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Optimal Decision Making in Fractional Multi-commodity Flow Problem in Uncertainty Environment

Salim Bavandi[a,b]*, Seyed Hadi Nasseri[a]

[a] *Department of Mathematics, University of Mazandaran, Babolsar, Iran*
[b] *Department of Civil, Chemical, Environmental and Materials Engineering-DICAM, University of Bologna, Viale del Risorgimento 2, 40136 Bologna, Italy.*

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ABSTRACT

This paper seeks to address the multi-commodity flow problem in uncertainty conditions, in which the objective function of the problem is of fractional type. The cost coefficients and capacities of the problem are uncertain. The purpose of using uncertainty theory is to deal with unknown factors in the uncertain network. After stating the optimality conditions, the problem is transformed into a certain fractional multi-commodity flow problem by applying the uncertain chance-constrained programming approach. Then, the variable transformation approach is used to transform the nonlinear objective function to its linear form. Finally, two numerical examples are evaluated to verify the efficiency of the proposed formulation.

1. Introduction

In general, a network can transfer a variety of commodities, in which case it is called a Multi-Commodity Flow (MCF) problem. The MCF problem attempts to minimize total costs when different types of commodities are transferred through the same network. In other words, the MCF problem can be defined as a network optimization problem, in which several commodities are transferred from the source node to the destination nodes [12]. In this network, the commodities are uniquely connected by using arcs. In this way, the arc capacity refers to the number of commodities that can pass through these arcs. Commodities may be distinguished by physical features or simply based on specific features. The MCF problem has been widely used in the transportation industry, especially in the aviation industry, to synchronize crew and fleet assignment. It is worth noting that several objective functions are used in such cases, which include time, risk, cost, and environmental issues. Therefore, multi-objective flow models are more suitable for real-world models and their conditions compared to single-objective models. However, considering multiple objective functions in uncertainty

* Correspondig author
E-mail address: salimbavandi@yahoo.com (Salim Bavandi)

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conditions makes highly difficult the process of solving the MCF problem with respect to traditional methods [3]. By considering only two objective functions, for example, minimizing costs and maximizing reliability or maximizing profits, it is possible to optimize the ratio of these objective functions. This type of problem, which is recognized as fractional programming, is a specific type of nonlinear programming problem with linear objective functions and constraints. So far, various methods have been proposed to solve such problems. The fractional programming method was first introduced by Marlow and Isbell in 1956 [14]. Charnes and Cooper [7] proposed a variable transformation to solve such problems. Later, Bitran et al. [6] developed this method for solving linear fractional programming. On the other hand, a large body of research has been carried out in uncertain situations, among which we can refer to the approach proposed by Nasseri et al. [20], for the linear fractional programming problem with random coefficients. In another study, Nasseri et al. [21] considered a fuzzy approach to the fuzzy stochastic linear fractional programming problem. Moreover, fractional programming occurs in many practical problems and can be used in network analysis (Ahmad et al. [1], Schaible [23], Schaible et al. [25]). Mahmoodirad et al. [19] studied the flow problems in the network with the minimum fuzzy fractional cost. Liu [18] evaluated a fractional transportation problem, in which the cost and the right-hand side coefficients are fuzzy parameters. A fractional minimal cost flow problem under belief degree-based uncertainty was studied by Niroomand [22]. In this paper, the uncertain fractional minimal cost flow problem was converted to a crisp form using a chance-constrained approach and its non-linear objective function was linearized by a variable changing approach. Ding et al. [9] proposed the uncertain minimum cost multi-commodity flow problem. To this aim, he formulated an (α,β)-minimum cost multi-commodity flow model and then, proposed an algorithm to solve the problem. However, a few research has been conducted on the Fractional Multi-Commodity Flow (FMCF) problem. In this regard, Fakhri et al. [10] investigated the FMCF problem and defined the duality of this problem based on its linear programming representation to evaluate some of the duality features.

This paper uses the uncertainty theory proposed by Liu [17] to deal with unknown factors in the uncertain network. Generally, there is not enough reliable historical data to estimate the probability distribution function in many situations [4]. For example, when commodities or products are transported through newly constructed roads, it is not possible to obtain the probability distribution function of transportation cost, due to the shortage of historical information [10]. However, collecting data and recording observations in some other cases such as evaluating a structure's resistance to natural disasters require conducting high-cost tests. Further, it is not even possible to test and collect data in some cases, such as examining the resistance of a bridge to the forces applied to the bridge. In such cases, it is necessary to ask the experts in the relevant field, as one of the solutions, to express their opinions based on the available evidence, documents, and their experiences. Despite the low study background, the uncertainty theory has been used in interesting sciences and applied problems such as economics, optimal control, and risk analysis, in addition to combined optimization problems such as the shortest path problem [11], vertex cover problem [8], and maximum flow problem [13].

