0.1667 0.5000	0.1667 0	1.0000 1.0000

Table 5: The new index and order of efficient MUs

DMU	New index	*Rank*
C	-1.4157	
D	-0.7498	
E	-1.3344	

Real data

We, now, describe the advantages of our proposed method by using the inputs and outputs of 20 Iranian bank branches which are presented in Table 6. The data reported here are taken from Amirteimoori and Kordrostami (2005). Note that the data are scaled. As can be seen in the last column of Table 6, DMUs 1, 4, 7, 12, 15, 17 and 20 are CCR efficient. Table 7 presents the results of Sexton ranking using some of the optimal weights obtained from (3) that summarized as follows:

DMU1:

A) w_1=0, w_2=0, w_3=6.4516130, μ_1=1.7062960, μ_2=1.1176250, μ_3=0.3191844

B) w_1=0, w_2=0, w_3=6.4516130, μ_1=2.6539210, μ_2=0.9515451, μ_3=0

C) w_1=0.2374179, w_2=0, w_3=4.9964710, μ_1=2.1199880, μ_2=1.0406650, μ_3=0.1877674

DMU4:

A) w_1=1.0951080, w_2=0, w_3=0.2511046, μ_1=0, μ_2=0, μ_3=1.0000000

B) w_1=0, w_2=0, w_3=4.7619050, μ_1=0, μ_2=0, μ_3=1.0000000

C) w_1=1.1143090, w_2=0, w_3=0.1720109, μ_1=0.7570253, μ_2=0, μ_3=0.8538941

DMU7:

A) =1.3908210, =0, =0, =0, =1.1111110, =0

B) =0, =1.6666670, =0, =0, =0.7679909, =0.4312964

C) w_1=1.1725710, w_2=0, w_3=0.4483462, μ_1=0, μ_2=0.4611661, μ_3=0.8169701

DMU12:

A) w_1=1.0084500, w_2=0, w_3=0.7143018, μ_1=0, μ_2=1.0834240, μ_3=0

B) w_1=0.2900087, w_2=0, w_3=2.9992270, μ_1=0, μ_2=0.8494963, μ_3=0.3438135

C) w_1=0.8656812, w_2=0, w_3=1.1683630, μ_1=0, μ_2=0.7140431, μ_3=0.5428952

DMU15:

A) w_1=0, w_2=1.0526320, w_3=0, μ_1=1.0000000, μ_2=0, μ_3=0

B) w_1=1.4598540, w_2=0, w_3=0, μ_1=1.0000000, μ_2=0, μ_3=0

C) w_1=1.3254440, w_2=0, w_3=0.2046027, μ_1=0.9004628, μ_2=0, μ_3=1.0156860

DMU17:

A) w_1=0.7412812, w_2=0, w_3=1.2620430, μ_1=0, μ_2=1.0000000, μ_3=0

B) w_1=0, w_2=0, w_3=4.8780490, μ_1=0, μ_2=1.0000000, μ_3=0

C) w_1=0.6516252, w_2=0, w_3=1.6993890, μ_1=0.972008

3, μ_2=0.9125193, μ_3=0

DMU20:

A) w_1=1.7152660, w_2=0, w_3=0, μ_1=0.9340329, μ_2=0.6871794, μ_3=0.6212579

B) w_1=1.7152660, w_2=0, w_3=0, μ_1=0, μ_2=0, μ_3=1.3089010

C) w_1=1.7152660, w_2=0, w_3=0, μ_1=1.0616650, μ_2=0, μ_3=1.1560430

Table 6: Real data and their CCR efficiencies

Branch	Inputs			Outputs			CCR efficiency
	Staff	Computer terminals	Space (m2)	Deposits	Loans	Charge	
DMU1	0.950	0.700	0.155	0.190	0.521	0.293	1.000
DMU2	0.796	0.600	1.000	0.227	0.627	0.462	0.833
DMU3	0.798	0.750	0.513	0.228	0.970	0.261	0.991
DMU4	0.865	0.550	0.210	0.193	0.632	1.000	1.000
DMU5	0.815	0.850	0.268	0.233	0.722	0.246	0.899
DMU6	0.842	0.650	0.500	0.207	0.603	0.569	0.748
DMU7	0.719	0.600	0.350	0.182	0.900	0.716	1.000
DMU8	0.785	0.750	0.120	0.125	0.234	0.298	0.798
DMU9	0.476	0.600	0.135	0.080	0.364	0.244	0.789
DMU10	0.678	0.550	0.510	0.082	0.184	0.049	0.289
DMU11	0.711	1.000	0.305	0.212	0.318	0.403	0.604
DMU12	0.811	0.650	0.255	0.123	0.923	0.628	1.000
DMU 13	0.659	0.850	0.340	0.176	0.645	0.261	0.817
DMU14	0.976	0.800	0.540	0.144	0.514	0.243	0.470
DMU15	0.685	0.950	0.450	1.000	0.262	0.098	1.000
DMU16	0.613	0.900	0.525	0.115	0.402	0.464	0.639
DMU17	1.000	0.600	0.205	0.090	1.000	0.161	1.000
DMU18	0.634	0.650	0.235	0.059	0.349	0.068	0.473
DMU19	0.372	0.700	0.238	0.039	0.190	0.111	0.408
DMU20	0.583	0.550	0.500	0.110	0.615	0.764	1.000

Table 7: Results of Sexton ranking based on alternative optimal weights

Branch	Ranking based on weight A	Ranking based on weight B	Ranking based on weight C
1	7	5	6
4	3	1	1
7	1	3	2
12	2	2	3
15	6	7	5
17	5	4	7
20	4	6	4

As can be seen, the existence of alternative optimal weights for these efficient DMUs leads to multiple cross-efficiency scores and hence the ranking of units is not possible and this is the main problem of this method. For example, in the second column, DMU7 achieves the top ranking, whilst in the third column, it has the 3th rank amongst DMUs. DMU17, which is ranked in the last position by the fourth column, gets the fourth rank according to third column and so on. The question is: which of these efficient DMUs must be located in the higher position. As a result, in this case the decision maker (DM) cannot decide by considering different ranks.

Now, we rank these seven efficient DMUs according to the AP and changing the Reference set methods. Table 8 presents the ranking related scores assigned to DMUs by these methods. In Table 9, the suggested method ranks the DMUs. As can be seen in Table 9, the first position is assigned to DMU15. It can be seen in Table 8 that this unit gets the most ranking related score by the above mentioned methods. DMU1, which is located in the last position by the proposed method, gets the lowest ranking related score by using earlier methods. It is obvious that the ranking results for these methods are consistent with our methodology results.

Table 8: Ranking related scores assigned to DMUs by ranking methods

Branch	AP	Changing the reference set
1	1.100470	0.1958082
4	1.933270	0.205381
7	1.172494	0.2024802
12	1.110409	0.1963884
15	4.902439	0.2291681
17	1.347673	0.1963884
20	1.184034	0.1972586

Table 9: Results of ranks by proposed method

Branch	New index	The proposed Rank
1	-19.298	7
4	-4.3356	3
7	-4.2443	2
12	-5.7494	5
15	-2.8645	1
17	-4.731	4
20	-8.1474	6

CONCLUSION

The existence of alternative optimal weights for the efficient DMUs leads to multiple cross-efficiency scores and hence the ranking of units is not possible with the Sexton method, as can be seen in Table 7. First, we have introduced a new index which is obtained from basic definitions of models and then compared the proposed method with the methods developed by Andersen and Petersen (1993) and Jahanshahloo et al. (2007). It has been shown whereas the ranking results for the AP and changing the reference set methods are consistent with the present study results, the new proposal enables an efficiency ranking for both extreme and, in particular, non-extreme efficient DMUs. What's more, the infeasibility problem that may arise in mentioned methods is eliminated as well. For this reason, our method is superior to these methods in removing their deficiencies and therefore makes a new contribution to DEA ranking. What we should point out here is that the computational complexity of our method can increase with increase of the number of DMUs, so how to provide better method is an interesting issue for future research.

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