In this paper, we first present the certain FMCF problem and then, examine its generalization under uncertainty conditions. The rest of this paper is organized as follows: Section 2 introduces some fundamental definitions and concepts related to this field. Section 3 defines the FMCF model. By using the uncertain chance-constrained approach and variable transformation proposed by Charnes-Cooper, in the remainder of this Section elaborates on the transformation of the model presented to a certain equivalent form. Finally, a numerical example is presented in Section 4 to verify the effectiveness of the proposed formulation.

2. Preliminaries

This section briefly presents some concepts and theorems of uncertainty theory. Readers can refer to reference Liu [15], which provided more details about this theory.

Assume that Γ is a nonempty set. The collection L of the subsets Γ is called an algebra on Γ if the following three conditions are met:

1. $\Gamma \in L$
2. If Λ, then $\Lambda^C \in L$.
3. If $\Lambda_1,...,\Lambda_n \in L$ then $\bigcup_{i=1}^{n} \Lambda_i \in L$.

The collection L is called an σ-algebra on Γ if the third condition is satisfied for a countable number Λ_i. In this case, (Γ, L) is called a measurable space and each $\Lambda \epsilon L$ is called a measurable set or an event. The least σ-algebra that includes all of the open intervals ℝ is called Borel algebra and represented by $\mathcal{B}$. Each $B \in \mathcal{B}$ is a Borel set. The function M on the L is an uncertain measurable function if it satisfies the four normality, duality, subadditivity, and product axioms. The triple $(\Gamma, L, \mathcal{M})$ is uncertain space. It is worth noting that uncertain measure is interpreted as the belief degree in the occurrence of an event based on the observations and personal experiences of an expert. Consequently, changing the temporal and spatial conditions of an expert would change the belief in the occurrence of an event. Therefore, the results will be valid as long as the belief function has not changed.

The function $f : (\Gamma, L) \to \mathbb{R}$ is called measurable whenever the inverse image of the function f is a member of σ-algebra on the L for each Borel set B of $\mathbb{R}$.

Definition 1. [16] the uncertain variable ξ is a function of the uncertain space $(\Gamma, L, \mathcal{M})$ to the $\mathbb{R}$ so that the set $\{x \in \mathbb{R} \mid \xi(x) \in B\}$ is an event for each Borel set B of $\mathbb{R}$.

In other words, an uncertain variable is an uncertain measurable function on an uncertain space $(\Gamma, L, \mathcal{M})$ **Definition 2.** [16] uncertain variables $\xi_1, \xi_2, ..., \xi_n$ are called independent whenever for each Borel set $B_1, B_2, ..., B_n$ we have:

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} \{\xi_i \in B_i\}\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

Definition 3. [16] For the uncertain variable ξ, the uncertainty distribution function Φ for each $x \in \mathbb{R}$ is defined as follows:

$$\Phi(x) = \mathcal{M}\{\xi \le x\}$$

The uncertain variable ξ is called linear whenever its uncertainty distribution function is as follows:

$$\Phi(x) = \begin{cases} 0 & \text{if} \quad x \le a \\ \dfrac{x-a}{b-a} & \text{if} \quad a \le x \le b \\ 1 & \text{if} \quad x > b \end{cases}$$

This type of variable is shown by $L(a,b)$.

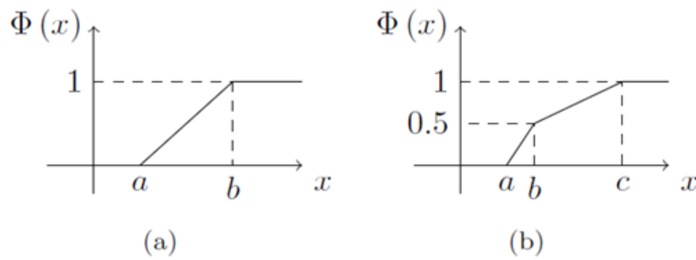

Figure 1. (a) Linear uncertain variable (b) Zigzag uncertain variable

The uncertain variable ξ is called zigzag and represented by $Z\left(a,b,c\right)$ whenever its uncertainty distribution function is as follows:

$$\varphi\left(x\right)=\begin{cases}0 & if \quad x\le a\\ \dfrac{x-a}{2\left(b-a\right)} & if \quad a\le x\le b\\ \dfrac{x+c-2b}{2\left(c-b\right)} & if \quad b\le x\le c\\ 1 & if \quad x>c\end{cases}$$

Figure 1 shows a diagram of the linear and zigzag uncertainty distribution functions, respectively.

Definition 4. [16] The uncertainty distribution $\Phi\left(x\right)$ is called regular whenever it is continuous and strictly increasing relative to x satisfying the relation $0<\Phi\left(x\right)<1$. Furthermore, the distribution should satisfy in the conditions $\lim_{x\to-\infty}\Phi\left(x\right)=0$ and $\lim_{x\to+\infty}\Phi\left(x\right)=1$.

The zigzag linear uncertainty distributions are examples of regular uncertainty distribution functions. In actual applications, it is always assumed that all uncertain variables have a regular distribution. Otherwise, it is required to apply some disturbances to obtain its regular form [14]. If the values domain of the regular uncertainty distribution function $\Phi\left(x\right)$ is limited on the set $\left(0,1\right)$, then, there is an inverse uncertainty distribution function $\Phi^{-1}(\alpha)$ in the interval $\left(0,1\right)$. The inverse uncertainty distribution function $L\left(a,b\right)$ is computed as $\Phi^{-1}\left(\alpha\right)=\left(1-\alpha\right)a+\alpha b$ while the zigzag inverse uncertainty distribution function $Z\left(a,b,c\right)$ is calculated as follows:

$$\Phi^{-1}\left(\alpha\right)=\begin{cases}\left(1-2\alpha\right)a+2\alpha b & if \quad \alpha<0.5\\ \left(2-2\alpha\right)b+\left(2\alpha-1\right)c & if \quad \alpha\ge 0.5\end{cases}$$

The following theorem describes one of the fundamental and widely used features of uncertainty theory.

Theorem 1. [16] Assume that $\xi_1,\xi_2,\ldots,\xi_n$ are uncertain variables and f is a true value measurable. In this case, the function $f\left(\xi_1,\xi_2,\ldots,\xi_n\right)$ is also an uncertain variable.

Since there is no specific way to practically obtain the uncertainty distribution function of an uncertain variable, the next theorem allows obtaining the inverse uncertainty distribution function in certain conditions, without having direct knowledge on the uncertainty distribution function.

Theorem 2. [17] Suppose $\xi_1,\xi_2,\ldots,\xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\ldots,\Phi_n$. If the function $f\left(\xi_1,\xi_2,\ldots,\xi_n\right)$ is strictly increasing on $x_1,\ldots,x_m$ and strictly decreasing on $x_{m+1},\ldots,x_n$, then the uncertain variable $\xi=f\left(\xi_1,\xi_2,\ldots,\xi_n\right)$ has an inverse uncertainty distribution as follow:

$$\Psi^{-1}\left(\alpha\right)=f\left(\Phi_1^{-1}\left(\alpha\right),\ldots,\Phi_m^{-1}\left(\alpha\right),\Phi_{m+1}^{-1}\left(1-\alpha\right),\ldots,\Phi_n^{-1}\left(1-\alpha\right)\right)$$

Mathematical expectation, as one of the basic and important concepts in the uncertainty theory, is an uncertain variable, which is defined as follows:

$$E[\xi]=\int_{0}^{+\infty}\mathcal{M}\{\xi\geq x\}\,dx-\int_{-\infty}^{0}\mathcal{M}\{\xi\leq x\}\,dx$$

where at least one of the integrals should be finite.

The next theorem expresses the method of calculating the uncertain expected value of a linear function of independent uncertain variables.

Theorem 3. [17] Suppose ξ and η are two independent uncertain variables with a finite uncertain expected value. Further, assume that both a and b are real numbers. In this case,

$$E[a\xi+b\eta]=aE[\xi]+bE[\eta]$$

3. Mathematical formulation

This section explains a mathematical model for the fractional multi-commodity flow problem. For this purpose, consider the directed network $G(N,A)$, in which the set of nodes and arches are represented by $N=\{1,2,...,n\}$ and $A=\{(i,j)|i,j\in N\}$. Suppose K is a set of commodities $K=\{1,2,...,|K|\}$, and c_{ij}^k and d_{ij}^k are coefficients for arc (i,j) cost. The other variables and parameters are defined as follows:

x_{ij}^k : The amount of commodity type k, which passes through the arch (i,j).

u_{ij} : The upper bound of the total flow for all commodities on the arc (i,j).

s_k : The number of resources of commodity k.

t_k : The number of destinations of commodity k.

b_k : The amount of commodity k, which should be sent from the source s_k to its destination t_k.

Additionally, assume that:

$$b_i^k=\begin{cases} b_k, & \text{If } i \text{ is the source node above the commodity } k\\ -b_k, & \text{If } i \text{ is the destination node above the commodity } k\\ 0, & \text{otherwise}\end{cases}$$

where b_i^k represents the flow balance of the commodity k in the node i. In fact, b_i^k indicates the demands or supplies of commodity k in node i. Note that p and q are given constants. Now, this problem can be formulated as follows:

<u>Problem1</u>: (FMCF)

$$Min\ \ f(x)=\frac{\sum_{k=1}^{K}\sum_{(i,j)\in A}c_{ij}^k x_{ij}^k+p}{\sum_{k=1}^{K}\sum_{(i,j)\in A}d_{ij}^k x_{ij}^k+q}$$

$$s.t.\quad \begin{cases} \sum_{(i,j)\in A}x_{ij}^k-\sum_{(j,i)\in A}x_{ji}^k=b_i^k, \quad \forall i\in N,\ \forall k\in K & (1)\\ \sum_{k=1}^{K}x_{ij}^k\leq u_{ij}, \quad \forall (i,j)\in A & (2)\\ x_{ij}^k\geq 0, \quad \forall (i,j)\in A,\ \forall k\in K & (3)\end{cases}$$

where the constraints (1) are balance constraints that express the conservation relations of the network flow. Constraints (2) are called bundle constraints that indicate the total flow of commodities in each arch (i,j) is smaller or equal to the given capacity. Constraints (3) are non-negativity constraints. On the other hand, the condition $\sum_{k=1}^{K}\sum_{(i,j)\in A} d_{ij}^k x_{ij}^k + q > 0$ is considered for the feasibility of the problem.

It is noteworthy that the objective function of the *problem 1* is a pseudo-linear function, and each local optimal solution of *problem 1* is a global optimal solution.

Concerning the certain multi-commodity flow, the capacity of each arc is a non-negative crisp value. However, capacities are often not constant in practice. If enough data is available, we may create a probability distribution of arc capacities. Nevertheless, it is sometimes not possible to obtain the probability distribution of arc capacities, due to, for example, the density, accidents, weather conditions, and even the novelty of the studied case in some cases, so that there is no access to historical data. Thus, introducing the belief degree to describe the distribution of arches capacities is the only hope in these situations, which is performed by specialists and experts. The uncertainty theory is a tool that can be used to model human uncertainty. This article uses this theory to confront the degree of belief. In effect, decision-makers often determine the level of confidence and then, find the optimal solution based on that degree. Hence, it seems appropriate to use the chance constraints to ensure the desired level of management and formulate an uncertain programming model to indicate the inherent uncertainty of the network.

To generalize the FMCF problem to its uncertain form, the parameters c_{ij}^k, d_{ij}^k, and u_{ij} are assumed independent uncertain variables, which are represented by ξ_{ij}^k, η_{ij}^k, and $u_{ij}^{\mathcal{U}}$, respectively. Therefore, the FMCF problem can be written in the form of an Uncertain Fractional Multi-Commodity Flow (UFMCF) problem as follow:

<u>Problem 2</u>: (UFMCF)

$$Min\ \ f(x;\xi,\eta)=\frac{\sum_{k=1}^{K}\sum_{(i,j)\in A}\xi_{ij}^k x_{ij}^k + p}{\sum_{k=1}^{K}\sum_{(i,j)\in A}\eta_{ij}^k x_{ij}^k + q}$$

$$s.t.\quad \begin{cases} \sum_{(i,j)\in A} x_{ij}^k - \sum_{(j,i)\in A} x_{ji}^k = b_i^k, & \forall i\in N,\ \forall k\in K \quad (4)\\ \sum_{k=1}^{K} x_{ij}^k \le u_{ij}^{\mathcal{U}}, & \forall (i,j)\in A \quad (5)\\ x_{ij}^k \ge 0, & \forall (i,j)\in A,\ \forall k\in K \quad (6) \end{cases}$$

Although the changes in the uncertain variables ξ_{ij}^k and η_{ij}^k are independent of the value of the certain variable x, the value of the objective function $f(x;\xi,\eta)$ varies by changing the values of these variables. Regarding the presence of an uncertain variable in this objective function, it is not possible to calculate a specified value for any given value x. It can be concluded that conventional methods fail to provide a method for determining the uncertain variable. On the other hand, constraints of the *problem 2* do not specify a certain feasible area, because of the constraint (5). Therefore, instead of this constraint, it is reasonable to use the belief degree that the total flow of commodities in each arc (i,j) is not greater than the mutual capacity of that arch with the minimum level of β, where β is a number between zero and one.

Based on the above-mentioned discussion, the UFMCF model is transformed into its certain form. For this purpose, we should consider the two main criteria of the expected value and critical value [16]. According to these criteria, there are three different certain forms called the expected value model, the chance-constrained-expected model, and chance-constrained mode. This article uses the chance-constrained model, which is used to ensure the certainty of uncertain constraints. Therefore, the belief function of any constraint should be greater than the confidence level that is pre-defined before the interval $(0,1]$. Now, the *problem 2* can be written as follows:

Problem 3:

$$\begin{aligned}
&Min \quad \bar{f} \\
&s.t.\begin{cases}
\mathcal{M}\left\{\dfrac{\sum_{k=1}^{K}\sum_{(i,j)\in A}\xi_{ij}^{k}x_{ij}^{k}+p}{\sum_{k=1}^{K}\sum_{(i,j)\in A}\eta_{ij}^{k}x_{ij}^{k}+q}\le \bar{f}\right\}\ge\alpha & \qquad (7)\\
\sum_{(i,j)\in A}x_{ij}^{k}-\sum_{(j,i)\in A}x_{ji}^{k}=b_{i}^{k}, \qquad \forall i\in N,\ \forall k\in K & \qquad (8)\\
\mathcal{M}\left\{\sum_{k=1}^{K}x_{ij}^{k}\le u_{ij}^{\mathcal{U}}\right\}\ge\beta, \qquad \forall (i,j)\in A & \qquad (9)\\
x_{ij}^{k}\ge 0, \qquad \forall (i,j)\in A, \forall k\in K & \qquad (10)
\end{cases}
\end{aligned}$$

where α and β are pre-determined parameters. In this model, the decision-maker hopes to obtain the least amount of $\bar{f}$ so that the uncertain variable $f\,(x;\xi,\eta)$ is less than or equal to $\bar{f}$ with a confidence level of $\alpha\in(0,1)$. The constraint (9) ensures that the total flow of commodities on each arch (i,j) is less than or equal to the mutual capacity with the minimum confidence level of β.

Definition 5. An x flow is feasible if it satisfies in constraints (7) − (10).

According to this definition, the solution that satisfies the constrain (7) can be interpreted as the provider of a flow that ensures keeping the degree of belief less than the level α. Further, the solution that satisfies the constraint (9), can be interpreted as a provider of flow passing through an arch, which ensures that the belief degree of the decision-maker is at least β.

The next theorem allows converting the latter problem into its certain equivalent form.

Theorem 4. Assume that the independent uncertainty variables ξ_{ij}^{k}, η_{ij}^{k}, and $u_{ij}^{\mathcal{U}}$ have regular uncertainty distributions Φ_{ij}^{k}, Ψ_{ij}^{k}, and Θ_{ij}, respectively. In this case, the model presented in the form of *problem 3* is equivalent to the following certain model:

Problem 4:

$$\begin{aligned}
&Min \quad \frac{\sum_{k=1}^{K}\sum_{(i,j)\in A}\Phi_{ij}^{k^{-1}}(\alpha)\,x_{ij}^{k}+p}{\sum_{k=1}^{K}\sum_{(i,j)\in A}\Psi_{ij}^{k^{-1}}(1-\alpha)\,x_{ij}^{k}+q} \\
&s.t.\begin{cases}
\sum_{(i,j)\in A}x_{ij}^{k}-\sum_{(j,i)\in A}x_{ji}^{k}=b_{i}^{k}, \qquad \forall i\in N,\ \forall k\in K & \qquad (11)\\
\sum_{k=1}^{K}x_{ij}^{k}\le\Theta_{ij}^{-1}(1-\beta), \qquad \forall (i,j)\in A & \qquad (12)\\
x_{ij}^{k}\ge 0, \qquad \forall (i,j)\in A,\ \forall k\in K & \qquad (13)
\end{cases}
\end{aligned}$$

Proof. We first show that the constraint (7) of the problem (3) is equivalent to the objective function of the *problem 4*. To this aim, assume that the following certain variable

$$\xi = \frac{\sum_{k=1}^{K} \sum_{(i,j)\in A} \xi_{ij}^{k} x_{ij}^{k} + p}{\sum_{k=1}^{K} \sum_{(i,j)\in A} \eta_{ij}^{k} x_{ij}^{k} + q}$$

has an uncertainty distribution of Υ. By considering the definition (3), we have:

$$\mathcal{M}\left\{ \frac{\sum_{k=1}^{K} \sum_{(i,j)\in A} \xi_{ij}^{k} x_{ij}^{k} + p}{\sum_{k=1}^{K} \sum_{(i,j)\in A} \eta_{ij}^{k} x_{ij}^{k} + q} \le \overline{f} \right\} \ge \alpha \Leftrightarrow \Upsilon\left(\overline{f}\right) \ge \alpha$$

Now, since $\xi = \frac{\sum_{k=1}^{K} \sum_{(i,j)\in A} \xi_{ij}^{k} x_{ij}^{k} + p}{\sum_{k=1}^{K} \sum_{(i,j)\in A} \eta_{ij}^{k} x_{ij}^{k} + q}$ is strictly increasing relative to ξ_{ij}^{k} and strictly decreasing relative to η_{ij}^{k}, according to the theorem (2), we have:

$$\Upsilon\left(\overline{f}\right) \ge \alpha \;\Leftrightarrow\; \overline{f} \ge \Upsilon^{-1}(\alpha) \;\Leftrightarrow\; \overline{f} \ge \frac{\sum_{k=1}^{K} \sum_{(i,j)\in A} \Phi_{ij}^{k^{-1}}(\alpha) x_{ij}^{k} + p}{\sum_{k=1}^{K} \sum_{(i,j)\in A} \Psi_{ij}^{k^{-1}}(1-\alpha) x_{ij}^{k} + q} \tag{14}$$

Next, we prove that the constraint (9) of the problem (3) is equivalent to the constraint (12) of the problem (4). Suppose $u_{ij}^{\mathcal{U}}$ has an uncertainty distribution Θ_{ij}. Given the Definition 3 and Theorem 2:

$$\begin{aligned} \mathcal{M}\left\{ \sum_{k=1}^{K} x_{ij}^{k} \le u_{ij}^{\mathcal{U}} \right\} \ge \beta \;\Leftrightarrow\; & 1 - \mathcal{M}\left\{ \sum_{k=1}^{K} x_{ij}^{k} > u_{ij}^{\mathcal{U}} \right\} \ge \beta \\ \Leftrightarrow\; & 1 - \Theta_{ij}\left(\sum_{k=1}^{K} x_{ij}^{k} \right) \ge \beta \\ \Leftrightarrow\; & \Theta_{ij}\left(\sum_{k=1}^{K} x_{ij}^{k} \right) \le 1 - \beta \\ \Leftrightarrow\; & \sum_{k=1}^{K} x_{ij}^{k} \le \Theta_{ij}^{-1}(1-\beta) \end{aligned} \tag{15}$$

Thus, the theorem is proved according to relations (14) and (15).

The model presented in problem (4) is a factional programming problem in a certain environment. By comparing problem (1) and problem (4), it can be found that $\Phi_{ij}^{k^{-1}}(\alpha)$ and $\Psi_{ij}^{k^{-1}}(1-\alpha)$ are the coefficients for the arch cost (i,j) for commodity k and $\Theta_{ij}^{-1}(1-\beta)$ is the mutual capacity of that arch.

Now, using the Charnes and Cooper variable transformation [6]:

$$t=\left(\sum_{k=1}^{K}\sum_{(i,j)\in A}\Psi_{ij}^{k^{-1}}\left(1-\alpha\right)x_{ij}^{k}+q\right)^{-1},\ y_{ij}^{k}=\left(\sum_{k=1}^{K}\sum_{(i,j)\in A}\Psi_{ij}^{k^{-1}}\left(1-\alpha\right)x_{ij}^{k}+q\right)^{-1}x_{ij}^{k}$$

It is possible to rewrite the problem (4) as follows:

Problem 5:

$$Min\ \sum_{k=1}^{K}\sum_{(i,j)\in A}\Phi_{ij}^{k^{-1}}\left(\alpha\right)y_{ij}^{k}+pt$$

$$s.t.\ \begin{cases}\sum_{(i,j)\in A}y_{ij}^{k}-\sum_{(j,i)\in A}y_{ji}^{k}=b_i^k t, & \forall i\in N,\ \forall k\in K \quad (16)\\ \sum_{k=1}^{K}y_{ij}^{k}\le t\Theta_{ij}^{-1}\left(1-\beta\right), & \forall (i,j)\in A \quad (17)\\ \sum_{k=1}^{K}\sum_{(i,j)\in A}\Psi_{ij}^{k^{-1}}\left(1-\alpha\right)y_{ij}^{k}+qt=1 & (18)\\ y_{ij}^{k}\ge 0,\ t\ge 0 & \forall (i,j)\in A,\ \forall k\in K \quad (19)\end{cases}$$

The following lemma explains the relationship between the problems (4) and (5).

Lemma 1. (See [23]) If $x_{ij}^{k},\ \forall(i,j)\in A,\forall k\in K$ is the optimal solution of the problem (4), then $\left(y_{ij}^{k},t\right)$ is an optimal solution for problem (5). On the other hand, if $\left(y_{ij}^{k},t\right)$ is an optimal solution for the problem (5), then $x_{ij}^{k}=\frac{y_{ij}^{k}}{t}$ is an optimal solution for the problem (4). In both cases, the optimal values are equal.

Now, we express the process of solving the UFMCF problem in the form of the following algorithm.

UFMCF Algorithm:

Step 1: Ask the decision-maker to determine the values α and β based on his/her subjective experiences.

Step 2: Calculate $\Phi_{ijk}^{-1}\left(\alpha\right)$, $\Psi_{ijk}^{-1}\left(1-\alpha\right)$ and $\Theta_{ijk}^{-1}\left(1-\beta\right)$, $\forall(i,j)\in A,\forall k\in K$.

Step 3: Set the coefficients of costs c_{ijk} and d_{ijk} and the mutual capacity u_{ij} equal to $\Phi_{ijk}^{-1}\left(\alpha\right)$, $\Psi_{ijk}^{-1}\left(1-\alpha\right)$ and $\Theta_{ijk}^{-1}\left(1-\beta\right)$, $\forall(i,j)\in A,\forall k\in K$, respectively, and formulate the FMCF problem according to the corresponding certain network.

Step 4: Adopt the Charnes and Cooper variable transformation approach for the problem formulated in **Step 3** and solve the corresponding linear problem.

The next section provides an example to describe the algorithm.

4. Numerical example

Consider the network shown in Figure 3, which contains 7 nodes [2]. We want to move two commodities from nodes 1 and 2 to nodes 5, 6, and 7. It is assumed that there are no direct paths from the origin nodes to the destination nodes. Instead, the paths are connected by hubs in nodes 3 and 4 (See Figure 2). The values listed next to the nodes indicate the supply and demand in the tone unit. In the following, the decision-maker seeks to obtain basic data such as cost and profit per unit of commodities and the mutual capacity of the arch. However, we generally fail to obtain these values accurately in the real world. Therefore, based on personal experience, we assume that these values have a zigzag uncertainty distribution, the details of which are given in Table 1.

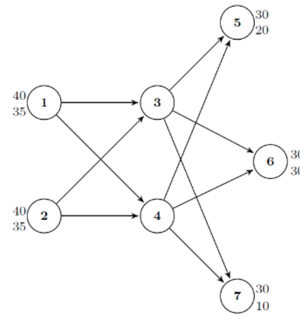

Figure 2. The network proposed for the MCF problem of the example

Table 1. Data related to the example

Arc(i,j)	Φ_{ij}^{k}		Ψ_{ij}^{k}		Θ_{ij}
	$k=1$	$k=2$	$k=1$	$k=2$	
(1,3)	$Z(5,10,20)$	$Z(10,20,30)$	$Z(2,3,5)$	$Z(5,6,7)$	$Z(15,25,35)$
(1,4)	$Z(8,15,25)$	$Z(18,28,40)$	$Z(3,4,6)$	$Z(10,12,18)$	$Z(20,25,30)$
(2,3)	$Z(7,15,20)$	$Z(12,18,25)$	$Z(6,8,10)$	$Z(10,18,20)$	$Z(10,20,30)$
(2,4)	$Z(4,8,10)$	$Z(8,15,20)$	$Z(2,5,10)$	$Z(5,8,10)$	$Z(15,20,25)$
(3,5)	$Z(1,5,10)$	$Z(3,8,12)$	$Z(1,3,5)$	$Z(1,3,5)$	$Z(20,25,30)$
(3,6)	$Z(5,10,15)$	$Z(8,12,18)$	$Z(2,4,6)$	$Z(5,6,7)$	$Z(10,20,25)$
(3,7)	$Z(8,15,20)$	$Z(10,20,30)$	$Z(5,8,10)$	$Z(8,12,15)$	$Z(5,10,20)$
(4,5)	$Z(3,6,12)$	$Z(8,10,15)$	$Z(2,3,4)$	$Z(5,8,10)$	$Z(10,15,25)$
(4,6)	$Z(4,10,15)$	$Z(6,12,18)$	$Z(2,4,6)$	$Z(4,10,15)$	$Z(15,25,40)$
(4,7)	$Z(4,12,18)$	$Z(6,10,15)$	$Z(2,3,5)$	$Z(2,5,8)$	$Z(20,30,35)$

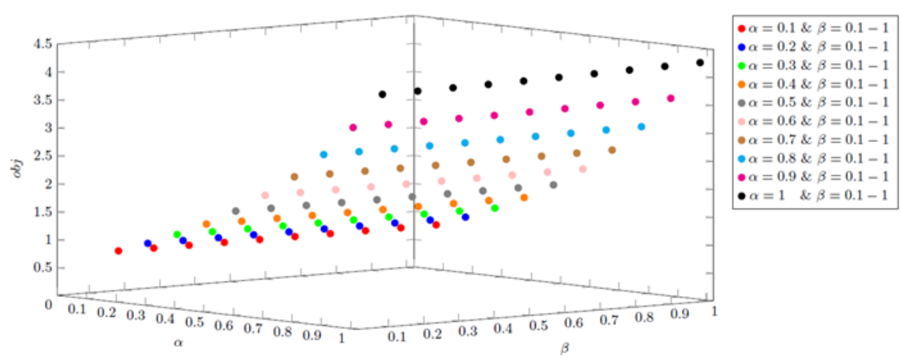

Figure 3. The optimal objective function for example

In fact, we want to determine how much of each commodity should be routed with each arc to minimize the total cost of the movement and maximize the overall profit. By using the data in Table 1, the problem is solved by coding a linear problem corresponding to the *problem 5* in GAMS solver and implementing the code on a PC or an Intel 2GHz Processor and 4.00 GB RAM. Table 2 reports the results of solving this problem for different amounts for α and β. The values of the optimal objective function for this problem are shown in Figure 3.

Table 2. The results obtained for the example

Exp.	α	β	x_{13}^1	x_{13}^2	x_{14}^1	x_{14}^2	x_{23}^1	x_{23}^2	x_{241}^1	x_{24}^2	x_{35}^1	x_{35}^2
1	0.1	0.1	40	0	0	35	0	25	50	0	30	20
2	0.2	0.2	40	0	0	35	0	25	50	0	30	15
3	0.3	0.3	40	0	0	35	0	25	50	0	10	15
4	0.4	0.4	40	0	0	35	0	25	50	0	10	15
5	0.5	0.5	40	0	0	35	0	25	50	0	10	15
6	0.6	0.6	40	0	0	35	0	25	50	0	10	0
7	0.7	0.7	40	0	0	35	50	25	0	0	30	0
8	0.8	0.8	40	0	0	35	50	25	0	0	30	0
9	0.9	0.9	40	0	0	35	50	25	0	0	30	0
10	1	1	10	0	30	35	50	25	0	0	0	0
Exp.	α	β	x_{36}^1	x_{36}^2	x_{37}^1	x_{37}^2	x_{45}^1	x_{45}^2	x_{46}^1	x_{46}^2	x_{47}^1	x_{47}^2
1	0.1	0.1	0	0	10	5	0	0	30	30	20	5
2	0.2	0.2	0	0	10	10	0	5	30	30	20	0
3	0.3	0.3	0	0	30	10	20	5	30	30	0	0
4	0.4	0.4	0	0	30	10	20	5	30	30	0	0
5	0.5	0.5	0	0	30	10	20	5	30	30	0	0
6	0.6	0.6	0	15	30	10	20	20	30	15	0	0
7	0.7	0.7	30	15	30	10	0	20	0	15	0	0
8	0.8	0.8	30	15	30	10	0	20	0	15	0	0
9	0.9	0.9	30	15	30	10	0	20	0	15	0	0
10	1	1	30	15	30	10	30	20	0	150	0	0

As you can see in Figures 3, the optimal value of the objective function is generally increased by increasing the value of α and β. The best solution is occurred, based on the presented model in $\alpha = 0.1$ and $\beta = 0.1$. Generally $\alpha \in (\frac{1}{2}, 1)$ is a constant that determined by the decision-maker. The results obtained from this example show that the proposed model can be very efficient if it has proper inputs. As you can, for different probability levels, different but close solutions are obtained. On the other hand, in this model, we make decisions whose feasibility is guaranteed as much as possible. In fact, constraints are not violated except in emergency and unpredictable situations. It is worth noting that multi-objective programming approaches can be used for solving this problem. However, there is no optimal solution that simultaneously optimizes objectives in most cases. Therefore, optimizing the ratio of the two objectives is a simpler way in this regard.

5. Conclusion

The present study sought to examine the uncertainty version of the fractional multi-commodity flow problem. This research studied this type of problem under the degree of belief based on uncertainty for the first time. In this way, cost coefficients and capacities of this problem were considered uncertain. The problem was transformed into a fractional multi-commodity flow problem using a chance-constrained programming approach. Then, we used the Charnes and Cooper variable transformation approach to convert the nonlinear objective function to its linear form. Generally, dealing with unknown factors in the uncertainty network is regarded as one of the most important advantages of using uncertainty theory, which was a highly motivating factor for studying uncertainty in such problems. For future research, we seek to study or generalize solution algorithms for fractional multi-commodity flow problems by adopting the results of this research.

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